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Analyzing Feedback Interconnections of Maximal Monotone Systems Using Dissipativity Approach

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Abstract: We consider interconnections of two dynamical systems in feedback configuration. The dynamics of the individual systems are modeled by a differential inclusion, and the corresponding set-valued mapping is (anti-) maximal monotone with respect to the state of the system for each fixed value of the external signal that defines the interconnection. We provide conditions on these mappings under which the dynamics of the resulting interconnected system are (anti-) maximal monotone. An interpretation of our main result is provided: firstly, by considering dynamical systems defined by the gradient of a saddle function, and secondly, by considering an interconnection of incrementally passive systems. In the same spirit, when we associate more structure to the individual systems by considering linear complementarity systems, we allow for more flexibility in describing the interconnections and derive more specific sufficient conditions in terms of system matrices that result in the overall system being described by (anti-) maximal monotone operator.

Keywords: Maximal monotone mappings, Interconnections, Passivity, Linear complementarity systems.

1. INTRODUCTION

In the study of dynamical systems, one can often break-down the complexity of analysis by viewing the dynamics as an interconnection of more than one subsystems coupled to each other. This way, the analysis of overall system can be decomposed into studying relevant properties of the smaller sized subsystems. In this regard, depending upon the system dynamics and the nature of interconnections, several tools and methods have been developed in the literature. Our focus in this article is on studying the interconnection of certain nonsmooth dynamical systems, modeled by differential inclusions with maximal monotone mappings, and we use tools from dissipativity theory and variational analysis for our purposes.

The first of these tools, dissipativity, comes from the pioneering work in (Willems, 1972) and continues to develop as a fundamental building block for analysis and design of control systems, as evident by several monographs. We see its utility in control design (Sepulchre et al., 1997; Ortega et al., 2013), analyzing interconnections (Arcak and Martins, 2021; Brogliato et al., 2020), input-output gains (van der Schaft, 2017), robustness (Scherer, 2022), optimal control (Grüne, 2021) and many more system-theoretic properties.

Our second set of tools is primarily motivated by the class of dynamical systems considered in this paper. In partic-

ular, we consider ordinary differential equations coupled with static set-valued relations, which under certain structural assumptions can be embedded within differential inclusions having maximal monotone operators. Such systems are particularly useful in modeling trajectories with constraints, or the evolution of optimization algorithms (Schumacher, 2004; Attouch and Peypouquet, 2019). An extensive overview of such systems has been presented in (Brogliato and Tanwani, 2020), where one sees the utility of tools from variational analysis (Rockafellar and Wets, 1998) and set-valued analysis (Aubin and Frankowska, 1990) for understanding the basic system properties. Most of the earlier works on the aforementioned system class have focused on proving existence of solutions. The book (Brézis, 1973) nicely captures the results for autonomous differential inclusions with static maximal monotone operators. On the other hand, studying existence of solutions for differential inclusions driven by an external signals continues to attract attention of the researchers; one may refer to (Moreau, 1977; Monteiro Marques, 1993) for earlier results and a recent article (Camlibel et al., 2022) generalizes some of these results.

All these advancements on the mathematical side about proving the existence of solutions have naturally contributed to a better understanding of the system behavior. Researchers in systems theory have therefore been able to study the relevance of these models for certain physical devices (van der Schaft and Schumacher, 1998), and develop algorithms for numerical simulations (Acary et al., 2011). In more recent works, we also see an emphasis on studying

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system design problems, such as, observers (Brogliato and Heemels, 2009; Tanwani et al., 2014), and output regulation (Tanwani et al., 2018). At an abstract level, in most of these design problems, as well as in certain optimization algorithms, one sees an interconnection of two nonsmooth dynamical systems and the well-posedness and stability of overall system is often concluded by showing that the overall interconnected system is described by a maximal monotone operator with a negative sign.

Motivated by this observation where system design problems, and certain optimization algorithms, involve interaction between different nonsmooth dynamical systems, we consider a somewhat abstract, but related, problem of analyzing such interactions. In our problem setting, we start with two dynamical systems, each of which is described by a differential inclusion. Each of these differential inclusions involves a set-valued mapping that depends on the two arguments: first one being the state of system for which the differential inclusion describes the set of possible vector fields, and the second one being the variable that describes the interconnection. We assume that this mapping is maximal monotone with respect to state of the system, for each fixed value of the coupling variable. With appropriate regularity assumptions on each of these mappings, we can show that the overall system is indeed maximal monotone. With similar ideas, we also study interconnections of linear complementarity systems, which can be written as differential inclusions with maximal monotone mappings. However, the particular structure allows us to define more generic interconnections and we provide sufficient conditions for the interconnected system to be maximal monotone.

2. OVERVIEW AND MOTIVATION

The class of dynamical systems considered in this paper are broadly described by

$$\dot{x} \in -\mathcal{M}(x) \quad (1)$$

where $\mathcal{M} : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is *maximal monotone*, that is, there exists $\alpha \geq 0$ such that for every $(x_i^*, x_i) \in \text{graph}(\mathcal{M})$, $i = 1, 2$, it holds that

$$\langle x_2^* - x_1^*, x_2 - x_1 \rangle \geq \alpha \|x_1 - x_2\|^2, \quad (2)$$

and moreover, for every $(x_1^*, x_1) \notin \text{graph}(\mathcal{M})$, there exists $(x_2^*, x_2) \in \text{graph}(\mathcal{M})$ such that

$$\langle x_2^* - x_1^*, x_2 - x_1 \rangle < 0. \quad (3)$$

If $\alpha > 0$, we call \mathcal{M} *strongly maximal monotone*. The property (2) is referred to as *monotonicity*, and the property in (3) is called *maximality*. It is useful to recall that the *effective domain* of \mathcal{M} is defined as $\text{dom}(\mathcal{M}) := \{x \in \mathbb{R}^n \mid \mathcal{M}(x) \neq \emptyset\}$.

The following elegant result describes some properties of the solution of (1) and appears in (Brézis, 1973, Theorem 3.1).

Theorem 1. Consider system (1) with \mathcal{M} being maximal monotone. For ever $x_0 \in \text{dom}(\mathcal{M})$, there exists a unique solution $x : [0, \infty) \rightarrow \text{dom}(\mathcal{M})$, with $x(0) = x_0$. Moreover, if x_1, x_2 represent two solutions of system (1), then

$$\|x_2(t) - x_1(t)\|^2 \leq e^{-\alpha t} \|x_2(0) - x_1(0)\|^2. \quad (4)$$

In other words, if system (1) has an equilibrium point then the equilibrium is Lyapunov stable. In case, \mathcal{M} is strongly

maximal monotone, the equilibrium point is exponentially stable.

2.1 Motivation

Our interest in this paper lies in studying interconnections of two dynamical systems of the form (1) and is motivated by the examples of the following form:

Example 1. Consider the constrained optimization problem: $\min_{x \in \mathcal{X}} g(x)$, where $\mathbb{R}^n \supseteq \mathcal{X} \triangleq \Omega \cap \{x \mid Ax \leq b\}$, for some $A \in \mathbb{R}^{m \times n}$. We assume that $g : \mathbb{R}^n \rightarrow \mathbb{R}$ is strictly convex and $\Omega \subseteq \mathbb{R}^n$ is convex, hence \mathcal{X} is convex. Using the Karush-Kuhn-Tucker (KKT) conditions, the optimization problem is solved by finding a pair (x^*, λ^*) such that, $0 \in \nabla g(x^*) + \mathcal{N}_\Omega(x^*) + A^\top \lambda^*$, and $0 \in b + \mathcal{N}_{\mathbb{R}_+^m}(\lambda^*) - Ax^*$, where the notation $\mathcal{N}_X(x)$ is used to denote the outward normal cone to a closed convex set X at $x \in X$. We can study this problem by considering the solution of the following dynamical system and analyzing the convergence of the state variables to the origin,

$$\begin{aligned} \dot{x} &\in -(\nabla g(x) + \mathcal{N}_\Omega(x)) - A^\top \lambda \\ \dot{\lambda} &\in -(b + \mathcal{N}_{\mathbb{R}_+^m}(\lambda)) + Ax. \end{aligned} \quad (5)$$

Note that, due to convexity assumption, the operators $(\nabla g(x) + \mathcal{N}_\Omega(x))$ and $(b + \mathcal{N}_{\mathbb{R}_+^m}(\lambda))$ are both maximal monotone. System (5) therefore defines an interconnection of x - and λ -dynamics, and each of these dynamics contain a maximal monotone operator and a coupling term. Using the properties of the operators that describe subsystem dynamics, and the interconnection term, we would like to know if the overall system is maximal monotone. For this example, the answer is indeed affirmative as we will see from later developments.

2.2 Outline of the paper

Motivated by this example, in Section 3, we study interconnected dynamical systems with (anti-) maximal monotone mappings:

$$\dot{y} \in -\mathcal{M}_y(y, z), \quad \dot{z} \in -\mathcal{M}_z(y, z)$$

where $\mathcal{M}_y(\cdot, z) : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is (strongly) maximal monotone for each $z \in \mathbb{R}^m$, and $\mathcal{M}_z(y, \cdot)$ is (strongly) maximal monotone for each $y \in \mathbb{R}^n$. We will develop conditions in this note under which the mapping $\mathcal{M} := (\mathcal{M}_y, \mathcal{M}_z)$ is (strongly) maximal monotone. The formal description of such systems appears in Section 3, and we provide conditions on the component dynamics and coupling terms such that the overall system is (anti-) maximal monotone. This allows us to invoke Theorem 1, showing that the trajectories of the interconnected system exponentially converge to each other. As an illustration of our conditions, we consider some well-studied systems from the literature where our results can be applied.

In Section 4, we consider a particular class of maximal monotone systems called the *Linear Complementarity Systems* (LCS), which has found several applications and are computational friendly. The dynamics of such systems are described by

$$\dot{x} = Ax + B\lambda \quad (6a)$$

$$w = Cx + D\lambda \quad (6b)$$

$$0 \leq \lambda \perp w \geq 0 \quad (6c)$$

with state $x \in \mathbb{R}^n$ and the complementarity variables $\lambda, w \in \mathbb{R}^m$. The notation $0 \leq \lambda \perp w \geq 0$ encodes three static relations, that is, $w \geq 0$, $\lambda \geq 0$ and $\langle \lambda, w \rangle = 0$, and the inequalities are interpreted as componentwise. Using some identities from convex analysis, it is possible to rewrite system (6) as a differential inclusion:

$$\dot{x} \in -\mathcal{F}(x) \triangleq Ax - B(D + \mathcal{N}_{\mathbb{R}_+^m}^{-1})^{-1}(Cx).$$

The following result, from (Camlibel and Schumacher, 2016), uses passivity of the quadruple matrix (A, B, C, D) to rewrite the LCS as a differential inclusion with a maximal monotone operator.

Theorem 2. Suppose that (A, B, C, D) in (6) is passive with storage function $x \mapsto x^\top x$, and $\text{Im}(C) \cap \text{rint}(\text{Im}(\mathcal{N}_{\mathbb{R}_+^m}^{-1} + D)) \neq \emptyset$. Then, the mapping $\mathcal{F}(\cdot)$ is maximal monotone.

The interconnections of LCS involve coupling of (x, λ) variables. Because of the additional coupling due to the complementarity variables λ , these interconnections need to be studied differently than the approach adopted in Section 3. In our treatment, we provide sufficient conditions involving passivity of the matrix quadruples associated to each subsystem, which allow us to establish maximal monotonicity of the overall system.

3. INTERCONNECTED SYSTEMS WITH MAXIMAL MONOTONE MAPPINGS

We now consider the interconnections of the form:

$$\dot{y} \in -\mathcal{M}_y(y, z) \quad (7a)$$

$$\dot{z} \in -\mathcal{M}_z(y, z) \quad (7b)$$

where $\mathcal{M}_y(\cdot, z) : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$, and $\mathcal{M}_z(y, \cdot) : \mathbb{R}^m \rightrightarrows \mathbb{R}^m$ define set-valued mappings for a given $z \in \mathbb{R}^m$, and a given $y \in \mathbb{R}^n$, respectively. We impose the following conditions on these mappings:

(M1) There exist $\alpha_y \geq 0$ and a function $\gamma_y : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}$ such that, for every $y_1^* \in \mathcal{M}_y(y_1, z_1)$, $y_2^* \in \mathcal{M}_y(y_2, z_2)$, we have

$$\begin{aligned} \langle y_1^* - y_2^*, y_2 - y_1 \rangle &\leq -\alpha_y \|y_2 - y_1\|^2 \\ &\quad + \gamma_y(y_1, z_2) - \gamma_y(y_1, z_1) \\ &\quad + \gamma_y(y_2, z_1) - \gamma_y(y_2, z_2). \end{aligned} \quad (8)$$

(M2) There exist $\alpha_z \geq 0$ and a function $\gamma_z : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}$ such that, for every $z_1^* \in \mathcal{M}_z(y_1, z_1)$, $z_2^* \in \mathcal{M}_z(y_2, z_2)$, we have

$$\begin{aligned} \langle z_1^* - z_2^*, z_2 - z_1 \rangle &\leq -\alpha_z \|z_2 - z_1\|^2 \\ &\quad + \gamma_z(y_2, z_1) - \gamma_z(y_1, z_1) \\ &\quad + \gamma_z(y_1, z_2) - \gamma_z(y_2, z_2). \end{aligned} \quad (9)$$

(M3) For each $z \in \mathbb{R}^m$, and every $(y_1^*, y_1) \in \mathbb{R}^n \times \mathbb{R}^n$ that satisfies $y_1^* \notin \mathcal{M}_y(y_1, z)$, there exists (y_2^*, y_2) with $y_2^* \in \mathcal{M}_y(y_2, z)$ such that

$$\langle y_2^* - y_1^*, y_2 - y_1 \rangle < 0.$$

(M4) For each $y \in \mathbb{R}^n$, and every $(z_1^*, z_1) \in \mathbb{R}^m \times \mathbb{R}^m$ that satisfies $z_1^* \notin \mathcal{M}_z(y, z_1)$, there exists (z_2^*, z_2) with $z_2^* \in \mathcal{M}_z(y, z_2)$ such that

$$\langle z_2^* - z_1^*, z_2 - z_1 \rangle < 0.$$

(M5) For each $z \in \mathbb{R}^m$, $\text{dom}(\mathcal{M}_y(\cdot, z)) \neq \emptyset$, and for each $y \in \mathbb{R}^n$, $\text{dom}(\mathcal{M}_z(y, \cdot)) \neq \emptyset$.

The conditions **(M1)** and **(M2)** basically describe the (strong) monotonicity of the mappings $\mathcal{M}_y(\cdot, z)$ and $\mathcal{M}_z(y, \cdot)$, respectively, for a given $z \in \mathbb{R}^m$, $y \in \mathbb{R}^n$. We will present some case studies in this section which would clarify these inequalities further. At this moment, in **(M1)**, it could be readily seen that if $y_i^* \in \mathcal{M}_y(y_i, z)$ for some $z \in \mathbb{R}^m$ and $i = 1, 2$, then the inequality (8) reduces to the inequality in (2). The same observation can be made for **(M2)**. Conditions **(M3)** and **(M4)** basically refer to the maximality of $\mathcal{M}_y(\cdot, z)$ and $\mathcal{M}_z(y, \cdot)$, respectively, for a given $z \in \mathbb{R}^m$, $y \in \mathbb{R}^n$. In this paper, we use **(M5)** for simplifying some technical arguments used in the proof of the main result and it would be natural to think about relaxing this assumption in future work.

Theorem 3. Consider the interconnected dynamical system (7) under the conditions **(M1)**, **(M2)**, **(M3)**, **(M4)**, **(M5)**. Suppose that the function $\gamma_y + \gamma_z$ is bilinear, so that there exist $C_\gamma \geq 0$ that satisfies

$$\|(\gamma_y + \gamma_z)(y, z)\| \leq C_\gamma (\|y\|^2 + \|z\|^2), \quad (10)$$

for all $(y, z) \in \mathbb{R}^n \times \mathbb{R}^m$. If, it holds that,

$$\min\{\alpha_y, \alpha_z\} \geq C_\gamma \quad (11)$$

then $(\mathcal{M}_y, \mathcal{M}_z) \triangleq \mathcal{M} : \mathbb{R}^{n+m} \rightrightarrows \mathbb{R}^{n+m}$ in system (7) is maximal monotone. If the inequality in (11) is strict, then \mathcal{M} is strongly maximal monotone.

Proof. Let $x_1 \triangleq (y_1, z_1)$, and $x_2 \triangleq (y_2, z_2)$, where $y_i \in \text{dom} \mathcal{M}_y(\cdot, z_i)$ and $z_i \in \text{dom} \mathcal{M}_z(y_i, \cdot)$, for $i = 1, 2$. Also, let $\alpha \triangleq \min\{\alpha_y, \alpha_z\}$, and $\gamma \triangleq \gamma_y + \gamma_z$. In the sequel, we will prove monotonicity and maximality separately.

Monotonicity: To check the monotonicity of the mapping $\mathcal{M}(\cdot)$, we consider $x_1^* \in \mathcal{M}(x_1)$ and $x_2^* \in \mathcal{M}(x_2)$. We can then write $x_1^* = (y_1^*, z_1^*)$ with $y_1^* \in \mathcal{M}_y(y_1, z_1)$, and $z_1^* \in \mathcal{M}_z(y_1, z_1)$. Similarly, we can write $x_2^* = (y_2^*, z_2^*)$ with $y_2^* \in \mathcal{M}_y(y_2, z_2)$, and $z_2^* \in \mathcal{M}_z(y_2, z_2)$. It then follows that

$$\begin{aligned} \langle x_2^* - x_1^*, x_2 - x_1 \rangle &= \langle y_2^* - y_1^*, y_2 - y_1 \rangle + \langle z_2^* - z_1^*, z_2 - z_1 \rangle \\ &\geq \alpha_y \|y_2 - y_1\|^2 + \alpha_z \|z_2 - z_1\|^2 + \gamma(y_1, z_1) - \gamma(y_1, z_2) \\ &\quad + \gamma(y_2, z_2) - \gamma(y_2, z_1) \\ &= \alpha_y \|y_2 - y_1\|^2 + \alpha_z \|z_2 - z_1\|^2 + \gamma(y_1 - y_2, z_1 - z_2) \\ &\geq \alpha \|y_2 - y_1\|^2 + \alpha \|z_2 - z_1\|^2 \\ &\quad - C_\gamma (\|y_1 - y_2\|^2 + \|z_1 - z_2\|^2) \\ &\geq (\alpha - C_\gamma) (\|y_2 - y_1\|^2 + \|z_2 - z_1\|^2) \end{aligned}$$

and hence, under the condition (11), the mapping $\mathcal{M}(\cdot)$ is monotone, or strongly monotone if the inequality in (11) is strict.

Maximality: Next, we use **(M3)**, **(M4)** and **(M5)** to show the maximality of \mathcal{M} . Towards this end, fix any $(y_1, z_1) \in \mathbb{R}^n \times \mathbb{R}^m$ and any $(y_1^*, z_1^*) \in \mathbb{R}^n \times \mathbb{R}^m$ such that $(y_1^*, z_1^*) \notin \mathcal{M}(y_1, z_1)$. One of the two statements is true: either $y_1^* \notin \mathcal{M}_y(y_1, z_1)$, or $z_1^* \notin \mathcal{M}_z(y_1, z_1)$. For definiteness, let us suppose the former, that is, $y_1^* \notin \mathcal{M}_y(y_1, z_1)$. From **(M5)**, $\text{dom}(\mathcal{M}_y(\cdot, z_1)) \neq \emptyset$. It then follows from **(M3)** that there exists $(y_2^*, y_2) \in \text{graph}(\mathcal{M}_y(\cdot, z_1))$ such that

$$\langle y_2^* - y_1^*, y_2 - y_1 \rangle < 0.$$

Consider the point (y_2, z_1) , and once again, using **(M5)**, we have that $\text{dom}(\mathcal{M}_z(y_2, \cdot)) \neq \emptyset$. If $z_1 \in \text{dom}(\mathcal{M}_z(y_2, \cdot))$, then we pick $z_2 = z_1$ and $z_2^* \in \mathcal{M}_z(y_2, z_1)$, so that

$$\langle z_2^* - z_1^*, z_2 - z_1 \rangle = 0.$$

Else, if $z_1 \notin \text{dom}(\mathcal{M}_z(y_2, \cdot))$, then using **(M4)**, there exists $(z_2^*, z_2) \in \text{graph}(\mathcal{M}_z(y_2, \cdot))$ such that

$$\langle z_2^* - z_1^*, z_2 - z_1 \rangle < 0.$$

We thus have $(y_2^*, z_2^*) \in \mathcal{M}(y_2, z_2)$ such that

$$\langle y_2^* - y_1^*, y_2 - y_1 \rangle + \langle z_2^* - z_1^*, z_2 - z_1 \rangle < 0,$$

which shows the maximality of the mapping $\mathcal{M}(\cdot)$. \square

3.1 Saddle-Point Dynamics

As a first instance of the illustration of the interconnection considered in (7), and the result presented in Theorem 3, let us consider a saddle function $f : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R} \cup \{-\infty, +\infty\}$, and the dynamics defined by its subdifferential. In particular, we assume that $f(\cdot, \cdot)$ satisfies the following:

- (SF1)** For each $z \in \mathbb{R}^m$, the function $f(\cdot, z)$ is concave, and upper semicontinuous.
- (SF2)** For each $y \in \mathbb{R}^n$, the function $f(y, \cdot)$ is convex, and lower semicontinuous.
- (SF3)** The function $f(\cdot, \cdot)$ is finite-valued everywhere on $\mathbb{R}^n \times \mathbb{R}^m$.

For such functions, we let $\partial_z f(y, z)$ denote the subdifferential of the convex function $f(y, \cdot)$ at $z \in \mathbb{R}^m$, which is well-defined due to **(SF2)**. Similarly, for each $z \in \mathbb{R}^m$, we let $\partial_y f(y, z)$ denote the superdifferential of the concave function $f(\cdot, z)$ at $y \in \mathbb{R}^n$, which is well-defined due to **(SF1)**. Condition **(SF3)** is somewhat restrictive in the context of saddle functions as considered in (Rockafellar, 1970), but it simplifies the analysis. We can now introduce the saddle-point dynamics as follows (Goebel, 2017):

$$\dot{y} \in \partial_y f(y, z) \quad (12a)$$

$$\dot{z} \in -\partial_z f(y, z) \quad (12b)$$

Proposition 4. Consider the dynamical system (12) with $f(\cdot, \cdot)$ satisfying **(SF1)**, **(SF2)**, **(SF3)**. Then, the set-valued mapping $(x, y) \mapsto (-\partial_y f(y, z), \partial_z f(y, z))$ is maximal monotone.

Proof. The proof basically relies on showing that the conditions **(M1)**–**(M5)** hold, and so the result follows due to Theorem 3.

Consider the points $(y_i, z_i) \in \mathbb{R}^n \times \mathbb{R}^m$ and $(y_i^*, z_i^*) \in -\partial_y f(y_i, z_i) \times \partial_z f(y_i, z_i)$. By definition,

$$\langle y_1^*, y - y_1 \rangle - f(y_1, z_1) \leq -f(y, z_1), \quad \forall y \in \mathbb{R}^n$$

$$\langle y_2^*, y - y_2 \rangle - f(y_2, z_2) \leq -f(y, z_2), \quad \forall y \in \mathbb{R}^n$$

This results in

$$\langle y_1^* - y_2^*, y_2 - y_1 \rangle \leq f(y_1, z_1) - f(y_1, z_2) - f(y_2, z_1) + f(y_2, z_2) \quad (13)$$

so that **(M1)** holds with $\alpha_y = 0$ and $\gamma_y(\cdot, \cdot) = -f(\cdot, \cdot)$.

Similarly, using the definition of the subdifferential of the convex function, we get

$$\langle z_1^*, z - z_1 \rangle + f(y_1, z_1) \leq f(y_1, z), \quad \forall z \in \mathbb{R}^m$$

$$\langle z_2^*, z - z_2 \rangle + f(y_2, z_2) \leq f(y_2, z), \quad \forall z \in \mathbb{R}^m$$

This results in

$$\langle z_1^* - z_2^*, z_2 - z_1 \rangle \leq f(y_1, z_2) - f(y_1, z_1) + f(y_2, z_1) - f(y_2, z_2) \quad (14)$$

so that **(M2)** holds with $\alpha_z = 0$ and $\gamma_z(\cdot, \cdot) = f(\cdot, \cdot)$.

Recalling the fact that the subdifferential of a proper, convex, lower semicontinuous function, and the superdifferential of a proper, concave, upper semicontinuous function are maximal monotone, the statements in **(M3)**, **(M4)**, **(M5)** follow from **(SF1)**, **(SF2)**, and **(SF3)**. \square

3.2 Dissipative Interconnection

Consider the dynamical system with state x , input u and output ω , described by

$$\begin{aligned} \dot{x} &= F(x, u) \\ \omega &= h(x) \end{aligned} \quad (15)$$

for some continuous functions $F : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$, and $h : \mathbb{R}^n \rightarrow \mathbb{R}^m$. We say that the system (15) is *incrementally passive* if there exists a continuously differentiable storage function $V, \mathbb{R}^{2n} \ni (x_1, x_2) \mapsto V(x_1, x_2) \in \mathbb{R}_+$ such that, $\frac{\partial V}{\partial x_1} F(x_1, u_1) + \frac{\partial V}{\partial x_2} F(x_2, u_2) \leq \langle \omega_1 - \omega_2, u_1 - u_2 \rangle$, where $\omega_1 = h(x_1)$ and $\omega_2 = h(x_2)$.

Let us now consider a negative feedback interconnection of two incrementally passive systems. The first system is described as

$$\dot{y} = F_y(y, u_y), \quad \omega_y = y \quad (16)$$

with $F_y : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ continuous. The dynamics of the second system are given by

$$\dot{z} = F_z(z, u_z), \quad \omega_z = z \quad (17)$$

with $F_z : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ continuous. The interconnection between these systems is described via the negative feedback as:

$$u_y = -\omega_z, \quad u_z = \omega_y. \quad (18)$$

The interconnection of incrementally passive systems has been widely studied in the literature, see for example (Zames, 1966), (Desoer and Vidyasagar, 1975, p. 184), (Pavlov and Marconi, 2008), (van der Schaft, 2017, Prop. 2.2.21). Here we show that such interconnections are actually maximally monotone. We choose to work with continuous single-valued functions so that the maximality holds (Rockafellar and Wets, 1998, Example 12.7).

Proposition 5. Suppose that the system (16) is incrementally passive with output ω_y , input u_y , the storage function $V_y(y_1, y_2) = \|y_1 - y_2\|^2$, and that the system (17) is incrementally passive with output ω_z , input u_z , the storage function $V_z(z_1, z_2) = \|z_1 - z_2\|^2$. Then, the system defined by their interconnection via negative feedback (18) is maximal monotone.

Proof. The proof follows once again by an application of Theorem 3, where we can show that **(M1)**–**(M5)** hold. The statement **(M1)** basically follows from the incremental passivity of (16) with $\alpha_y = 0$ and $\gamma_y(y, z) = \langle y, z \rangle$. Similarly, **(M2)** follows from the incremental passivity of (17) with $\alpha_z = 0$ and $\gamma_z(y, z) = -\langle y, z \rangle$. Thus, $\gamma_y + \gamma_z = 0$. Statements **(M3)**, **(M4)**, **(M5)** basically follow from the continuity of the mappings $F_y(\cdot, \cdot)$ and $F_z(\cdot, \cdot)$ over the entire space $\mathbb{R}^n \times \mathbb{R}^n$. \square

4. INTERCONNECTED LINEAR COMPLEMENTARITY SYSTEMS

Let us now consider the interconnections of slightly different, but closely related, class of nonsmooth systems where the dynamics are described by linear complementarity systems (LCS). In particular, the dynamics of the first system are described by the LCS:

$$\dot{y} = A_1 y + B_1 \lambda_y + L_1 u_y \quad (19a)$$

$$w_y = C_1 y + D_1 \lambda_y + E_1 u_y \quad (19b)$$

$$0 \leq \lambda_y \perp w_y \geq 0 \quad (19c)$$

and the dynamics of the second system are described by the LCS:

$$\dot{z} = A_2 z + B_2 \lambda_z + L_2 u_z \quad (20a)$$

$$w_z = C_2 z + D_2 \lambda_z + E_2 u_z \quad (20b)$$

$$0 \leq \lambda_z \perp w_z \geq 0 \quad (20c)$$

with $A_i \in \mathbb{R}^{n_i \times n_i}$, $L_i \in \mathbb{R}^{n_i \times m_i}$, $B_i \in \mathbb{R}^{n_i \times n_{u_i}}$, $C_i \in \mathbb{R}^{m_i \times n_i}$, $D_i \in \mathbb{R}^{m_i \times m_i}$, $E_i \in \mathbb{R}^{m_i \times n_{u_i}}$, and we let, $n := n_1 + n_2$, $m := m_1 + m_2$. The interconnection between (19) and (20) is defined as

$$\begin{pmatrix} u_y \\ u_z \end{pmatrix} = \begin{pmatrix} \omega_z \\ \omega_y \end{pmatrix} = \begin{pmatrix} H_2 z + G_2 \lambda_z \\ H_1 y + G_1 \lambda_y \end{pmatrix}. \quad (21)$$

The outputs ω_y and ω_z can be seen as the measured outputs for each subsystem, while w_y and w_z are internal variables (which can be considered as outputs of a passive internal subsystem as seen below). This yields the following interconnected system:

$$\begin{pmatrix} \dot{y} \\ \dot{z} \end{pmatrix} = \underbrace{\begin{pmatrix} A_1 & L_1 H_2 \\ L_2 H_1 & A_2 \end{pmatrix}}_{\triangleq \tilde{A}} \underbrace{\begin{pmatrix} y \\ z \end{pmatrix}}_{\triangleq x} + \underbrace{\begin{pmatrix} B_1 & L_1 G_2 \\ L_2 G_1 & B_2 \end{pmatrix}}_{\triangleq \tilde{B}} \underbrace{\begin{pmatrix} \lambda_y \\ \lambda_z \end{pmatrix}}_{\triangleq \lambda}$$

$$\begin{pmatrix} w_y \\ w_z \end{pmatrix} = \underbrace{\begin{pmatrix} C_1 & E_1 H_2 \\ E_2 H_1 & C_2 \end{pmatrix}}_{\triangleq \tilde{C}} \underbrace{\begin{pmatrix} y \\ z \end{pmatrix}}_{\triangleq x} + \underbrace{\begin{pmatrix} D_1 & E_1 G_2 \\ E_2 G_1 & D_2 \end{pmatrix}}_{\triangleq \tilde{D}} \underbrace{\begin{pmatrix} \lambda_y \\ \lambda_z \end{pmatrix}}_{\triangleq \lambda}$$

$$0 \leq \lambda \perp w \geq 0 \quad (22)$$

The interconnection matrices are L_1 , L_2 , E_1 and E_2 .

Assumption 1. The quadruples (A_i, B_i, C_i, D_i) , $i = 1, 2$, define a passive system. Equivalently, there exist $P_i = P_i^\top \succcurlyeq 0$ such that:

$$\begin{pmatrix} -A_i^\top P_i - P_i A_i & P_i B_i - C_i^\top \\ (P_i B_i - C_i^\top)^\top & D_i + D_i^\top \end{pmatrix} \succcurlyeq 0, \quad (23)$$

where the notation $P \succcurlyeq 0$ (resp. $P \succ 0$) is used to denote P being positive semidefinite (resp. positive definite).

This means that the passivity of the subsystem (19) (resp. (20)) holds with supply rates $\lambda_y^\top w_y$ (resp. $\lambda_z^\top w_z$), and not $u_y^\top \omega_y$ (resp. $u_z^\top \omega_z$) as usual. The passivity is preserved under the interconnection constraint, if and only if there exists an $n \times n$ matrix $P = P^\top \succcurlyeq 0$ such that:

$$\tilde{Q} \triangleq \begin{pmatrix} -\tilde{A}^\top P - P \tilde{A} & P \tilde{B} - \tilde{C}^\top \\ (P \tilde{B} - \tilde{C}^\top)^\top & \tilde{D} + \tilde{D}^\top \end{pmatrix} \succcurlyeq 0. \quad (24)$$

Then the interconnected system is passive with the supply rate $w^\top \lambda$ and storage functions $V(x) = x^\top P x$.

As stated earlier in Theorem 2, it is possible to rewrite (19), (20), (21), and hence the interconnection (22) in the form of differential inclusions as follows:

$$\begin{aligned} \dot{y} &\in A_1 y - B_1 (D_1 + \mathcal{N}_{\mathbb{R}^{m_1}}^{-1})^{-1} (C_1 y + E_1 H_2 z \\ &\quad + E_1 G_2 \lambda_z) + L_1 (H_2 x_2 + G_2 \lambda_z) \\ \dot{z} &\in A_2 z - B_2 (D_2 + \mathcal{N}_{\mathbb{R}^{m_2}}^{-1})^{-1} (C_2 z + E_2 H_1 y \\ &\quad + E_2 G_1 \lambda_y) + L_2 (H_1 x_1 + G_1 \lambda_y) \\ \lambda &\in -(\tilde{D} + \mathcal{N}_{\mathbb{R}^m}^\top)^{-1} (\tilde{C} x) \end{aligned} \quad (25)$$

If $G_1 = 0$ and $G_2 = 0$, (25) can be recast into (7) under Assumption 1 and suitable constraints qualification, which guarantee the maximal monotonicity. The purpose of the next analysis is to treat the general case. Our next result, Theorem 6, analyzes conditions under which the LMI in (24) is satisfied under Assumption 1, and for various cases (passive, strongly passive). The link with maximal monotonicity is made later in the section in Corollary 8. In what follows, we use $\lambda_{\max}(M)$ (resp. $\lambda_{\min}(M)$) to denote the largest (resp. smallest) eigenvalue of a matrix M , and $\sigma_{\max}(M)$ for the largest singular value of M .

Theorem 6. Consider systems (19) and (20) under Assumption 1, and their interconnection in (22). Then the following assertions are true:

1. *Preservation of passivity:* Assume that both subsystems are passive. Then the quadruple $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$ is passive with storage matrix $P = \text{diag}(P_1, P_2)$, if $E_1 G_2 = -G_1^\top E_2^\top$, $P_1 L_1 G_2 - H_1^\top E_2^\top = 0$, $P_2 L_2 G_1 - H_2^\top E_1^\top = 0$, and $P_1 L_1 H_2 + (P_2 L_2 H_1)^\top = 0$.

2. *Preservation of strong passivity:* Assume that both subsystems are strongly passive, then the quadruple $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$ is also strongly passive with $P = \begin{pmatrix} P_1 & P_{12} \\ P_{12}^\top & P_2 \end{pmatrix}$ where P_1 and P_2 are solutions of (23), if conditions (26), (27) and (29) are satisfied, where **(a)**:

$$\min(\lambda_{\min}(D_1 + D_1^\top), \lambda_{\min}(D_2 + D_2^\top)) \geq \lambda_{\max}^{\frac{1}{2}}(E_1 G_2 G_2^\top E_1^\top) + \lambda_{\max}^{\frac{1}{2}}(E_2 G_1 G_1^\top E_2^\top), \quad (26)$$

(then $\tilde{D} + \tilde{D}^\top \succ 0$), and **(b)**

$$\sigma_{\max}(\tilde{Q}) \leq \min(\lambda_{\min}(Q_1), \lambda_{\min}(Q_2)), \quad (27)$$

where, we let

$$\tilde{Q} = \begin{pmatrix} P_{12} L_2 H_1 + H_1^\top L_2^\top P_{12}^\top & \tilde{Q}_{12} \\ \tilde{Q}_{12}^\top & P_{12}^\top L_1 H_2 + H_2^\top L_1^\top P_{12} \end{pmatrix}, \quad (28)$$

with $\tilde{Q}_{12} \triangleq P_1 L_1 H_2 + P_{12} A_2 + A_1^\top P_{12}^\top + H_1^\top L_2^\top P_2$; P_{12} is such that $\sigma_{\max}(P_{12}) < \lambda_{\min}(P_1) \lambda_{\min}(P_2)$, $Q_1 \triangleq -A_1^\top P_1 - P_1 A_1 \succ 0$ and $Q_2 \triangleq -A_2^\top P_2 - P_2 A_2 \succ 0$. These conditions hold if $P_{12} = 0$ and $P_1 L_1 H_2 = -(P_2 L_2 H_1)^\top$ (then $\tilde{Q} = 0$ and $-\tilde{A}^\top P - P \tilde{A} \succ 0$), and finally **(c)**:

$$\begin{aligned} \sigma_{\max} \left(\begin{pmatrix} P_{12} L_2 G_1 + P_1 B_1 - C_1^\top & P_1 L_1 G_2 + P_{12} B_2 - H_1^\top E_2^\top \\ P_{12}^\top B_1 + P_2 L_2 G_1 - H_2^\top E_1^\top & P_{12}^\top L_1 G_2 + P_2 B_2 - C_2^\top \end{pmatrix} \right) \\ \leq \left(\lambda_{\min}(\text{diag}(D_1 + D_1^\top, D_2 + D_2^\top)) + \lambda_{\min}(R) \right)^{\frac{1}{2}} \\ \left(\lambda_{\min}(\text{diag}(Q_1, Q_2)) + \lambda_{\min}(\tilde{Q}) \right)^{\frac{1}{2}} \end{aligned} \quad (29)$$

where

$$R = \begin{pmatrix} 0 & E_1 G_2 + G_1^\top E_2^\top \\ E_2 G_1 + G_2^\top E_1^\top & 0 \end{pmatrix}.$$

The proof is omitted due to space constraints. It basically uses the fact that the LMI in (24) is equivalent to following

four inequalities (Bernstein, 2009, Fact 8.2.4): (a) $\tilde{D} + \tilde{D}^\top \succcurlyeq 0$; (b) $P\tilde{B} - \tilde{C}^\top = (P\tilde{B} - \tilde{C}^\top)(\tilde{D} + \tilde{D}^\top)^\dagger(\tilde{D} + \tilde{D}^\top)$; (c) $\tilde{A}^\top P + P\tilde{A} \preccurlyeq 0$; and (d) $-\tilde{A}^\top P - P\tilde{A} \succcurlyeq (P\tilde{B} - \tilde{C}^\top)(\tilde{D} + \tilde{D}^\top)^\dagger(\tilde{B}^\top P - \tilde{C})$.

Remark 7. Notice that the conditions in item 1, are different from those in item 2, because preservation of passivity is more stringent than preservation of strong passivity: while the former calls for strong structural conditions, the latter holds if perturbations are small enough.

As mentioned in Theorem 2, under some basic conditions, the negative feedback interconnection of a passive system with a maximal monotone static nonlinearity (possibly set-valued) defines a differential inclusion

$$\dot{x} \in -\mathcal{F}(x) = \tilde{A}x - \tilde{B}(\tilde{D} + \mathcal{N}_{\mathbb{R}_+^m}^{-1})^{-1}(\tilde{C}x)$$

with $\mathcal{F}(\cdot)$ a maximal monotone operator with $\text{dom}(\mathcal{F}) = \{x \in \mathbb{R}^n \mid \tilde{C}x \in \text{dom}(\tilde{D} + \mathcal{N}_{\mathbb{R}_+^m}^{-1})^{-1} = \text{Im}(\tilde{D} + \mathcal{N}_{\mathbb{R}_+^m}^{-1})\} = \tilde{C}^{-1}(\text{Im}(\tilde{D} + \mathcal{N}_{\mathbb{R}_+^m}^{-1}))$. This motivates the next result.

Corollary 8. Consider systems (19) and (20) under Assumption 1, and their interconnection in (22). Let $\text{Im}(\tilde{C}) \cap \text{rint}(\text{Im}(\mathcal{N}_{\mathbb{R}_+^m}^{-1} + \tilde{D})) \neq \emptyset$. If the conditions of item 1 (resp. item 2) in Theorem 6 are satisfied, then the LCS in (22) has a maximal monotone (resp. single-valued Lipschitz strongly monotone) right-hand side mapping $\mathcal{F}(\cdot)$.

Proof. If the conditions of item 1 hold, then $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$ is passive, and the mapping, $x \mapsto \mathcal{F}(x)$ is maximal monotone.

Let the quadruple $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$ be strongly passive, then there exists $\mu > 0$ such that $(\tilde{A} - \mu I_n, \tilde{B}, \tilde{C}, \tilde{D})$ is passive (this holds for all $0 < \mu < \lambda_{\min}(\tilde{Q})$ with \tilde{Q} in (24)). Therefore, the mapping

$$x \mapsto \tilde{\mathcal{F}}(x) = -(\tilde{A} + \mu I_n)x + \tilde{B}(\tilde{D} + \mathcal{N}_{\mathbb{R}_+^m}^{-1})^{-1}(\tilde{C}x)$$

is maximal monotone, and $\langle x_1^* - x_2^*, x_1 - x_2 \rangle \geq \mu \|x_1 - x_2\|^2$ for all $x_i \in \text{dom}(\tilde{\mathcal{F}})$, $x_i^* \in \tilde{\mathcal{F}}(x_i)$, $i = 1, 2$. In addition strong passivity implies $\tilde{D} \succ 0$, hence applying (Brogliato and Goeleven, 2011, Proposition 1) the mapping $(\tilde{D} + \mathcal{N}_{\mathbb{R}_+^m}^{-1})^{-1}$ is single-valued, well-defined and Lipschitz continuous with constant $1/\lambda_{\min}(\tilde{D} + \tilde{D}^\top)$. \square

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