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## Equivalent Conditions for the Synchronization of Identical Linear Systems

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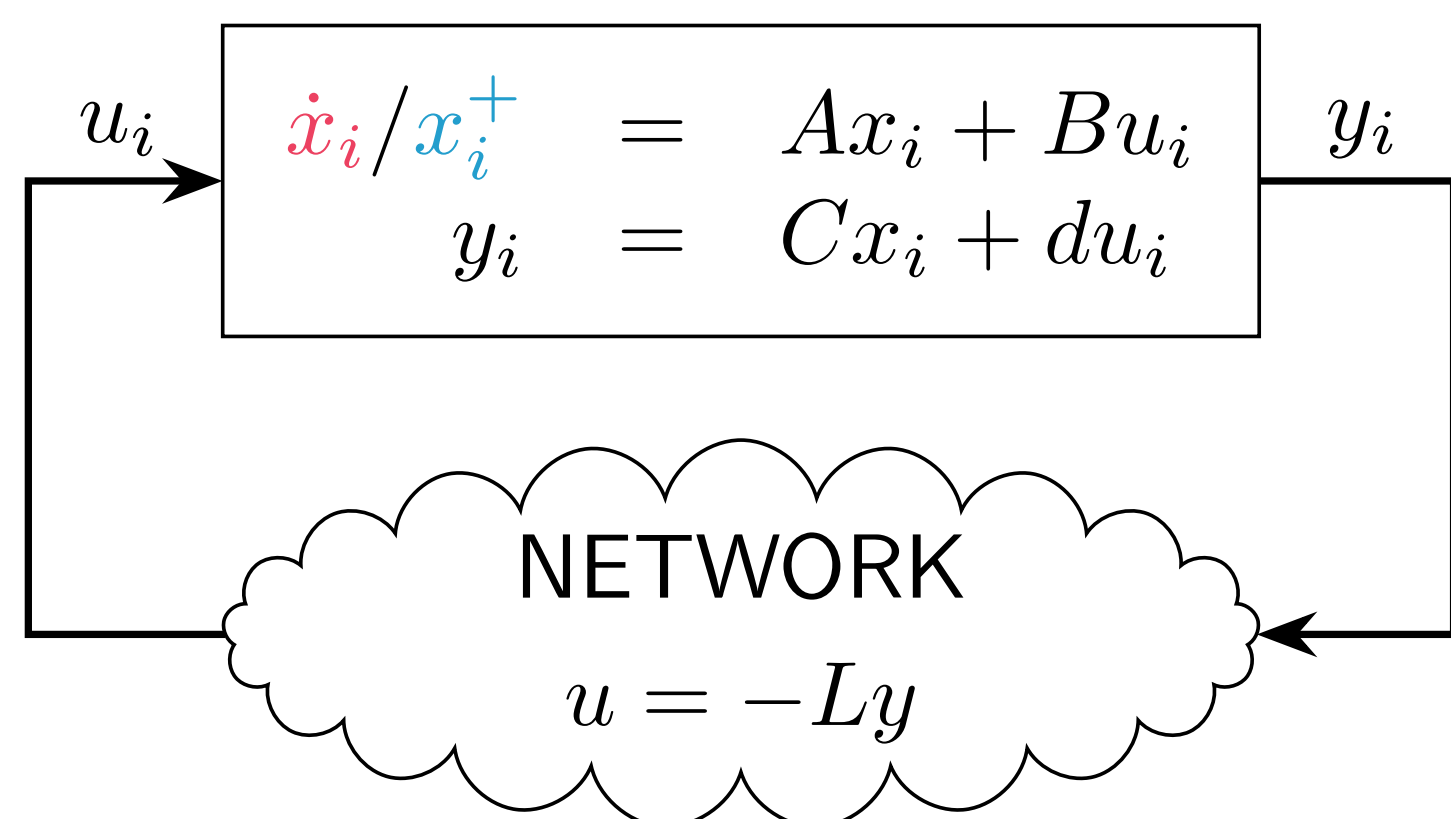


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## Motivation

Several conditions for linear systems synchronization have been proposed along the years. We show their equivalence as necessary and sufficient conditions for uniform global exponential synchronization of  $N$  identical single-input-single-output (SISO) systems, connected through a directed graph (without any assumption). Both the **continuous-time** and the **discrete-time** case are considered.

## Problem formulation



Let us consider  $N$  identical SISO dynamical systems of arbitrary order, with state  $x_i \in \mathbb{R}^n$  evolving as

$$\begin{aligned} \dot{x}_i/x_i^+ &= Ax_i + Bu_i \\ y_i &= Cx_i + du_i \end{aligned} \quad i = 1, \dots, N \quad (1)$$

and with scalar inputs  $u_i \in \mathbb{R}$  and outputs  $y_i \in \mathbb{R}$ , interconnected through a directed graph  $\mathcal{G}$  with Laplacian  $L \in \mathbb{R}^{N \times N}$  as follows:

$$u = -Ly = -L((I_N \otimes C)x + du), \quad (2)$$

where  $u = [u_1 \dots u_N]^T \in \mathbb{R}^N$ ,  $y = [y_1 \dots y_N]^T \in \mathbb{R}^N$  and  $x := [x_1^T \dots x_N^T]^T \in \mathbb{R}^{Nn}$ .

For the sake of generality, we also allow for the presence of a direct input-output link  $d \in \mathbb{R}$  in (1). With  $d \neq 0$ , the implicit equation (2) can be uniquely solved as

$$u = -(L_d \otimes C)x := -((I_N + dL)^{-1}L \otimes C)x \quad (3)$$

if and only if matrix  $I_N + dL$  is invertible, i.e.,  $-d^{-1}$  is not an eigenvalue of  $L$ . When  $d = 0$ , we retrieve the classical case  $L_d := L$ . Overall, the linear distributed interconnection (1), (2) can be written as

$$\dot{x}/x^+ = (I_N \otimes A)x - (L_d \otimes BC)x. \quad (4)$$

The synchronization set, or *attractor*, is defined as

$$\mathcal{A} := \{x : x_i - x_j = 0, \forall i, j \in \{1, \dots, N\}\}. \quad (5)$$

## Definitions

**( $\alpha$ -UGES)** For  $\alpha > 0$  there exist  $M > 0$  such that

$$|x(t)|_{\mathcal{A}} \leq M e^{-\alpha t} |x(0)|_{\mathcal{A}}, \quad \forall t \in \mathbb{R}_{\geq 0}$$

$$\text{(resp. } |x(t)|_{\mathcal{A}} \leq M \alpha^t |x(0)|_{\mathcal{A}}, \quad \forall t \in \mathbb{N}\text{)}.$$

**( $\alpha$ -synchronization)** Every  $i$ -th sub-system satisfies

$$|x_i(t) - \tilde{x}_o(t)| \leq M e^{-\alpha t} \sum_{i=1}^N |x_i(0) - \tilde{x}_o(0)|, \quad \forall t \in \mathbb{R}_{\geq 0}$$

$$\left( \text{resp. } |x_i(t) - \tilde{x}_o(t)| \leq M \alpha^t \sum_{i=1}^N |x_i(0) - \tilde{x}_o(0)|, \quad \forall t \in \mathbb{N} \right),$$

$i = 1, \dots, N$ , for a suitable target trajectory  $\tilde{x}_o$ .

## Synchronization conditions

Before stating the main result, we note that, trivially, all the zero eigenvalues of  $L$  are also zero eigenvalues of  $L_d = (I_N + dL)^{-1}L$ . Then we denote by  $0 = \lambda_0, \lambda_1, \dots, \lambda_\nu$  the eigenvalues of  $L_d$ , where the complex conjugate pairs are only counted once (so that  $\nu := N - 1 - n_c$ , with  $n_c$  being the number of complex conjugate pairs).

Consider the **continuous-time** (resp. **discrete-time**) system in (1), (2), the attractor  $\mathcal{A}$  in (5) and the parameter  $\alpha^* \geq 0$  (resp.  $\alpha^* \in (0, 1]$ ).

The following statements are equivalent:

- [Complex condition] The **spectral abscissa** (resp. **spectral radius**) of the complex-valued matrices

$$A_k := A - \lambda_k BC, \quad k = 1, \dots, \nu, \quad (6)$$

is smaller than  $-\alpha^*$  (resp.  $\alpha^*$ ).

- [Real condition] The **spectral abscissa** (resp. **spectral radius**) of the real-valued matrices

$$A_{e,k} := \begin{bmatrix} A - \text{Re}(\lambda_k)BC & \text{Im}(\lambda_k)BC \\ -\text{Im}(\lambda_k)BC & A - \text{Re}(\lambda_k)BC \end{bmatrix}, \quad k = 1, \dots, \nu, \quad (7)$$

is smaller than  $-\alpha^*$  (resp.  $\alpha^*$ ).

- [Lyapunov inequality] For each  $k = 1, \dots, \nu$ , there exist real-valued matrices  $P_k = P_k^T \succ 0$  and  $\Pi_k^T = -\Pi_k$  such that one of the following identities holds:

$$(P_k + j\Pi_k)A_k + A_k^*(P_k + j\Pi_k) \prec -2\alpha^*(P_k + j\Pi_k) \quad (8a)$$

$$\left( \text{resp. } A_k^*(P_k + j\Pi_k)A_k \prec (\alpha^*)^2(P_k + j\Pi_k) \right), \quad (8b)$$

or

$$\begin{bmatrix} P_k & \Pi_k \\ -\Pi_k & P_k \end{bmatrix} A_{e,k} + A_{e,k}^T \begin{bmatrix} P_k & \Pi_k \\ -\Pi_k & P_k \end{bmatrix} \prec -2\alpha^* \begin{bmatrix} P_k & \Pi_k \\ -\Pi_k & P_k \end{bmatrix} \quad (9a)$$

$$\left( \text{resp. } A_{e,k}^T \begin{bmatrix} P_k & \Pi_k \\ -\Pi_k & P_k \end{bmatrix} A_{e,k} \prec (\alpha^*)^2 \begin{bmatrix} P_k & \Pi_k \\ -\Pi_k & P_k \end{bmatrix} \right). \quad (9b)$$

- [Lyapunov function] There exist  $\alpha > \alpha^*$  (resp.  $0 < \alpha < \alpha^*$ ), positive constants  $c_1, c_2$  and  $c_3$ , and a strict quadratic Lyapunov function  $V(x)$  satisfying:

$$c_1 |x|_{\mathcal{A}}^2 \leq V(x) \leq c_2 |x|_{\mathcal{A}}^2, \quad (10a)$$

$$\dot{V}(x) \leq -2\alpha V(x) \quad (10b)$$

$$\left( \text{resp. } V(x^+) \leq \alpha^2 V(x) \right). \quad (10c)$$

- [UGES] There exists  $\alpha > \alpha^*$  (resp.  $0 < \alpha < \alpha^*$ ) such that the closed attractor  $\mathcal{A}$  in (5) is  $\alpha$ -UGES for the closed loop (1), (2).

- [IVP] There exists  $\alpha > \alpha^*$  (resp.  $0 < \alpha < \alpha^*$ ) such that, for the closed loop (1), (2), the sub-states  $x_i$  uniformly globally  $\alpha$ -exponentially synchronize to the unique solution of the following initial value problem:

$$\dot{\tilde{x}}_o/\tilde{x}_o^+ = A\tilde{x}_o, \quad \tilde{x}_o(0) = \frac{1}{p^T \mathbf{1}_N} \sum_{k=1}^N p_k x_k(0), \quad (11)$$

where  $p := [p_1 \dots p_N]^T \in \mathbb{R}^N$  is a left eigenvector corresponding to the zero eigenvalue of  $L$ .

## References

[1]



Equivalent Conditions for the Synchronization of Identical Linear Systems over Arbitrary Interconnections  
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