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Robustness of Coupled ODEs and PDEs, state-of-the-art and perspectives

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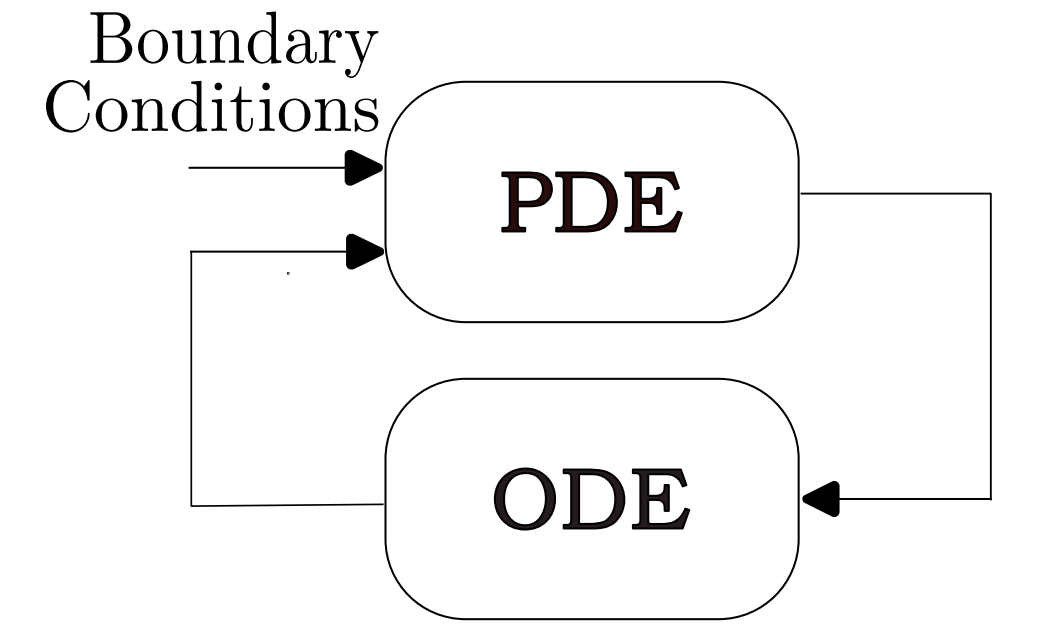
Motivation & Thesis Goals

Objective: Study of infinite dimensional systems, modeled by partial derivative equations (PDEs), connected through boundary conditions to finite dimensional systems, modeled by ordinary differential equations (ODEs).

Motivation: Allows modeling many physical phenomena, including transport processes, and interaction between heat diffusion and chemical reactions, with ODEs acting as sensors, controllers, and actuators. Limited research on the robust input-output approach for these systems.

Innovative Aspects:

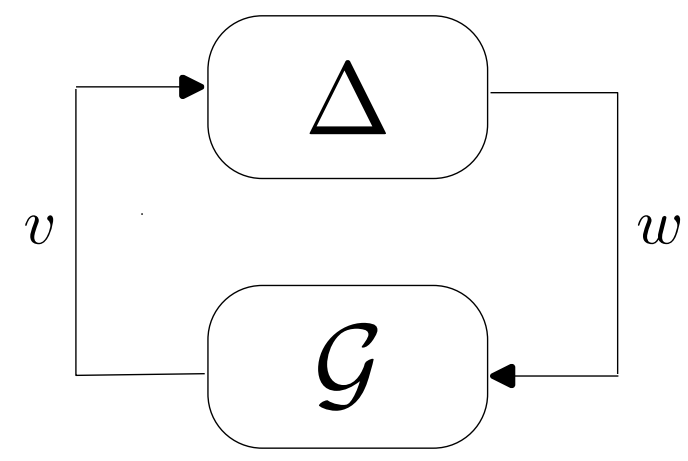
- Robust control like **modeling** of such heterogeneous systems, moving towards multi-dimensional space and multi-variable input-output systems
- Methodological tools like IQC or a new type of separator to produce numerically efficient **analysis** results for heterogeneous systems
- Methods for the **synthesis** of controllers or observers within the chosen formalism



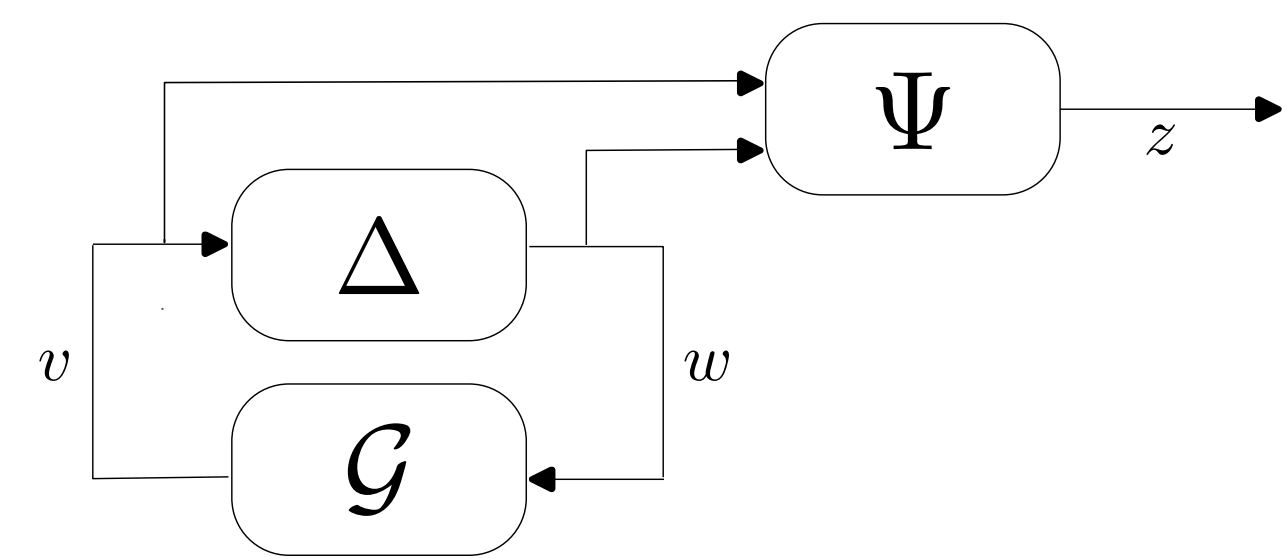
Stability & Control Methods

Stability of PDEs is mainly addressed through:

- **Modal approach:** Effective calculation of the spectrum of an infinite-dimensional operator associated with the PDE
 - ⊕ No robustness,
 - ⊕ Spectrum is difficult to calculate
 - ⊕ Limited to linear systems;
- **Lyapunov function approach:** Lyapunov certificate $\dot{V}(x) < 0$ to be found, often derived from an energy function, evaluated through a state norm
 - ⊕ Applicable to nonlinear systems,
 - ⊕ Heavily depends on the type of PDE-ODE coupling considered;
- **Input-Output approach:** Tools from the robust control community, such as small gain theorem and Integral Quadratic Constraints (IQC)
 - ⊕ Partially extended to the infinite-dimensional framework, ⊕ Need to choose a multiplier to cope with a structured matrix of uncertainties Δ ,
 - ⊕ No explicit property on the states of the PDE



Input-Output Methods for PDE-ODE



Integral Quadratic Constraints: IQCs provide a framework for abstracting the complexities inherent in dynamic system models, facilitating rigorous analysis of robust stability and performance metrics. Two signals v, w satisfy the soft IQC defined by Π if

$$\left\langle \begin{pmatrix} v \\ w \end{pmatrix}, \Pi \begin{pmatrix} v \\ w \end{pmatrix} \right\rangle = \int_{-\infty}^{\infty} \begin{bmatrix} \hat{v}(j\omega) \\ \hat{w}(j\omega) \end{bmatrix}^* \Pi(j\omega) \begin{bmatrix} \hat{v}(j\omega) \\ \hat{w}(j\omega) \end{bmatrix} d\omega \geq 0 \quad (1)$$

$w = \Delta(v)$ satisfy the IQC defined by Π if (1) holds and if $\exists \epsilon > 0$ such that also

$$\begin{bmatrix} \mathcal{G}(j\omega) \\ I \end{bmatrix}^* \Pi(j\omega) \begin{bmatrix} \mathcal{G}(j\omega) \\ I \end{bmatrix} \leq -\epsilon I \quad \forall \omega \in \mathbb{R} \quad (2)$$

with $v = \mathcal{G}w$ holds, then the feedback loop composed of \mathcal{G} and Δ is stable.

Small gain: Particular case of IQC where (1) + (2) boils down to $\|\mathcal{G}\| \cdot \|\Delta\| < 1$ when $\Pi = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}$.

Filters can be added to the I-O system to deal with structured Δ : $z := \Psi \begin{bmatrix} v \\ w \end{bmatrix}$ and $\Pi = \Psi^* M \Psi$.

Define state-space representation of the system, $\Psi \begin{bmatrix} \mathcal{G} \\ I \end{bmatrix} = D + C(sI - A)^{-1}B$. **LMIs** can be written using the

KYP lemma: $\exists \epsilon > 0$ s.t (2) holds iff $\exists P = P^T$ such that $\begin{bmatrix} A^T P + PA & PB \\ B^T P & 0 \end{bmatrix} + \begin{bmatrix} C^T \\ D^T \end{bmatrix} M \begin{bmatrix} C & D \end{bmatrix} < 0$

Dissipation analysis: Quantifies dissipative properties of dynamical systems, defined by a storage function $V(x) = x^T P x$ and supply rate $s(z) = z^T M z$, characterized by $V(x(t))$ and $\dot{V}(x(t)) \leq s(z(t))$.

Partial Integral Equations

PIE are infinite-dimensional state-space systems of the form

$$\mathcal{T}\dot{x}(t) = \mathcal{A}x(t) \quad x(0) = x_0$$

where \mathcal{T} and \mathcal{A} are Partial Integral (PI) operators [1].

3-PI: can be used to represent PDE and coupled PDEs

$$(\mathcal{P}_{\{R_0, R_1, R_2\}}x) := R_0(s)x(s) + \int_a^s R_1(s, \theta)x(\theta)d\theta + \int_s^b R_2(s, \theta)x(\theta)d\theta \quad \mathcal{P}_{\{R_i\}} : L_2 \rightarrow L_2$$

4-PI: are used to represent coupling of ODE-PDEs

$$\left(\mathcal{P} \begin{bmatrix} x \\ \mathbf{x} \end{bmatrix} \right) (s) := \begin{bmatrix} Px + \int_a^b Q_1(s) \mathbf{x}(s) ds \\ Q_2(s)x + (\mathcal{P}_{\{R_i\}}\mathbf{x})(s) \end{bmatrix} \in \mathbb{R}^n \times L_2^n[a, b]$$

Infinite dimensional IQCs can be written using the PI operator formalism:

Theorem: Suppose \mathcal{G} is a PIE system, Δ is a set of nonlinear systems (uncertainties), and their interconnection is well-posed. Suppose there exist a bounded operator Ψ , and \mathcal{K} such that:

- Δ satisfies the hard IQC defined by Ψ, \mathcal{K}
- For any v, w we have that $\left\langle \Psi_T \begin{bmatrix} \mathcal{G}w \\ w \end{bmatrix}, M_{\mathcal{K}} \Psi_T \begin{bmatrix} \mathcal{G}w \\ w \end{bmatrix} \right\rangle_L \leq -\epsilon \|u_T\|_L^2$

Then the feedback system defined by $[\mathcal{G}, \Delta]$ is stable [2]

- ⊕ The state of the system is kept explicit,
- ⊕ Difficult to use this method analytically,
- ⊕ Boundary constraints are hidden, we would like to keep them explicit until the end to easily study how stability changes when we connect ODE-PDE differently

Current work & Perspectives

Projection method for heat equation: IQC can be built using the projection method. We need to extend the vector of boundary condition [3]

$B_m(u) = \text{col}(u(0), u(1), \dots, \partial_x^{(m-1)}u(0), \partial_x^{(m-1)}u(1))$ if we consider the heat equation.

$$\mathbf{B}_{m_{ext}}(u) = \begin{bmatrix} \mathbf{B}_m(u) \\ \partial_t \mathbf{B}_m(u) \end{bmatrix}$$

This method allows us to build a filter Ψ . Hence we can find a class of multipliers M , specific to the PDE considered, and the terminal cost Z . The hard finite horizon IQC with terminal cost is

$$\int_0^T z^T(t) M z(t) dt + \xi^T(T) Z \xi(T) \geq 0$$

with ξ, z state and output of the filter Ψ .

- ⊕ Boundary conditions can be easily changed
- ⊕ The PDE state is hidden.

Perspectives:

- Write PIE where border conditions are kept explicit
- Extended version of IQCs for the heat equation

References

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- [2] Aleksandr Talitskii, Matthew Peet, Peter Seiler: *Integral Quadratic Constraints with Infinite-Dimensional Channels*, American Control Conference (2023)
- [3] Matthieu Barreau, Carsten Scherer, Frédéric Gouaisbaut and Alexandre Seuret: *Integral Quadratic Constraints on Linear Infinite-dimensional Systems for Robust Stability Analysis*, IFAC-PapersOnLine (2020)