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Impulsive switching signals with functional inequalities: Stability analysis using hybrid systems framework

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Abstract

In this work, we introduce a class of impulsive switching signals described via functional inequalities which govern the switching among different modes with state resets. By choosing the parameters of the inequalities appropriately, we can recover several known classes of switching signals and also allow for signals that depend on time, mode or state of the system. Signals from this class can also be generated online via the use of an auxiliary timer while the dynamical system is running. Via a multiple Lyapunov functions approach, we provide sufficient conditions on the functional parameters of the switching signal which ensure that the equilibrium is globally asymptotically stable (GAS) for autonomous impulsive switched system. In case of inputs, similar methodology is used to provide sufficient conditions for input-to-state stability (ISS) and integral-input-to-state stability (iISS) uniformly over the proposed class of impulsive switching signals. As case studies, we consider switched systems which *do not* satisfy ISS (respectively, iISS) property for switching signals with arbitrarily large dwell-times but they are shown to be ISS (resp. iISS) for our proposed class of impulsive switchings signals described via functional inequalities.

Keywords: Impulsive switched systems, hybrid systems, global asymptotic stability, input-to-state stability

1. Introduction

Inheriting the merits of both switched systems [19, 33] and impulsive systems [2, 9], impulsive switched systems are particularly useful in the study of complex systems which consist of both continuous and discrete dynamics. An impulsive switched system is composed of three parts: a family of continuous processes characterizing the flow behavior of the system between impulses or switches, a family of discrete processes characterizing the jump behavior during impulses or switches, and a governing law which determines the impulse times and orchestrates the switching of the system dynamics. The stability analysis of such systems is rather intricate due to the interaction of continuous and discrete dynamics and this paper focuses on providing sufficient conditions which can cover a broad class of such systems. For systems without inputs, global asymptotic stability (GAS) of the equilibrium is a property of fundamental interest [17], and for the systems with inputs, we consider input-to-state stability (ISS) and integral input-to-state stability (iISS) introduced in [32] and [1] respectively, which characterize the effect of external signals on the stability of the state trajectories. For a switched

system to be GAS, ISS or iISS uniformly over arbitrary switching signals, not only all the subsystems are required to have these properties, but in addition all subsystems must share a common Lyapunov function. While the existence of a common Lyapunov function turns out to be the necessary and sufficient condition for GAS, ISS or iISS switched systems under arbitrary switching [5, 11], such conditions are very restrictive in practice and there are well-known classes of switched systems which are GAS, ISS or iISS uniformly over some set of switching signals but not all switching signals [19]. For such systems, different approaches can be used to describe the sets of switching signals over which the desired stability property is ensured uniformly. The most straightforward method is to introduce slow switching. Intuitively, the actions of switching are treated as the source of instability and hence, as long as switching occurs sufficiently infrequently, the stability of the switched systems can be ensured. Well-known slow switching conditions include *dwell-time* (DT) condition and *average dwell-time* (ADT) condition. An autonomous switched system with a common asymptotically stable equilibrium for all subsystems is shown to be GAS, uniformly for switching signals with sufficiently large DT or ADT [24, 15]. For switched systems with inputs, ISS can also be studied similarly with respect to slow switching signals subjected to DT or ADT conditions [34]. Building on these results, there has been significant research to generalize the class of switching signals for which one can establish GAS, ISS, or iISS uniformly. These generalizations are developed by modifying the framework from aforementioned references in certain directions: a) by

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modifying the definition of ADT, b) by allowing some unstable subsystems and/or stabilizing discrete jump maps, and c) by relaxing the assumptions on the supply functions in the dissipation inequalities associated to each subsystem.

For the first case, the notion of ADT is generalized by replacing the affine function of time in the definition of ADT by a more generic function in [18]. Similar generalization is also seen in the recent work [12]. Meanwhile, in [23], the authors propose impulse sequences whose impulse frequencies are eventually uniformly bounded, but iISS-related questions have not been investigated for such a class of signals. For the second case, one may consider unstable subsystems with stabilizing impulsive maps and frequent switching, so that the impulsive maps dominate the continuous flow and stabilize the system [35, 36]. Conditions describing how fast such stabilizing impulsive effects must occur are captured under the notion of *reverse average dwell-time* (rADT), introduced in [14]. Alternatively, since the activation of unstable subsystems is treated as the source of instability, stability of a switched system with unstable subsystems can be preserved by limiting the time that the system dwells in the unstable modes. Following this approach, *average activation time* (AAT) conditions are introduced, which are usually imposed together with ADT conditions in the characterization of switching signals, uniformly over which an impulsive switched system is shown GAS, ISS or iISS [37, 25, 22]. The third research direction comes from relaxing the linear decay on the Lyapunov functions of individual subsystems to nonlinear decay functions. The papers [7, 21] provide counterexamples of switched systems with all stable subsystems where the dissipation inequalities have nonlinear supply functions and the resulting switched system is not asymptotically stable no matter how large the DT or ADT is chosen. Sufficient conditions for impulsive systems with nonlinear decay functions in dissipation inequalities associated to continuous and discrete dynamics appear in [29, 4]. These were generalized to switched systems in [38] while using a Lyapunov function for the stability analysis, and the construction of this Lyapunov function builds on some earlier work done in [27]. Inspired by [38], the recent work [22] allows for unstable systems and develops conditions for GAS, ISS, iISS which relate ADT and AAT in a rather elegant manner.

Despite the fact that all the aforementioned approaches of switching are well established and acknowledged in the literature, they have their own limitations. On one hand, slow switching or rapid switching are purely time-dependent, and as shown in [20, 10] that when the Lyapunov functions for individual subsystems are naively chosen, the lower bounds on the DT or ADT may be too conservative and sometimes impractical for providing stability guarantees. It is also observed in [7, 21] that for some switched systems with stable nonlinear subsystems, there always exists a divergent solution no matter how slowly the system switches. On the other hand, AAT condition only characterizes subsystems to be either “stable” or “unstable” and confines the total activation time of all “unstable” modes. This is clearly restrictive, as dwelling in a “more unstable” mode has a more negative im-

pact on the stability compared with dwelling in a “less unstable” mode. Therefore, the restrictions of the activation time of the two modes should be different, or put differently, one should allow for mode dependent AAT. The scope of this work is to overcome these limitations by proposing a class of switching signals which refines and unifies different classes of switching signals. In a nutshell, the major contribution of this paper lies in the identification of certain classes of impulsive switching signals such that

- for the systems which are uniformly stable with respect to the impulsive switching signals satisfying some well-known conditions in the literature, our results show that stability is guaranteed uniformly with respect to a newly proposed broader class of impulsive switching signals;
- for the systems which are not uniformly stable with respect to the impulsive switching signals satisfying those well-known conditions in the literature, our results show that stability can still be guaranteed uniformly with respect to a smaller class of impulsive switching signals.

These objectives are achieved by introducing a class of impulsive switching signals, which is defined via functional inequalities. The parameters in these inequalities are possible functions of time, modes, and state of the system. In addition to providing sufficient conditions for non-emptiness of such classes of impulsive switching signals and for ruling out Zeno signals, we also provide a constructive method to generate such impulsive switching signals online. This is done with the help of an auxiliary timer in the hybrid systems framework involving flow and jump dynamics. Our results provide sufficient conditions for GAS, ISS, and iISS in terms of the parameters of functional inequalities and the functions appearing in the dissipation inequalities of individual subsystems. As an illustration of our theoretical results, we consider several case studies. The first case study shows that autonomous impulsive switched systems are guaranteed to be GAS uniformly over a broader class of switching signals than those satisfying time-dependent conditions. The other two case studies contain switched systems which are known to be not ISS (respectively not iISS) under ADT switching no matter how slowly the system switches, but they are ISS (resp. iISS) uniformly over the class of impulsive switching signals studied in this paper.

The rest of the paper is organized as follows. Section 2 reviews the necessary mathematical tools for this paper. Section 3 introduces the novel class of impulsive switching signals. Section 4 and Section 5 provide the main stability theorems and case studies of autonomous impulsive switched systems and impulsive switched systems with inputs, respectively. Finally Section 6 concludes the paper.

2. Preliminaries

In this section, we will first introduce the basic notations needed. We will then formally define impulsive switched systems and provide some stability definitions for such systems.

A motivational example is also presented at the end of this section to highlight the central theme of our research.

2.1. Mathematical notations

Let \mathbb{N} be the set of natural numbers including 0, $\mathbb{R}, \mathbb{R}_{\geq 0}, \mathbb{R}_{> 0}$ be the set of all real numbers, non-negative real numbers and positive real numbers, respectively. For any vector $x \in \mathbb{R}^n$, denote its 2-norm by $|x|$. Moreover, for any set $\mathcal{A} \subset \mathbb{R}^n$, let $|x|_{\mathcal{A}} := \inf_{y \in \mathcal{A}} |x - y|$. With some abuse of notation, when \mathcal{T} is a countable set, $|\mathcal{T}|$ means its cardinality. For a function $f : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}^n$ whose left limit exists at $t > 0$, denote this left limit by $f(t^-) := \lim_{s \rightarrow t^-} f(s)$.

A function $\alpha : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ is said to be of class \mathcal{PD} (positive definite) if it is continuous, $\alpha(0) = 0$ and $\alpha(s) > 0$ for all $s > 0$. If α is also strictly increasing, then it is said to be of class \mathcal{K} . In addition, if α is also unbounded then it is said to be of class \mathcal{K}_∞ . A function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ is said to be of class \mathcal{KL} if $\beta(\cdot, t)$ is of class \mathcal{K} for each $t \in \mathbb{R}_{\geq 0}$, $\beta(s, \cdot)$ is continuously decreasing and $\beta(s, t) \rightarrow 0$ as $t \rightarrow \infty$ for each fixed $s \in \mathbb{R}_{\geq 0}$; see [17, Chapter 4] for their use in formulation of common stability notions. In addition, we require class \mathcal{KLL} functions: A function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ is a class \mathcal{KLL} function if $\beta(\cdot, \cdot, j)$ is a class \mathcal{KL} function for each $j \geq 0$ and $\beta(\cdot, t, \cdot)$ is a class \mathcal{K} function for each $t \geq 0$.

2.2. Definition of impulsive switched systems

Let $\mathcal{S} \subset \mathbb{N}$ be a finite set. An *impulsive switching signal* is a pair (σ, \mathcal{T}) , where $\sigma : \mathbb{R}_{\geq 0} \rightarrow \mathcal{S}$ is a right-continuous, piecewise constant function and $\mathcal{T} := \{t_1, t_2, \dots\}$ is a countable subset of $\mathbb{R}_{> 0}$ such that the following conditions are satisfied:

- (Compatibility) all discontinuities of σ belong to \mathcal{T} ;
- (Non-Zeno) for any bounded interval I , $|I \cap \mathcal{T}|$ is finite.

Let $\bar{\Sigma}$ be the set of all admissible impulsive switching signals. In particular, for any $(i, j) \in \mathcal{S}^2$, let $\mathcal{T}_{i,j} := \{t \in \mathcal{T} : \sigma(t^-) = i, \sigma(t) = j\}$. With this definition, $\mathcal{T}_{i,j}$ is the collection of all switching instants from mode i to mode j when $i \neq j$, and $\mathcal{T}_{i,i}$ is the collection of all impulses while maintaining the mode to be i . It is easy to see that when $\mathcal{T}_{i,i} = \emptyset$ for all $i \in \mathcal{S}$, the impulsive switching signal reduces to the usual switching signal and when \mathcal{S} is a singleton, the impulsive switching signal reduces to the usual impulse signal.

For a given impulsive switching signal $(\sigma, \mathcal{T}) \in \bar{\Sigma}$ and depending on whether the system is autonomous (i.e., has no exogenous input) or it has an input, we consider the following two types of impulsive switched systems:

$$\dot{x}(t) = f_{\sigma(t)}(x(t)) \quad \text{for almost all } t \notin \mathcal{T}, \quad (1a)$$

$$x(t) = g_{i,j}(x(t^-)) \quad t \in \mathcal{T}_{i,j}, (i, j) \in \mathcal{S}^2, \quad (1b)$$

or

$$\dot{x}(t) = f_{\sigma(t)}(x(t), w(t)) \quad \text{for almost all } t \notin \mathcal{T}, \quad (2a)$$

$$x(t) = g_{i,j}(x(t^-), w(t)) \quad t \in \mathcal{T}_{i,j}, (i, j) \in \mathcal{S}^2. \quad (2b)$$

Here, $x : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}^n$ is the state trajectory, $w : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}^m$ is the input signal and it is assumed to be Lebesgue integrable, locally essentially bounded on $\mathbb{R}_{\geq 0}$ and bounded on \mathcal{T} . We denote by \mathcal{M}_w the set of all such admissible input signals. Moreover, $f_i : \mathbb{R}^n \rightarrow \mathbb{R}^n$ as in (1a) and $f_i : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$ as in (2a) are assumed to be locally Lipschitz in their arguments, and $g_{i,j} : \mathbb{R}^n \rightarrow \mathbb{R}^n$ as in (1b) and $g_{i,j} : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$ as in (2b) are assumed to be continuous in their arguments. With these assumptions, the solutions of differential equations (1a), (2a) are well-defined and absolutely continuous [13, Chapter 1]. The discontinuities introduced by the equations (1b), (2b) make the state-trajectories piecewise continuous, and we work with solutions that are right-continuous. We call (1a), (2a) the *continuous flow* and (1b), (2b) the *discrete jumps*. We denote the solution of (1) with initial state x_0 and impulsive switching signal (σ, \mathcal{T}) by $x(\cdot; x_0, \sigma, \mathcal{T})$. Similarly, we denote the solution of (2) with initial state x_0 , input w and impulsive switching signal (σ, \mathcal{T}) by $x(\cdot; x_0, w, \sigma, \mathcal{T})$. When $x_0, w, (\sigma, \mathcal{T})$ are clear from the context, the solution is also abbreviated by x , same as the state trajectory.

2.3. Stability definitions for impulsive switched systems

In this paper, we will focus on one internal stability property for the autonomous system (1) and two external stability properties for the system with an input (2). Moreover, we consider these stability notions with respect to a compact set $\mathcal{A} \subset \mathbb{R}^n$, and provide conditions that establish these properties uniformly with respect to the impulsive switching signals in a subset $\Sigma \subset \bar{\Sigma}$. In what follows, the essential supremum norm of w over a set I (that is, the supremum of w on I except for a set of measure zero) is denoted by $\text{ess sup}_{s \in I} |w(s)|$.

Definition 1. An impulsive switched system (1) is *globally asymptotically stable* (GAS) uniformly over Σ if there exist $\beta \in \mathcal{KL}$ such that

$$|x(t; x_0, \sigma, \mathcal{T})|_{\mathcal{A}} \leq \beta(|x_0|_{\mathcal{A}}, t) \quad (3)$$

for all $t \geq 0, x_0 \in \mathbb{R}^n$ and $(\sigma, \mathcal{T}) \in \Sigma$.

Definition 2. An impulsive switched system (2) is *input-to-state stable* (ISS) uniformly over Σ if there exist $\beta \in \mathcal{KL}, \gamma_1, \gamma_2 \in \mathcal{K}$ such that

$$|x(t; x_0, w, \sigma, \mathcal{T})|_{\mathcal{A}} \leq \beta(|x_0|_{\mathcal{A}}, t) + \gamma_1(\text{ess sup}_{s \in (0, t]} |w(s)|) + \gamma_2(\sup_{s \in (0, t] \cap \mathcal{T}} |w(s)|) \quad (4)$$

for all $t \geq 0, x_0 \in \mathbb{R}^n, w \in \mathcal{M}_w$ and $(\sigma, \mathcal{T}) \in \Sigma$.

Definition 3. An impulsive switched system (2) is *integral input-to-state stable* (iISS) uniformly over Σ if there exist $\alpha_0 \in \mathcal{K}_\infty, \beta \in \mathcal{KL}, \gamma_1, \gamma_2 \in \mathcal{K}$ such that

$$\alpha_0(|x(t; x_0, w, \sigma, \mathcal{T})|_{\mathcal{A}}) \leq \beta(|x_0|_{\mathcal{A}}, t) + \int_0^t \gamma_1(|w(s)|) ds$$

$$+ \sum_{s \in (0, t] \cap \mathcal{T}} \gamma_2(|w(s)|) \quad (5)$$

for all $t \geq 0, x_0 \in \mathbb{R}^n, w \in \mathcal{M}_w$ and $(\sigma, \mathcal{T}) \in \Sigma$.

Our definitions of ISS and iISS are adopted from [3] and [26]. Notice that both ISS and iISS require the unforced system, i.e., system (2) with $w(t) \equiv 0$, to be GAS. Also notice that the last terms in (4), and in (5) do not appear in standard definitions of ISS and iISS for non-switched systems; these additional terms are needed in our framework to capture the growth of state trajectories at switching/impulse instants due to the presence of inputs in the discrete jump maps (2b).

2.4. Motivational example

Consider a one-dimensional switched system with 3 modes:

$$\dot{x}(t) = \begin{cases} x(t) & \sigma(t) = 1, \\ -x(t) & \sigma(t) = 2, \\ -2x(t) & \sigma(t) = 3, \end{cases} \quad t \notin \mathcal{T},$$

$$x(t) = \begin{cases} 2x(t^-) & t \in \mathcal{T}_{1,3} \cup \mathcal{T}_{2,3}, \\ x(t^-) & \text{otherwise.} \end{cases} \quad t \in \mathcal{T}.$$

To ensure GAS of the set $\mathcal{A} = \{0\}$, for a class of switching signals which allow occasional activation of the unstable mode 1, the system must consistently switch to either mode 2 or mode 3, both of which are stable. While [25] provides an upper bound on the average activation time for mode 1 based on switching frequency, a less conservative stability condition should consider the specific switches, particularly acknowledging that switches to mode 3 amplify the state and thus have a more destabilizing effect than the other switches. Conversely, as dwelling in mode 3 has a more stabilizing impact than mode 2, a less conservative stability condition should also take into account the activation time of each mode. In the context of this one-dimensional system with linear flow and jumps, establishing a class of impulsive switching signals which guarantee stability is already intricate. This complexity intensifies when dealing with non-linear flows, jumps, and external inputs – a challenge that forms the core focus of our research. In particular, the results of Section 4 provide a class of switching signals for which this system is GAS, while taking into consideration the issues raised here.

3. Impulsive switching signals defined via functional inequalities

Given a constant $T > 0$, a function $c : \mathcal{S} \rightarrow \mathbb{R}$ and a function $d : \mathcal{S}^2 \times \mathbb{R}^n \rightarrow \mathbb{R}$ such that $d(i, j, x)$ is continuous in x , we consider impulsive switching signals (σ, \mathcal{T}) satisfying the following functional inequality:

$$\int_s^t c(\sigma(\tau)) d\tau + \sum_{\tau \in (s, t] \cap \mathcal{T}} d(\sigma(\tau^-), \sigma(\tau), x(\tau^-)) \leq T, \quad (6)$$

for all $t > s \geq 0$. Define $\Sigma(c, d, T) := \{(\sigma, \mathcal{T}) \in \bar{\Sigma} : (6) \text{ holds}\}$. It is not difficult to see that the inequality (6) is a time/mode/state-dependent condition on the impulsive switching signals. However, full information of the state trajectory $x(\cdot)$ is unnecessary for checking (6). In fact, because σ is piece-wise constant, only the set \mathcal{T} , and $\sigma(t_i^-), x(t_i^-)$ for any $t_i \in \mathcal{T}$ are needed (note that $\sigma(t_i) = \sigma(t_{i+1}^-)$, where t_{i+1} is the first switching/impulse instant after t_i). In other words, we only need to know the set of switching/impulse instants, the values of the impulsive switching signal and the state before each switching/impulse instant and conclude whether $(\sigma, \mathcal{T}) \in \Sigma(c, d, T)$.

As discussed in Section 2, since GAS, ISS, iISS for impulsive switched systems are defined to be uniform with respect to a class of impulsive switching signals, we need to ensure that $\Sigma(c, d, T)$ is significantly large. In this section, we first show that depending on the functional parameters c, d appearing in the definition, $\Sigma(c, d, T)$ in fact encompasses many well-known classes of switching signals. We then show that when c, d satisfy some mild conditions, $\Sigma(c, d, T)$ is non-empty and the inequality (6) does not allow switching signals with Zeno phenomenon (that is, accumulation of infinitely many switching/impulse time instants over a finite interval). We also show that any $(\sigma, \mathcal{T}) \in \Sigma(c, d, T)$ can be generated online via the use of an auxiliary timer.

3.1. Comparison with known sets of switching signals

Here, we recall that, the paper [14, Theorem 1] considered stability of impulsive systems with impulsive sequences $\mathcal{S}[\mu, \lambda]$ that satisfy

$$\forall t > s \geq 0 : -d^* |(s, t] \cap \mathcal{T}| - (c^* - \lambda)(t - s) \leq \mu,$$

for some $c^*, d^* \in \mathbb{R}$, and $\mu, \lambda > 0$. This class of impulsive signals can be obtained from (6) by setting $T = \mu$, $c(i) \equiv \lambda - c^*$, and $d(i, j, x) \equiv -d^*$. Since the set $\mathcal{S}[\mu, \lambda]$ is equivalent to the class of *average dwell-time* (ADT) or *reversed average dwell-time* (rADT) switching signals depending on the signs of $c^* - \lambda, d^*$, the set $\Sigma(c, d, T)$ can also represent classes of ADT or rADT switching signals, denoted by Σ_{ADT} or Σ_{rADT} respectively, by selecting proper constant functions c, d . In addition to that, since we allow the functions c to be mode-dependent and d to be both mode- and state-dependent, the set of impulsive switching signals $\Sigma(c, d, T)$ can be made more general. In this subsection, we show that $\Sigma(c, d, T)$ can be the superset of many well-known classes of switching signals, some of which cannot be modeled by $\mathcal{S}[\mu, \lambda]$.

Assume that $d(i, j, x) \leq 0$ for all $(i, j) \in \mathcal{S}^2, x \in \mathbb{R}^n$ and there exist a partition $\mathcal{S} = \mathcal{S}_s \cup \mathcal{S}_u$ and $c_s^* < 0, c_u^* \geq 0$ such that $c(i) \leq c_s^*$ for all $i \in \mathcal{S}_s$ and $c(i) \leq c_u^*$ for all $i \in \mathcal{S}_u$. We have

$$\begin{aligned} \int_s^t c(\sigma(\tau)) d\tau &\leq c_s^* \int_s^t \mathbf{1}_{\sigma(\tau) \in \mathcal{S}_s} d\tau + c_u^* \int_s^t \mathbf{1}_{\sigma(\tau) \in \mathcal{S}_u} d\tau \\ &\leq c_s^*(t - s) + (c_u^* - c_s^*) \int_s^t \mathbf{1}_{\sigma(\tau) \in \mathcal{S}_u} d\tau, \end{aligned}$$

where $\mathbf{1}_{(\cdot)}$ is the indicator function. Hence, (6) holds if

$$\int_s^t \mathbf{1}_{\sigma(t) \in \mathcal{J}_u} d\tau \leq -\frac{c_s^*(t-s)}{c_u^* - c_s^*} + \frac{T}{c_u^* - c_s^*}. \quad (7)$$

This is the *average activation time* (AAT) condition of modes in the set \mathcal{J}_u [25], with AAT parameter $-\frac{c_s^*}{c_u^* - c_s^*}$ and chatter bound $\frac{T}{c_u^* - c_s^*}$. Denote the switching signals satisfying (7) by Σ_{AAT} . Meanwhile, it can also be computed that (6) holds if

$$\int_s^t \mathbf{1}_{\sigma(t) \in \mathcal{J}_s} d\tau \geq \frac{c_u^*(t-s)}{c_u^* - c_s^*} - \frac{T}{c_u^* - c_s^*}. \quad (8)$$

This is the *reversed average activation time* (rAAT) condition of modes in the set \mathcal{J}_s , with rAAT parameter $\frac{c_u^*}{c_u^* - c_s^*}$ and chatter bound $-\frac{T}{c_u^* - c_s^*}$. Denote the switching signals satisfying (8) by Σ_{rAAT} .

Finally, suppose that the system state does not jump after each switch. Assume $c(i) \leq 0$ for all $i \in \mathcal{J}$. Let $V_i : \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$ be some value functions for each $i \in \mathcal{J}$ and construct $d(i, j, x) := \ln V_j(x) - \ln V_i(x) + \epsilon$ for some $\epsilon > 0$. Then, it is not difficult to see that (6) holds if $d(\sigma(t^-), \sigma(t), x(t^-)) \leq 0$, which is equivalent to

$$V_{\sigma(t)}(x(t)) \leq e^{-\epsilon} V_{\sigma(t^-)}(x(t^-)) \quad \forall t \in \mathcal{T}, \quad (9)$$

which is a state-dependent condition. Denote switching signals satisfying (9) by Σ_{SD} .

So far, it is seen that under some particular conditions on the functions $c(i), d(i, j, x)$, the set of switching signals $\Sigma_{\text{ADT}}, \Sigma_{\text{rADT}}, \Sigma_{\text{AAT}}, \Sigma_{\text{rAAT}}, \Sigma_{\text{SD}}$ are all subsets of $\Sigma(c, d, T)$. As a result, an impulsive switched system is GAS, ISS or iISS uniformly over $\Sigma_{\text{ADT}}, \Sigma_{\text{rADT}}, \Sigma_{\text{AAT}}, \Sigma_{\text{rAAT}}$ or Σ_{SD} if it is also GAS, ISS or iISS uniformly over $\Sigma(c, d, T)$. It can be acknowledged that neither ADT nor rADT conditions are mode-dependent, and AAT, rAAT conditions are only partially mode-dependent, in the sense that they only differentiate modes i with either negative or non-negative values of $c(i)$. Meanwhile, the state-dependent condition (9) is not time-dependent. The comparisons show that the set of impulsive switching signals defined via the functional inequalities (6) is a generalization of the combination of all the aforementioned sets of switching signals.

3.2. Regularity considerations

Stability analysis in this paper will become vacuous if $\Sigma(c, d, T) = \emptyset$. This could happen, for example, when $c(i) > 0$ and $d(i, j, x) \geq 0$ for all $i, j \in \mathcal{J}, x \in \mathbb{R}^n$. Clearly in this case, (6) will be violated when $t - s$ is sufficiently large. The first proposition in this section provides sufficient conditions, under which $\Sigma(c, d, T)$ is non-empty.

Proposition 3.1. *If there exists $i \in \mathcal{J}$ such that $c(i) \leq 0$, or there exist $d^* < 0$ and k modes $i_1, i_2, \dots, i_k \in \mathcal{J}$ such that $d(i_l, i_{l+1}, x) \leq d^*$ for all $x \in \mathbb{R}^n$ and $l = 1, 2, \dots, k$, where $i_{k+1} := i_1$, then $\Sigma(c, d, T) \neq \emptyset$ for any $T > 0$.*

Proof. Clearly when $c(i) \leq 0$, the constant signal with $\mathcal{T} = \emptyset, \sigma(t) = i$ for all $t \geq 0$ satisfies (6), as its left-hand side is always non-positive.

When $c(i) > 0$ for all $i \in \mathcal{J}$ and the other conditions in Proposition 3.1 hold, consider the periodic impulsive switching signal with $\mathcal{T} = \{l\Delta t : l = 1, 2, \dots\}$, where $\Delta t := \frac{\min\{T, -d^*\}}{\max_{i \in \mathcal{J}} c(i)}$, and $\sigma(t) = i_{1+\text{mod}(l, k)}$ for all $t \in [l\Delta t, (l+1)\Delta t)$, $l \in \mathbb{N}$. For any interval $[s, t)$, let $j \in \mathbb{N}$ be such that $t - s \in [j\Delta t, (j+1)\Delta t)$. Then there can be at most j switches/impulses over the interval $[s, t)$ according to the definition of \mathcal{T} . We have

$$\begin{aligned} & \int_s^t c(\sigma(\tau)) d\tau + \sum_{\tau \in (s, t] \cap \mathcal{T}} d(\sigma(\tau^-), \sigma(\tau), x(\tau^-)) \\ & \leq (t-s) \max_{i \in \mathcal{J}} c(i) + jd^* \leq (j+1)\Delta t \max_{i \in \mathcal{J}} c(i) + jd^* \\ & = (j+1) \min\{T, -d^*\} + jd^* \leq T. \end{aligned}$$

Hence $(\sigma, \mathcal{T}) \in \Sigma(c, d, T)$. \square

We remark here that the set of switching signals $\Sigma_{\text{ADT}}, \Sigma_{\text{rADT}}, \Sigma_{\text{AAT}}, \Sigma_{\text{rAAT}}, \Sigma_{\text{SD}}$ as discussed in Section 3.1 all satisfy Proposition 3.1 and hence they are all non-empty.

Note that since $\Sigma(c, d, T)$ is defined to be a subset of $\bar{\Sigma}$, it naturally excludes Zeno signals. Nevertheless, Zeno signals may satisfy the condition (6). For example, when $d(i, j, x) < 0$ for all $i, j \in \mathcal{J}, x \in \mathbb{R}^n$, then (6) allows signals with infinitely many switches/impulses over a finite interval. This observation leads to the following straightforward sufficient condition, under which (6) automatically excludes Zeno signals:

Proposition 3.2. *When there exists $d^* > 0$ such that $d(i, j, x) \geq d^*$ for all $i, j \in \mathcal{J}, x \in \mathbb{R}^n$, then (σ, \mathcal{T}) satisfying (6) cannot be Zeno.*

Proof. Clearly in this case, infinitely many switches/impulses over a finite time interval will lead to the left-hand side of (6) being infinite. \square

Proposition 3.2 directly concludes that the switching signals satisfying the ADT condition are non-Zeno, since in this case, we have $d(i, j, x) = d^* > 0$ for all $i, j \in \mathcal{J}, x \in \mathbb{R}^n$. A less trivial sufficient condition guaranteeing non-Zeno signals is observed for autonomous impulsive switched systems with convergent jumps and bounded finite-time reachable set. This sufficient condition allows negative $d(i, j, x)$ and is stated below.

Proposition 3.3. *Consider the autonomous impulsive switched system modeled by (1). Assume that there exist $\epsilon > 0$, continuous functions $\psi_i : \mathbb{R}^n \rightarrow \mathbb{R}$ for all $i \in \mathcal{J}$ such that*

$$d(i, j, x) \geq \psi_j(g_{i,j}(x)) - \psi_i(x) + \epsilon. \quad (10)$$

Further assume that for any initial state $x_0 \in \mathbb{R}^n$ and any impulsive switching signal (σ, \mathcal{T}) satisfying (6), the following two properties hold:

- The jumps are convergent. That is, either \mathcal{T} is finite, or $\mathcal{T} = \{t_1, t_2, \dots\}$ such that $\lim_{k \rightarrow \infty} |x(t_k) - x(t_k^-)| = 0$.
- The finite-time reachable set is bounded. That is, for any $T^* \geq 0$, there exists a compact set $\mathcal{R}(x_0, T^*) \subset \mathbb{R}^n$, independent of (σ, \mathcal{T}) , such that $x(t; x_0, \sigma, \mathcal{T}) \in \mathcal{R}(x_0, T^*)$ for all $t \in [0, T^*]$.

Then, such (σ, \mathcal{T}) satisfying (6) cannot be Zeno.

Proof. Pick arbitrary initial state $x_0 \in \mathbb{R}^n$. Suppose there is a Zeno signal (σ, \mathcal{T}) satisfying (6). Denote the accumulation point of switches/impulses as T^* ; that is, $\lim_{k \rightarrow \infty} t_k = T^* < \infty$. It follows from the uniform continuity of ψ_i over $\mathcal{R}(x_0, T^*)$ that there exists $\delta > 0$ such that for any $t_a, t_b \in [0, T^*]$,

$$|x(t_a) - x(t_b)| \leq \delta \Rightarrow |\psi_i(x(t_a)) - \psi_i(x(t_b))| \leq \frac{\epsilon}{2}, \quad \forall i \in \mathcal{I}. \quad (11)$$

It also follows from the uniform continuity of f_i over $\mathcal{R}(x_0, T^*)$ that, for a given $p \geq 1$, there exists $\tau_1^* > 0$ such that for any $t_k, t_{k+1} \in [0, T^*] \cap \mathcal{T}$,

$$|t_{k+1} - t_k| \leq \tau_1^* \Rightarrow |x(t_{k+1}^-) - x(t_k)| \leq \frac{\delta}{2p}. \quad (12)$$

Additionally, it follows from the convergent jumps property that there exists $\tau_2^* > 0$ such that for any $t_k \in [0, T^*] \cap \mathcal{T}$,

$$t_k \geq T^* - \tau_2^* \Rightarrow |x(t_k) - x(t_k^-)| \leq \frac{\delta}{2p}. \quad (13)$$

Define $\tau^* := \min\{\tau_1^*, \tau_2^*\}$ and let $s_0, s_1, s_2, \dots, s_r \in [T^* - \tau^*, T^*] \cap \mathcal{T}$ be consecutive switch/impulse instants such that $\sigma(s_r) = \sigma(s_0)$, called a cycle. Since there are p modes in \mathcal{I} , without loss of generality we choose r such that $r \leq p$.

Consider the case $r = 1$. Since $s_1 \geq T^* - \tau_2^*$, it follows from (13) that $|x(s_1) - x(s_1^-)| \leq \frac{\delta}{2p} \leq \delta$. It then follows from (11) and the fact $\sigma(s_1) = \sigma(s_0)$ that

$$d(\sigma(s_0), \sigma(s_1), x(s_1^-)) \geq \psi_{\sigma(s_1)}(x(s_1)) - \psi_{\sigma(s_0)}(x(s_1^-)) + \epsilon \geq \frac{\epsilon}{2}.$$

When $r \geq 2$, we can first conclude that

$$\begin{aligned} & \sum_{k=1}^r d(\sigma(s_k^-), \sigma(s_k), x(s_k^-)) \\ & \geq \sum_{k=1}^r (\psi_{\sigma(s_k)}(x(s_k)) - \psi_{\sigma(s_{k-1})}(x(s_k^-)) + \epsilon) \\ & = r\epsilon + \sum_{k=1}^r (\psi_{\sigma(s_{k-1})}(x(s_{k-1})) - \psi_{\sigma(s_{k-1})}(x(s_k^-))) \\ & \quad + \sum_{k=1}^r (\psi_{\sigma(s_k)}(x(s_k)) - \psi_{\sigma(s_{k-1})}(x(s_{k-1}))) \\ & = r\epsilon + \sum_{k=0}^{r-1} (\psi_{\sigma(s_k)}(x(s_k)) - \psi_{\sigma(s_k)}(x(s_{k+1}^-))) \\ & \quad + (\psi_{\sigma(s_r)}(x(s_r)) - \psi_{\sigma(s_0)}(x(s_0))). \end{aligned}$$

On one hand, it follows from (11) and (12) that $\psi_{\sigma(s_k)}(x(s_k)) - \psi_{\sigma(s_k)}(x(s_{k+1}^-)) \geq -\frac{\epsilon}{2}$. On the other hand, we first obtain from (12) and (13) that

$$\begin{aligned} |x(s_r) - x(s_0)| & \leq \sum_{k=1}^r |x(s_k) - x(s_k^-)| + \sum_{k=0}^{r-1} |x(s_{k+1}^-) - x(s_k)| \\ & \leq \frac{r\delta}{2p} + \frac{r\delta}{2p} \leq \delta. \end{aligned}$$

Thus, it is further implied by (11) and the fact $\sigma(s_r) = \sigma(s_0)$ that $\psi_{\sigma(s_r)}(x(s_r)) - \psi_{\sigma(s_0)}(x(s_0)) \geq -\frac{\epsilon}{2}$. Therefore, we conclude that

$$\sum_{k=1}^r d(\sigma(s_k^-), \sigma(s_k), x(s_k^-)) \geq r\epsilon - \frac{r\epsilon}{2} - \frac{\epsilon}{2} = \frac{(r-1)\epsilon}{2} \geq \frac{\epsilon}{2}.$$

In other words, over a cycle, $\sum_{k=1}^r d(\sigma(s_k^-), \sigma(s_k), x(s_k^-))$ is bounded below by $\frac{\epsilon}{2}$. Since $[T^* - \tau^*, T^*] \cap \mathcal{T}$ consists of infinitely many switches/impulses and \mathcal{I} is finite, it must contain infinitely many consecutive cycles [16] and hence the left-hand side of (6) is infinite when the time interval $[s, t)$ contains these cycles. This leads to the violation of (6) for any $T > 0$. \square

By applying Proposition 3.3 to Σ_{SD} when the system has no jumps, we conclude that the state-dependent switching signals satisfying (9) are non-Zeno, provided that the solution does not escape in finite time.

Proposition 3.3 requires the properties of convergent jumps and bounded finite-time reachable set, especially in order to bound the term $|\psi_{\sigma(s_r)}(x(s_r)) - \psi_{\sigma(s_0)}(x(s_0))|$. Such properties do not hold in general for impulsive switched systems with arbitrary inputs. Beyond Proposition 3.2 and Proposition 3.3, which only provide sufficient conditions for excluding Zeno impulsive switching signals, less conservative conditions with more information on the system dynamics can certainly be investigated in the future research.

3.3. Hybrid model of the switching signals

It may appear that, since the switching signals satisfying (6) allow for dependence on state, such switching signals cannot be pre-defined before running the impulsive switched system. With given functions c, d and constant T , stability results which are uniform with respect to $\Sigma(c, d, T)$ are meaningful if we can generate a switching signal $\sigma \in \Sigma(c, d, T)$ concurrently while the system is running. We show that this can be done by designing an auxiliary timer and modeling the impulsive switched system in the framework of hybrid systems, under the assumption that the mode-to-switch is known at each switching/impulse instant.

To this end, we define the hybrid state by $z = (y^\top \xi a b)^\top \in \mathcal{X} := \mathbb{R}^{n+1} \times \mathcal{A}$, where the component $y \in \mathbb{R}^n$ is a copy of the state x of the impulsive switched system, the component $\xi \in \mathbb{R}$ is the timer for dealing with the condition (6) and the components $a, b \in \mathcal{A}$ are the current mode and the mode-to-go, respectively. We start with an autonomous switched impulsive system; i.e., it has dynamics

given by (1). We appeal to the hybrid model in [8], described compactly by the following inclusions:

$$\begin{aligned} \dot{z} &\in F(z), & z &\in C \\ z^+ &\in G(z), & z &\in D \end{aligned} \quad (14)$$

with data C, D, F, G given by

$$C = \mathbb{R}^n \times [0, T] \times \mathcal{A}^2, \quad (15a)$$

$$D = \{(y, \xi, a, b) \in \mathcal{X} : \xi \in [0, T - \max\{d(a, b, y), 0\}]\}, \quad (15b)$$

$$F(z) = \begin{pmatrix} f_a(y) \\ [c(a), \max\{c(a), 0\}] \\ 0 \\ 0 \end{pmatrix}, \quad (15c)$$

$$G(z) = \begin{pmatrix} g_{a,b}(y) \\ \Xi(z) \\ b \\ \mathcal{A} \end{pmatrix}, \quad (15d)$$

where

$$\Xi(z) := \begin{cases} \{\xi + d(a, b, y)\} & \text{if } d(a, b, y) \geq 0, \\ [\max\{0, \xi + d(a, b, y)\}, \xi] & \text{if } d(a, b, y) < 0. \end{cases} \quad (16)$$

When the switched impulsive system has an input; i.e., it has dynamics given by (2), we alternatively consider the hybrid model in [31], described compactly by the following inclusions:

$$\begin{aligned} \dot{z} &\in F(z, u), & (z, u) &\in C \\ z^+ &\in G(z, u), & (z, u) &\in D \end{aligned} \quad (17)$$

with data C, D, F, G given by

$$C = \mathbb{R}^n \times [0, T] \times \mathcal{A}^2 \times \mathbb{R}^m, \quad (18a)$$

$$D = \{(y, \xi, a, b) \in \mathcal{X} : \xi \in [0, T - \max\{0, d(a, b, y)\}]\} \times \mathbb{R}^m, \quad (18b)$$

$$F(z, u) = \begin{pmatrix} f_a(y, u) \\ [c(a), \max\{c(a), 0\}] \\ 0 \\ 0 \end{pmatrix}, \quad (18c)$$

$$G(z, u) = \begin{pmatrix} g_{a,b}(y, u) \\ \Xi(z) \\ b \\ \mathcal{A} \end{pmatrix} \quad (18d)$$

where $\Xi(z)$ is defined in (16). It is not difficult to verify that the hybrid systems (15), (18) are well-posed as defined in [8], [31], in the sense that

- C and D are closed subsets of \mathcal{X} for (15) (resp. closed subsets of $\mathcal{X} \times \mathbb{R}^m$ for (18));
- F is outer semicontinuous and locally bounded relative to C and $F(z)$ (resp. $F(z, u)$) is convex for every $z \in C$ (resp. $(z, u) \in C$);

- G is outer semicontinuous and locally bounded relative to D .

We now provide some explanation of the hybrid systems (15), (18). Indeed, the first components in (15c), (15d) or (18c), (18d) show that y mimics the dynamics of the state of the impulsive switched system. To capture the condition (6), we let the timer ξ flow with a flow map in the set $[c(a), \max\{c(a), 0\}]$ when $\xi \in [0, T]$. Meanwhile, ξ can also jump – corresponding to a switch/impulse for the impulsive switched system – with a jump map in the set $\Xi(z)$ when $\xi \in [0, T - \max\{d(a, b, y), 0\}]$. When jump occurs, the current mode a is updated to the mode-to-go b , whereas b can be changed to any mode in \mathcal{A} .

Let $\text{dom } z \subset \mathbb{R}_{\geq 0} \times \mathbb{N}$ denote the hybrid time domain of a hybrid solution z of (15) or a hybrid solution pair (z, u) of (18). We provide a few definitions regarding hybrid solutions.

Definition 4. A hybrid solution z or a hybrid solution pair (z, u) is called

- Complete, if $\text{dom } z$ is unbounded;
- Non-Zeno, if either it is not complete, or $\text{dom}_t z := \{t : \exists j \text{ s.t. } (t, j) \in \text{dom } z\}$ is unbounded;
- Double-jump-free, if for any $t \in \text{dom}_t z$, the set $\{j : (t, j) \in \text{dom } z\}$ contains at most 2 elements;
- Proper, if it is complete, non-Zeno and double-jump-free.

The following proposition states the equivalence between the impulsive switching signal/state pair of the impulsive switched system and the solution of the hybrid system.

Proposition 3.4. *There is a surjective map from the proper hybrid solutions of (15) or proper hybrid solution pairs of (18) with hybrid input $u(t, j_t) = w(t)$, $j_t := \max\{j : (t, j) \in \text{dom } z\}$ to the switching signals $(\sigma, \mathcal{T}) \in \Sigma(c, d, T)$, given by*

$$\sigma(t) = a(t, j_t), \quad (19a)$$

$$\mathcal{T} = \{t : \exists j \text{ s.t. } (t, j), (t, j+1) \in \text{dom } z\}. \quad (19b)$$

Moreover, the corresponding state of the impulsive switched system is given by

$$x(t) = y(t, j_t). \quad (20)$$

The implication of Proposition 3.4 is two-folded. On the one hand, surjectivity and the formula (20) imply that we can conclude stability results for the impulsive switched system, by analyzing the corresponding stability properties of the hybrid system. On the other hand, we can always obtain an impulsive switching signal $(\sigma, \mathcal{T}) \in \Sigma(c, d, T)$ via the formula (19), by generating a proper solution or solution pair of the hybrid system.

Proof. We only provide the proof for the case of hybrid system (15), as the proof for the case of hybrid system (18) is almost identical.

To show that each impulsive switching signal $(\sigma, \mathcal{T}) \in \Sigma(c, d, T)$ can be expressed as a proper solution of the hybrid

system (15) via the formula (19), let $(\sigma, \mathcal{J}) \in \Sigma(c, d, T)$, $x : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}^n$ be an arbitrary pair of impulsive switching signal and the state trajectory of the impulsive switched system (1). Let the set of switching instants be $\mathcal{J} := \{t_1, t_2, \dots\}$ and take $t_0 = 0$ as the convention. We consider the case when \mathcal{J} is infinite, and define a complete hybrid arc $z : \text{dom } z \rightarrow \mathcal{X}$ as follows:

- $\text{dom } z = \bigcup_{i \in \mathbb{N}} [t_i, t_{i+1}] \times \{i\}$.
- $y(t, j) = x(t)$ for all $t \in [t_j, t_{j+1})$ and $y(t_{j+1}, j) = \lim_{t \rightarrow t_{j+1}^-} x(t)$.
- $a(t, j) = \sigma(t_j)$ for all $t \in [t_j, t_{j+1}]$.
- $b(t, j) = \sigma(t_{j+1})$ for all $t \in [t_j, t_{j+1}]$.
- $\xi(t, j) : \text{dom } z \rightarrow [0, T]$, with $t \in (t_j, t_{j+1})$, is constructed as follows:

$$\xi(0, 0) = 0, \quad (21a)$$

$$\frac{\partial}{\partial t} \xi(t, j) = \begin{cases} 0 & \text{if } \xi(t, j) = 0 \text{ and } c(a(t, j)) < 0, \\ c(a(t, j)) & \text{otherwise} \end{cases} \quad (21b)$$

$$\xi(t_j, j) = \lim_{t \rightarrow t_j^-} \xi(t, j), \quad (21c)$$

$$\xi(t_j, j+1) = \max\{0, \xi(t_j, j) + d(a(t_j, j), b(t_j, j), y(t_j, j))\}. \quad (21d)$$

The complete hybrid arc z when \mathcal{J} is finite or empty can also be defined similarly, taking into the account that the mode-to-go $b(t, j)$ can be assigned arbitrarily and the timer $\xi(t, j)$ can be defined to be constant after the last jump. By construction, (19) and (20) hold for all $t \geq 0$. Also, this hybrid solution is proper due to the properties of the impulsive switching signal (σ, \mathcal{J}) . We are left to show that the hybrid arc z defined above is a solution to the hybrid system with dynamics (15).

It is easy to check that $y(t, j), a(t, j), b(t, j)$ and the dynamics of $\xi(t, j)$ as in (21) are compatible with the continuous flow $F(z)$ and discrete jump $G(z)$ in (15). We now need to check that continuous flow happens only when $z \in C$ and discrete jump only happens when $z \in D$ and we will prove this by contradiction. Note that because both (21b) and (21d) will not change $\xi(t, j)$ to be negative, incompatibility will only occur if there exists $(\bar{t}, \bar{j}) \in \text{dom } z$ such that $\xi(\bar{t}, \bar{j}) > T$. In this case, let

$$(\underline{t}, \underline{j}) := \sup \{(t, j) \in \text{dom } z : (t, j) \leq (\bar{t}, \bar{j}), \xi(t, j) = 0\}.$$

It is noted that the set above is non-empty as $\xi(0, 0) = 0$. Also $(\underline{t}, \underline{j}) \in \text{dom } z$, as $\{(t, j) \in \text{dom } z : (t, j) \leq (\bar{t}, \bar{j})\}$ is compact and both $\xi(t, j), d(a(t, j), b(t, j), y(t, j))$ are continuous in t over each interval $[t_j, t_{j+1}] \times \{j\}$. Hence $\xi(\underline{t}, \underline{j}) = 0$, and for any $(t, j) \in \text{dom } z$ with $(\underline{t}, \underline{j}) < (t, j) \leq (\bar{t}, \bar{j})$, we have $\xi(t, j) > 0$. Thus, it follows from (21b) that $\frac{\partial}{\partial t} \xi(t, j) = c(a(t, j))$ during the flow. Moreover, it also implies that $\xi(t, j+1) - \xi(t, j) = d(a(t, j), b(t, j), x(t, j))$ at the jumps, or

otherwise the dynamics (21d) suggests that $\xi(t_j, j+1) = 0$, which contradicts the supremum property of $(\underline{t}, \underline{j})$. As a result, we have

$$\begin{aligned} T &< \xi(\bar{t}, \bar{j}) - \xi(\underline{t}, \underline{j}) \\ &= \int_{\underline{t}}^{\bar{t}} \frac{\partial}{\partial t} \xi(t, j_t) dt + \sum_{\substack{(t, j) \in \text{dom } z \text{ s.t.} \\ (\underline{t}, \underline{j}) < (t, j) \leq (\bar{t}, \bar{j}), (t, j+1) \in \text{dom } z}} (\xi(t, j+1) - \xi(t, j)) \\ &= \int_{\underline{t}}^{\bar{t}} c(a(t, j_t)) dt + \sum_{\substack{(t, j) \in \text{dom } z \text{ s.t.} \\ (\underline{t}, \underline{j}) \leq (t, j) \leq (\bar{t}, \bar{j}), (t, j+1) \in \text{dom } z}} d(a(t, j), b(t, j), y(t, j)) \\ &= \int_{\underline{t}}^{\bar{t}} c(\sigma(\tau)) d\tau + \sum_{\tau \in (s, t] \cap \mathcal{J}} d(\sigma(\tau^-), \sigma(\tau), x(\tau^-)) \end{aligned}$$

which is a contradiction to (6).

To show that all proper solutions of the hybrid system (15) can be mapped to impulse switching signals in $\Sigma(c, d, T)$ via the formula (19), we let $z : \text{dom } z \rightarrow \mathcal{X}$ be an arbitrary proper solutions of the hybrid system (15). Clearly, it yields $\sigma : \mathbb{R}_{\geq 0} \rightarrow \mathcal{J}$, $\mathcal{J} \subset \mathbb{R}_{>0}$ by (19) and $x : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}^n$ by (20). The compatibility condition is satisfied by construction and non-Zeno condition is satisfied because z is non-Zeno.

We are left to show that (σ, \mathcal{J}) satisfies the condition (6). Note that because z is complete, $\xi(t, j) \in [0, T]$ for all $(t, j) \in \text{dom } z$. Thus for any $t > s \geq 0$, it follows from the dynamics of ξ in (15c), (15d), and the fact z is double-jump-free that

$$\begin{aligned} &\int_s^t c(\sigma(\tau)) d\tau + \sum_{\tau \in (s, t] \cap \mathcal{J}(\sigma)} d(\sigma(\tau^-), \sigma(\tau), x(\tau^-)) \\ &= \int_s^t c(a(\tau, j_\tau)) d\tau + \sum_{\substack{(\tau, k) \in \text{dom } z \text{ s.t.} \\ (s, j_s) \leq (\tau, k) \leq (t, j_t), \\ (\tau, k+1) \in \text{dom } z}} d(a(\tau, k), b(\tau, k), y(\tau, k)) \\ &\leq \int_s^t \frac{\partial}{\partial \tau} \xi(\tau, j_\tau) d\tau + \sum_{\substack{(\tau, k) \in \text{dom } z \text{ s.t.} \\ (s, j_s) \leq (\tau, k) \leq (t, j_t), \\ (\tau, k+1) \in \text{dom } z}} (\xi(\tau, k+1) - \xi(\tau, k)) \\ &= \xi(t, j_t) - \xi(s, j_s) \leq T. \end{aligned}$$

which is exactly (6) and this completes the proof. \square

4. Stability of autonomous impulsive switched systems

Having discussed some properties of the impulsive switching signal, we now turn to the study of uniform stability properties of the impulsive switched systems. We first provide stability result for the autonomous impulsive switched systems (1). Following the methodology employed in [6], we adopt a similar approach by considering an assumption which requires the existence of multiple Lyapunov functions, each corresponding to a subsystem.

Assumption 1. There exist smooth Lyapunov functions $V_i : \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$ for all $i \in \mathcal{J}$, satisfying the conditions:

- There exist $\underline{\alpha}, \bar{\alpha} \in \mathcal{K}_\infty$ such that

$$\underline{\alpha}(|x|_{\mathcal{A}}) \leq V_i(x) \leq \bar{\alpha}(|x|_{\mathcal{A}}), \quad \forall x \in \mathbb{R}^n, i \in \mathcal{I}. \quad (22)$$

- There exist a partition $\mathcal{I} = \mathcal{I}_s \cup \mathcal{I}_u$, functions $\alpha_i \in \mathcal{D}\mathcal{D}$ such that

$$\left\langle \frac{\partial}{\partial x} V_i(x), f_i(x) \right\rangle \leq -\alpha_i(V_i(x)) \quad \forall x \in \mathbb{R}^n, i \in \mathcal{I}_s, \quad (23a)$$

$$\left\langle \frac{\partial}{\partial x} V_i(x), f_i(x) \right\rangle \leq \alpha_i(V_i(x)) \quad \forall x \in \mathbb{R}^n, i \in \mathcal{I}_u. \quad (23b)$$

Remark 4.1. Note that the inequality in (23a) is the equivalent definition of GAS for continuous-time systems [17]. Thus, all subsystems in \mathcal{I}_s are GAS. However, the partition $\mathcal{I}_s, \mathcal{I}_u$ is not necessarily an indicator whether a subsystem is GAS or not. This is because, even if a subsystem is GAS, it may still lead to the inequality (23b) due to the improper choice of V_i . In this case, this subsystem needs to be categorized to the set \mathcal{I}_u .

To state the main stability result, we need one regularity assumption on the functions $\alpha_i, i \in \mathcal{I}$.

Assumption 2. The functions α_i are globally one-sided Lipschitz; that is, for each $i \in \mathcal{I}$, there exists $\bar{c}_i > 0$ such that

$$\alpha_i(t) - \alpha_i(s) \leq \bar{c}_i(t - s) \quad (24)$$

for all $t \geq s \geq 0, i \in \mathcal{I}$.

Assumption 2 is not restrictive. Please refer to [22, Lemma 6] to see how an arbitrary positive definite function can be majorized to be a globally one-sided Lipschitz function with prescribed Lipschitz constant \bar{c}_i .

4.1. Stability result for autonomous systems

We now state the first main theorem of this paper:

Theorem 4.2. Consider the autonomous impulsive switched system (1) and assume that Assumptions 1,2 hold on the subsystems. Pick arbitrary positive constants $\kappa_i, i \in \mathcal{I}$ and define functions $c : \mathcal{I} \rightarrow \mathbb{R}, d : \mathcal{I}^2 \times \mathbb{R}^n \rightarrow \mathbb{R}$ with $d(i, j, x)$ continuous in x , such that

$$c(i) > -\kappa_i, \quad \forall i \in \mathcal{I}_s, \quad (25)$$

$$c(i) > \kappa_i, \quad \forall i \in \mathcal{I}_u, \quad (26)$$

and for all $i, j \in \mathcal{I}, x \in \mathbb{R}^n$, it holds that,

$$d(i, j, x) > \int_1^{V_j(g_{i,j}(x))} \frac{\kappa_j}{\alpha_j(r)} dr - \int_1^{V_i(x)} \frac{\kappa_i}{\alpha_i(r)} dr. \quad (27)$$

Then, for any $T > 0$, the system (1) is GAS uniformly over $\Sigma(c, d, T)$.

Proof. The inequalities (25), (26) imply the existence of $\epsilon > 0$ such that

$$c(i) \geq -\kappa_i + \epsilon \quad \forall i \in \mathcal{I}_s, \quad (28a)$$

$$c(i) \geq \kappa_i + \epsilon \quad \forall i \in \mathcal{I}_u. \quad (28b)$$

On the other hand, because both $d(i, j, x), \int_1^{V_j(g_{i,j}(x))} \frac{\kappa_j}{\alpha_j(r)} dr - \int_1^{V_i(x)} \frac{\kappa_i}{\alpha_i(r)} dr$ are continuous in x for each pair $(i, j) \in \mathcal{I}^2$, it is induced by (27) that there exists a continuous function $\tilde{\rho} : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{> 0}$ such that, for all $i, j \in \mathcal{I}$ and $x \in \mathbb{R}^n$,

$$d(i, j, x) - \left(\int_1^{V_j(g_{i,j}(x))} \frac{\kappa_j}{\alpha_j(r)} dr - \int_1^{V_i(x)} \frac{\kappa_i}{\alpha_i(r)} dr \right) \geq \tilde{\rho}(|x|_{\mathcal{A}}). \quad (29)$$

Define the hybrid Lyapunov function by

$$\bar{V}(z) := e^{T-\xi} \varphi_a(V_a(y)), \quad (30)$$

where

$$\varphi_i(s) := \begin{cases} \exp\left(\int_1^s \frac{\kappa_i}{\alpha_i(r)} dr\right), & \text{if } s > 0, \\ 0, & \text{if } s = 0. \end{cases} \quad (31)$$

For system (14), let us consider the attractor

$$\mathcal{A}^* := \mathcal{A} \times [0, T] \times \mathcal{I}^2, \quad (32)$$

then $|z|_{\mathcal{A}^*} = |y|_{\mathcal{A}}$. It is implied by [22, Lemma 8] that $\varphi_i \in \mathcal{K}_\infty$ for all $i \in \mathcal{I}$. Thus it follows from the condition (22) and the fact $\xi \in [0, T]$ that

$$\check{\alpha}(|z|_{\mathcal{A}^*}) \leq \bar{V}(z) \leq \hat{\alpha}(|z|_{\mathcal{A}^*}) \quad \forall z \in \mathcal{X}, \quad (33)$$

where $\check{\alpha}(s) := \min_{i \in \mathcal{I}} \varphi_i(\underline{\alpha}(s)), \hat{\alpha}(s) := e^T \max_{i \in \mathcal{I}} \varphi_i(\bar{\alpha}(s))$ and both $\check{\alpha}, \hat{\alpha} \in \mathcal{K}_\infty$. We will then examine the flow of \bar{V} when $z \in C$, and the jump of \bar{V} when $z \in D$ individually.

Pick arbitrary $z \in C, f \in F(z)$. When $a \in \mathcal{I}_s$, it follows from (15c), (23a) and (28a) that

$$\begin{aligned} \left\langle \frac{\partial}{\partial z} \bar{V}(z), f \right\rangle &\leq e^{T-\xi} \varphi'_a(V_a(y)) \left\langle \frac{\partial}{\partial y} V_a(y), f_a(y) \right\rangle \\ &\quad - c(a) e^{T-\xi} \varphi_a(V_a(y)) \\ &\leq -e^{T-\xi} \frac{\kappa_a \varphi_a(V_a(y))}{\alpha_a(V_a(y))} \alpha_a(V_a(y)) - c(a) e^{T-\xi} \varphi_a(V_a(y)) \\ &= -(\kappa_a + c(a)) e^{T-\xi} \varphi_a(V_a(y)) \\ &= -\epsilon \bar{V}(z), \end{aligned}$$

where we have also used the fact that the derivative of φ_i is given by

$$\varphi'_i(s) = \frac{\kappa_i \varphi_i(s)}{\alpha_i(s)}. \quad (34)$$

Similarly, when $a \in \mathcal{I}_u$, it follows from (15c), (23b) and (28b) that

$$\begin{aligned} \left\langle \frac{\partial}{\partial z} \bar{V}(z), f \right\rangle &\leq e^{T-\xi} \frac{\kappa_a \varphi_a(V_a(y))}{\alpha_a(V_a(y))} \alpha_a(V_a(y)) \\ &\quad - c(a) e^{T-\xi} \varphi_a(V_a(y)) \\ &= (\kappa_a - c(a)) e^{T-\xi} \varphi_a(V_a(y)) \\ &= -\epsilon \bar{V}(z). \end{aligned}$$

As a result, by appealing to the first inequality in (33), we conclude that

$$\left\langle \frac{\partial}{\partial z} \bar{V}(z), f \right\rangle \leq -\alpha_c(|z|_{\mathcal{A}^*}) \quad \forall z \in C, f \in F(z), \quad (35)$$

where $\alpha_c(s) := \epsilon \check{\alpha}(s) \in \mathcal{K}_\infty \subset \mathcal{PD}$.

On the other hand, pick arbitrary $z \in D, g \in G(z)$. It follows from (15d) and (29) that

$$\begin{aligned} \bar{V}(g) &\leq e^{T-\max\{0, \xi+d(a,b,y)\}} \varphi_b(V_b(g_{a,b}(y))) \\ &\leq e^{T-\xi-d(a,b,y)} \varphi_b(V_b(g_{a,b}(y))) \\ &= e^{T-\xi} \varphi_a(V_a(y)) e^{-d(a,b,y)} \frac{\varphi_b(V_b(g_{a,b}(y)))}{\varphi_a(V_a(y))} \\ &= \bar{V}(z) e^{-d(a,b,y) + \int_1^{V_b(g_{a,b}(y))} \frac{\kappa_b}{\alpha_b(r)} dr - \int_1^{V_a(y)} \frac{\kappa_a}{\alpha_a(r)} dr} \\ &\leq \bar{V}(z) e^{-\bar{\rho}(|z|_{\mathcal{A}^*})}. \end{aligned}$$

Hence $\bar{V}(g) - \bar{V}(z) \leq -H(z)$ for all $z \in D, g \in G(z)$, where $H(z) = \bar{V}(z)(1 - e^{-\bar{\rho}(|z|_{\mathcal{A}^*})})$. Because $H(z)$ is continuous in z , $H(z) = 0$ for all $z \in \mathcal{A}^*$ and $H(z) > 0$ for all $z \notin \mathcal{A}^*$, there exists $\alpha_d \in \mathcal{PD}$ such that $H(z) \geq \alpha_d(|z|_{\mathcal{A}^*})$ and hence

$$\bar{V}(g) - \bar{V}(z) < -\alpha_d(|z|_{\mathcal{A}^*}) \quad \forall z \in D, g \in G(z). \quad (36)$$

Therefore, by [8, Theorem 3.18], the conditions (33),(35),(36) imply that the hybrid system (15) is uniformly globally pre-asymptotically stable with respect to the set \mathcal{A}^* ; i.e., there exists $\beta^* \in \mathcal{KL}$ such that

$$|z(t, j)|_{\mathcal{A}^*} \leq \beta^*(|z_0|_{\mathcal{A}^*}, t, j) \quad (37)$$

holds for all $(t, j) \in \text{dom } z$. It then follows from the relation between the solutions of the impulsive switched system (1) and the solutions of the hybrid system (15) given by Proposition 3.4 that by picking $\beta(s, t) = \beta^*(s, t, 0)$, we conclude (3), which shows that the impulsive switched system is GAS. \square

We remark here that when the jumps are convergent and the finite-time reachable set is bounded (which hold when $g_{i,j}$ are all identity maps), the conditions (25), (26) and a condition slightly stronger than (27), in the form of the existence $\epsilon > 0$ such that, for all $i, j \in \mathcal{S}, x \in \mathbb{R}^n$,

$$d(i, j, x) \geq \int_1^{V_j(g_{i,j}(x))} \frac{\kappa_j}{\alpha_j(r)} dr - \int_1^{V_i(x)} \frac{\kappa_i}{\alpha_i(r)} dr + \epsilon$$

not only guarantee that the system is GAS uniformly over $\Sigma(c, d, T)$ by Theorem 4.2, but also ensure that all signals satisfying (6) are non-Zeno and hence belong to $\Sigma(c, d, T)$ by Proposition 3.3.

It is noted in Theorem 4.2 that since the parameters κ_i can be chosen arbitrarily, they will give different functions c, d . For many impulsive switched systems with mathematically similar subsystems, the functions α_i are comparable; that is, they are constant multiples of each other. In this case, there is a natural choice of κ_i which can simplify the expression (27). This is stated by the following corollary.

Corollary 4.3. Consider the autonomous impulsive switched system (1) and assume that Assumption 1 holds on the subsystems. Further assume that there exists a globally one-sided Lipschitz function $\alpha^* \in \mathcal{PD}$ such that $\alpha_i = \kappa_i \alpha^*$ for some $\kappa_i > 0$ and all $i \in \mathcal{S}$. Define functions $c : \mathcal{S} \rightarrow \mathbb{R}$ satisfying (25), (26) and $d : \mathcal{S}^2 \times \mathbb{R}^n \rightarrow \mathbb{R}$ with $d(i, j, x)$ continuous in x , satisfying

$$d(i, j, x) > \int_{V_i(x)}^{V_j(g_{i,j}(x))} \frac{1}{\alpha^*(r)} dr \quad \forall i, j \in \mathcal{S}, x \in \mathbb{R}^n. \quad (38)$$

Then, for any $T > 0$, the system (1) is GAS uniformly over $\Sigma(c, d, T)$.

The condition for $d(i, j, x)$ can be further simplified when α_i are linear functions.

Corollary 4.4. Consider the autonomous impulsive switched system (1) and assume that Assumption 1 holds on the subsystems. Further assume that $\alpha_i(s) = \kappa_i s$ for some $\kappa_i > 0$ and all $s \geq 0, i \in \mathcal{S}$. Define functions $c : \mathcal{S} \rightarrow \mathbb{R}$ satisfying (25), (26) and $d : \mathcal{S}^2 \times \mathbb{R}^n \rightarrow \mathbb{R}$ with $d(i, j, x)$ continuous in x , satisfying

$$d(i, j, x) > \ln V_j(g_{i,j}(x)) - \ln V_i(x). \quad (39)$$

Then, for any $T > 0$, the system (1) is GAS uniformly over $\Sigma(c, d, T)$.

4.2. A case study of autonomous systems

Here we provide a case study of an n -dimensional autonomous impulsive switched system with linear subsystems and linear jumps

$$\dot{x}(t) = A_{\sigma(t)} x(t), \quad t \notin \mathcal{T}, \quad (40a)$$

$$x(t) = B_{i,j} x(t^-), \quad t \in \mathcal{T}_{i,j}, (i, j) \in \mathcal{S}^2, \quad (40b)$$

and show how Theorem 4.2 can be used to identify a class of impulsive switching signals, uniformly over which the system is GAS. Case studies of impulsive switched systems with particular nonlinear subsystems and nonlinear jumps will appear later in Section 5, where the example systems have inputs.

Suppose that there is a partition $\mathcal{S} = \mathcal{S}_s \cup \mathcal{S}_u$ such that only the subsystems with indices in \mathcal{S}_s are GAS. We can therefore assume the existence of $\lambda_i > 0$ for all $i \in \mathcal{S}$ such that

$$A_i^\top P_i + P_i A_i + \lambda_i P_i \leq 0, \quad \forall i \in \mathcal{S}_s, \quad (41a)$$

$$A_i^\top P_i + P_i A_i - \lambda_i P_i \leq 0, \quad \forall i \in \mathcal{S}_u, \quad (41b)$$

where ≤ 0 means the matrix on the left is negative semi-definite. By picking the Lyapunov functions $V_i(x) = x^\top P_i x$, we conclude that Assumption 1 holds with $\underline{\alpha}(s) := \min_{i \in \mathcal{S}} \Lambda(P_i)$, $\bar{\alpha}(s) := \max_{i \in \mathcal{S}} \Lambda(P_i)$ and $\alpha_i(s) := \lambda_i s$, where $\Lambda(P_i)$ denotes the spectrum of P_i . Since all α_i are linear, the functional parameters $c(i), d(i, j, x)$ can be constructed according to (25), (26) and (39) with some $\epsilon > 0$ such that

$$c(i) = \begin{cases} -\lambda_i + \epsilon & \text{if } i \in \mathcal{S}_s, \\ \lambda_i + \epsilon & \text{if } i \in \mathcal{S}_u, \end{cases}$$

$$d(i, j, x) = \ln x^\top B_{i,j}^\top P_j B_{i,j} x - \ln x^\top P_i x + \epsilon.$$

Furthermore, Corollary 4.4 implies that the system (40) is GAS uniformly over $\Sigma(c, d, T)$.

Now if we assume $\mathcal{S}_u = \emptyset$, define $\lambda := \min_{i \in \mathcal{S}} \lambda_i$, and pick $\mu \geq 0$ such that

$$B_{i,j}^\top P_j B_{i,j} - \mu P_i \leq 0 \quad \forall (i, j) \in \mathcal{S}^2, \quad (42)$$

then we have $c(i) \leq -\lambda + \epsilon$, $d(i, j, x) \leq \ln \mu + \epsilon$ and hence $\Sigma(-\lambda + \epsilon, \ln \mu + \epsilon, T) \subset \Sigma(c, d, T)$. Meanwhile, according to the discussion in Section 3.1, $\Sigma(-\lambda + \epsilon, \ln \mu + \epsilon, T)$ is a class of switching signals satisfying ADT condition, with ADT parameter $\frac{\ln \mu + \epsilon}{\lambda - \epsilon}$ and chatter bound $\frac{T}{\ln \mu + \epsilon}$. Consequently, the system (40) is also GAS with ADT switching signals. Note that by taking $\epsilon \rightarrow 0$, the infimum of ADT parameter $\frac{\ln \mu}{\lambda}$ coincides with the result in [15].

On the contrary, if $\mathcal{S}_u \neq \emptyset$, let $\lambda_s := \min_{i \in \mathcal{S}_s} \lambda_i$, $\lambda_u := \max_{i \in \mathcal{S}_u} \lambda_i$. If we still assume that (42) holds for some $\mu \geq 0$, then we can similarly conclude that $c(i) \leq -\lambda_s + \epsilon$ for all $i \in \mathcal{S}_s$, $c(i) \leq \lambda_u + \epsilon$ for all $i \in \mathcal{S}_u$ and $d(i, j, x) \leq \ln \mu + \epsilon$. Hence, the condition (6) is satisfied if,

$$\begin{aligned} (-\lambda_s + \epsilon) \int_s^t \mathbf{1}_{\sigma(\tau) \in \mathcal{S}_s} d\tau + (\lambda_u + \epsilon) \int_s^t \mathbf{1}_{\sigma(\tau) \in \mathcal{S}_u} d\tau \\ + (\ln \mu + \epsilon) |(s, t] \cap \mathcal{T}| \leq T, \end{aligned}$$

for all $t > s \geq 0$. After rearranging the terms, we obtain

$$\begin{aligned} (\lambda_u + \lambda_s) \int_s^t \mathbf{1}_{\sigma(\tau)} d\tau + (\ln \mu + \epsilon) |(s, t] \cap \mathcal{T}| \\ \leq (\lambda_s - \epsilon)(t - s) + T, \end{aligned}$$

which is satisfied if the switching signals satisfy ADT and AAT conditions, in the sense that

$$\begin{aligned} \int_s^t \mathbf{1}_{\sigma(\tau)} d\tau &\leq \eta(t - s) + T_0, \\ |(s, t] \cap \mathcal{T}| &\leq \frac{t - s}{\tau_a} + N_0, \end{aligned}$$

with some parameters η, T_0, τ_a, N_0 such that

$$(\lambda_u + \lambda_s)\eta + \frac{\ln \mu + \epsilon}{\tau_a} \leq \lambda_s - \epsilon, \quad (43)$$

$$(\lambda_u + \lambda_s)T_0 + (\ln \mu + \epsilon)N_0 \leq T. \quad (44)$$

Divide both sides of (43) by λ_s and take $\epsilon \rightarrow 0$, we obtain

$$\left(\frac{\lambda_u}{\lambda_s} + 1 \right) \eta + \frac{\ln \mu}{\lambda_s \tau_a} < 1, \quad (45)$$

which is exactly the formula guaranteeing GAS of a switched system with unstable subsystems as in [22].

As a numerical illustration, let $n = 2, \mathcal{S} = \{1, 2, 3\}$ and consider the matrices

$$A_1 = \begin{pmatrix} -0.1 & -1 \\ 2 & -0.1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} -0.1 & -2 \\ 1 & -0.1 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 0.1 & -1 \\ 2 & 0.1 \end{pmatrix}, \quad (46)$$

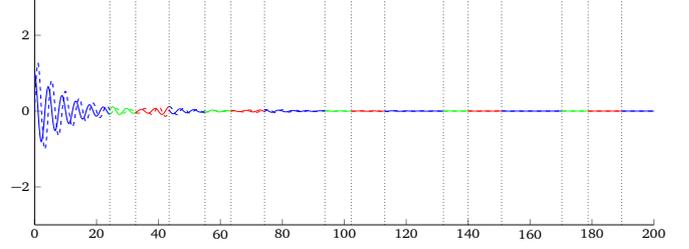


Figure 1: Plot of a solution of the system (40) with the given numerical parameters. The solid curve is $x_1(t)$ and dashed curve is $x_2(t)$. Different color represents the current activated mode. Blue: mode 1; green: mode 2; red: mode 3. The vertical dotted lines indicate the switching instants.

$$B_{i,j} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \forall i, j \in \mathcal{S}. \quad (47)$$

Notice that only A_1 and A_2 are Hurwitz matrices so $\mathcal{S}_s = \{1, 2\}$, $\mathcal{S}_u = \{3\}$. Meanwhile, $B_{i,j}$ has the property that whenever a switch/impulse occurs, the state x_1, x_2 will swap. It can be numerically verified that the inequalities (41) are satisfied with $\lambda_1 = \lambda_2 = 0.1335, \lambda_3 = 0.2665$ and matrices

$$P_1 = P_3 = \begin{pmatrix} 7.4876 & 0.1244 \\ 0.1244 & 3.7562 \end{pmatrix}, \quad P_2 = \begin{pmatrix} 3.7562 & -0.1244 \\ -0.1244 & 7.4876 \end{pmatrix}. \quad (48)$$

Furthermore, the inequality (42) is satisfied with $\mu = 2$. Pick $\epsilon = 0.01, T = 5$ and an initial state $x_0 = (1 \ 0)^\top$. With the help of HyEQ toolbox [30] in MATLAB, the solution trajectory of this impulsive switched system (40), subjected to an impulsive switching signal $(\sigma, \mathcal{T}) \in \Sigma(c, d, T)$ generated by the hybrid system (15), is plotted in Fig. 1. Indeed, this solution is convergent since the system is GAS, while there is an elapse of 12 units of time for each switch on average, with the unstable mode 3 being activated for 27% of the total time. On the other hand, when this system is subjected to ADT switching signals with ADT parameter $\tau_a = 12$, it can be computed by (45) that $\eta = 19\%$, smaller than 27% which is observed earlier when a switching signal from $\Sigma(c, d, T)$ is applied. This comparison shows that the results in this paper provide a broader class of impulsive switching signals, including those satisfying the ADT/AAT-mixed condition (45), uniformly over which the impulsive switched system is GAS.

5. Impulsive switched systems with inputs

We now switch our focus to impulsive switched systems with inputs (2). Similar stability results can also be concluded for such systems, based on an assumption requiring the existence of multiple Lyapunov functions corresponding to the individual subsystems.

Assumption 3. There exist \mathcal{C}^1 Lyapunov functions $V_i : \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$ for all $i \in \mathcal{S}$, satisfying the conditions:

- There exist $\underline{\alpha}, \bar{\alpha} \in \mathcal{K}_\infty$ such that

$$\underline{\alpha}(|x|_{\mathcal{S}}) \leq V_i(x) \leq \bar{\alpha}(|x|_{\mathcal{S}}), \quad \forall x \in \mathbb{R}^n, i \in \mathcal{S}. \quad (49)$$

- There exist a disjoint partition $\mathcal{S} = \mathcal{S}_s \cup \mathcal{S}_u$, functions $\alpha_i \in \mathcal{D}$ and $\gamma_c \in \mathcal{K}$ such that, for all $x \in \mathbb{R}^n, w \in \mathbb{R}^m$,

$$\left\langle \frac{\partial}{\partial x} V_i(x), f_i(x, w) \right\rangle \leq -\alpha_i(V_i(x)) + \gamma_c(|w|), \quad i \in \mathcal{S}_s, \quad (50a)$$

$$\left\langle \frac{\partial}{\partial x} V_i(x), f_i(x, w) \right\rangle \leq \alpha_i(V_p(x)) + \gamma_c(|w|), \quad i \in \mathcal{S}_u. \quad (50b)$$

- There exist continuous functions $\chi_{i,j} : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ for all $(i, j) \in \mathcal{S}^2$ and $\gamma_d \in \mathcal{K}$ such that

$$V_j(g_{i,j}(x, w)) \leq \chi_{i,j}(V_i(x)) + \gamma_d(|w|) \quad (51)$$

for all $x \in \mathbb{R}^n, w \in \mathbb{R}^m$.

Remark 5.1. Assumption 3 is the standard set of descriptions on the nonlinear dynamics of the subsystems and the possible jumps between them. For each subsystem in \mathcal{S}_s , the condition (50a) provide a necessary and sufficient condition for it to be iISS [1]. In addition if it is assumed that α_i in (50a) is \mathcal{K}_∞ , then each subsystem in \mathcal{S}_s is ISS. Meanwhile, the condition (51) describes how the input at switching/impulse instants may affect the jumps of the system state.

5.1. Stability result for systems with inputs

The stability results for iISS or ISS impulsive switched systems are summarized in the following theorem.

Theorem 5.2. *Consider the impulsive switched system with an input (2) and assume that Assumption 3 and Assumption 2 hold on the subsystems. Pick arbitrary positive constants $\kappa_i, i \in \mathcal{S}$ and define functions $c : \mathcal{S} \rightarrow \mathbb{R}$ satisfying (25), (26), $d : \mathcal{S}^2 \rightarrow \mathbb{R}$ with $d(i, j, x)$ continuous in x , such that, for all $i, j \in \mathcal{S}, x \in \mathbb{R}^n$,*

$$d(i, j, x) > \int_1^{\chi_{i,j}(V_i(x))} \frac{\kappa_j}{\alpha_j(r)} dr - \int_1^{V_i(x)} \frac{\kappa_i}{\alpha_i(r)} dr. \quad (52)$$

Then, for any $T > 0$, the system (2) is iISS uniformly over $\Sigma(c, d, T)$. Moreover, if it is assumed in Assumption 3 that $\alpha_i \in \mathcal{K}_\infty$, and instead of (52), there exists $\epsilon > 0$ such that, for all $i, j \in \mathcal{S}, x \in \mathbb{R}^n$,

$$d(i, j, x) \geq \int_1^{\chi_{i,j}(V_i(x))} \frac{\kappa_j}{\alpha_j(r)} dr - \int_1^{V_i(x)} \frac{\kappa_i}{\alpha_i(r)} dr + \epsilon. \quad (53)$$

Then, for any $T > 0$, the system (2) is ISS uniformly over $\Sigma(c, d, T)$.

Comparing Theorem 5.2 with Theorem 4.2, we observe the following two subtle differences:

- The condition (27) is almost identical to the condition (52), except that while the lower bound on $d(i, j, x)$ as in (27) relies on $V_j(g_{i,j}(x))$, which is the value of the new Lyapunov function after the jump, this term is relaxed by $\chi_{i,j}(V_i(x))$ in (52). This difference is due to the fact that when the arbitrary and unknown input signal w is present, the value of the new Lyapunov function $V_j(g_{i,j}(x, w))$ is not accessible before the jump and has to be estimated.

- The condition (53) is stronger than the condition (52) when further proving that an impulsive switched system is ISS uniformly over $\Sigma(c, d, T)$, in the sense that the difference between the left-hand side and right-hand side of (52) needs to be bounded uniformly with respect to x . Without this uniform condition (53), an impulsive switched system can be GAS with 0 input but not ISS. Such systems are recently studied in [28].

Proof. Similar to the proof of Theorem 4.2, there exists $\epsilon > 0$ (without loss of generality, can be assumed to be the same as the one for the inequality (53)) such that (28a), (28b) hold. Also, when (52) holds, there exists a continuous function $\tilde{\rho} : \mathbb{R}_{>0} \rightarrow \mathbb{R}_{>0}$ such that, for all $i, j \in \mathcal{S}, x \in \mathbb{R}^n$,

$$d(i, j, x) - \left(\int_1^{\chi_{i,j}(V_i(x))} \frac{\kappa_j}{\alpha_j(r)} dr - \int_1^{V_i(x)} \frac{\kappa_i}{\alpha_i(r)} dr \right) \geq \tilde{\rho}(|x|_{\mathcal{A}}). \quad (54)$$

Define the hybrid Lyapunov function by

$$\bar{W}(z) := \varphi_a^{-1}(\bar{V}(z)), \quad (55)$$

where \bar{V} is defined in (30) and φ_a is defined in (31). It can be directly seen that since $\xi \in [0, T]$, $\bar{W}(z) \geq V_a(y)$. Moreover, by defining $\theta(s) := \varphi_a^{-1}(e^{T-\xi} \varphi_a(s))$ with fixed $a \in \mathcal{S}, \xi \in [0, T]$, it can be concluded from Lemma A.2 that $\bar{W}(z) = \theta(V_a(y)) \leq e^{\frac{\xi}{\kappa_a}(T-\xi)}(V_a(y) - 0) + \theta(0) \leq e^{\frac{\xi}{\kappa_a} T} V_a(y)$. Define the set \mathcal{A}^* as in (32). Appeal to (49) and the fact that $|z|_{\mathcal{A}^*} = |y|_{\mathcal{A}}$, we have

$$\check{\alpha}(|z|_{\mathcal{A}^*}) \leq \bar{W}(z) \leq \hat{\alpha}(|z|_{\mathcal{A}^*}) \quad \forall z \in \mathcal{X}, \quad (56)$$

where $\check{\alpha}(s) := \underline{\alpha}(s)$, $\hat{\alpha}(s) := e^{\kappa T} \bar{\alpha}(s)$, and

$$\kappa := \max_{i \in \mathcal{S}} \frac{\bar{c}_i}{\kappa_i}, \quad (57)$$

and both $\check{\alpha}, \hat{\alpha} \in \mathcal{K}_\infty$. We will then examine the flow of \bar{W} when $(z, w) \in C$, and the jump of \bar{W} when $(z, w) \in D$ individually.

Pick arbitrary $(z, w) \in C, f \in F(z, w)$. When $a \in \mathcal{S}_s$, it follows from (18c), (50a) and (28a) that

$$\begin{aligned} \left\langle \frac{\partial}{\partial z} \bar{V}(z), f \right\rangle &\leq e^{T-\xi} \frac{\kappa_a \varphi_a(V_a(y))}{\alpha_a(V_a(y))} (-\alpha_a(V_a(y)) + \gamma_c(|w|)) \\ &\quad - c(a) e^{T-\xi} \varphi_a(V_a(y)) \\ &= -(\kappa_a + c(a)) e^{T-\xi} \varphi_a(V_a(y)) + e^{T-\xi} \frac{\kappa_a \varphi_a(V_a(y))}{\alpha_a(V_a(y))} \gamma_c(|w|) \\ &\leq -\epsilon \bar{V}(z) + e^T \frac{\kappa_a \bar{V}(z)}{\alpha_a(V_a(y))} \gamma_c(|w|). \end{aligned}$$

Similarly, when $a \in \mathcal{S}_u$, it follows from (18c), (50b) and (28b) that

$$\begin{aligned} \left\langle \frac{\partial}{\partial z} \bar{V}(z), f \right\rangle &\leq e^{T-\xi} \frac{\kappa_a \varphi_a(V_a(y))}{\alpha_a(V_a(y))} (\alpha_a(V_a(y)) + \gamma_c(|w|)) \\ &\quad - c(a) e^{T-\xi} \varphi_a(V_a(y)) \end{aligned}$$

$$\begin{aligned}
&= (c(a) - \kappa_a) e^{T-\xi} \varphi_a(V_a(y)) + e^{T-\xi} \frac{\kappa_a \varphi_a(V_a(y))}{\alpha_a(V_a(y))} \gamma_c(|w|) \\
&\leq -\epsilon \bar{V}(z) + e^T \frac{\kappa_a \bar{V}(z)}{\alpha_a(V_a(y))} \gamma_c(|w|).
\end{aligned}$$

In other words, in both cases, we have

$$\left\langle \frac{\partial}{\partial z} \bar{V}(z), f \right\rangle \leq -\epsilon \bar{V}(z) + e^T \frac{\kappa_a \bar{V}(z)}{\alpha_a(V_a(y))} \gamma_c(|w|). \quad (58)$$

Meanwhile, it follows from (34) that

$$(\varphi_i^{-1})'(s) = \frac{1}{\varphi_i'(\varphi_i^{-1}(s))} = \frac{\alpha_i(\varphi_i^{-1}(s))}{\kappa_a s}. \quad (59)$$

Hence,

$$\begin{aligned}
\left\langle \frac{\partial}{\partial z} \bar{W}(z), f \right\rangle &\leq \frac{\alpha_a(\varphi_a^{-1}(\bar{V}(z)))}{\kappa_a \bar{V}(z)} \left\langle \frac{\partial}{\partial z} \bar{V}(z), f \right\rangle \\
&\leq -\frac{\epsilon}{\kappa_a} \alpha_a(\bar{W}(z)) + e^T \frac{\alpha_a(\bar{W}(z))}{\alpha_a(V_a(y))} \gamma_c(|w|).
\end{aligned}$$

Finally, since $\varphi_a(\bar{W}(z)) = e^{T-\xi} \varphi_a(V_a(y)) \leq e^T \varphi_a(V_a(y))$, it follows from Lemma A.1 and (57) that $\alpha_a(\bar{W}(z)) \leq e^{\frac{\epsilon_a}{\kappa_a} T} \alpha_a(V_a(y))$. Thus, for every $(z, w) \in C, f \in F(z, w)$,

$$\left\langle \frac{\partial}{\partial z} \bar{W}(z), f \right\rangle \leq -\alpha_c^h(\bar{W}(z)) + \gamma_c(|w|) \quad (60)$$

where for any $s \geq 0$, $\alpha_c^h(s) := \epsilon \min_{i \in \mathcal{I}} \frac{\alpha_i(s)}{\kappa_i}$ and $\gamma_c(s) := e^{(1+\kappa)T} \gamma_c(s)$. We immediately observe that $\alpha_c^h \in \mathcal{P}\mathcal{D}$ and $\gamma_c \in \mathcal{X}$. Furthermore, there exists $\alpha_c \in \mathcal{P}\mathcal{D}$ such that $\alpha_c(|z|_{\mathcal{A}^*}) \leq \alpha_c^h(\bar{W}(z))$. Hence, for all $(z, w) \in C, f \in F(z, w)$,

$$\left\langle \frac{\partial}{\partial z} \bar{W}(z), f \right\rangle \leq -\alpha_c(|z|_{\mathcal{A}^*}) + \gamma_c(|w|). \quad (61)$$

Moreover, when all $\alpha_i \in \mathcal{X}_\infty$, we also have $\alpha_c \in \mathcal{X}_\infty$ (e.g., $\alpha_c(s) := \alpha_c^h(\check{\alpha}(s))$).

On the other hand, pick arbitrary $(z, w) \in D, g \in G(z, w)$. It follows from (18d), (51) that

$$\begin{aligned}
\bar{V}(g) &\leq e^{T-\max\{0, \xi+d(a,b,y)\}} \varphi_b(V_b(g_{a,b}(y))) \\
&\leq e^{T-\max\{0, \xi+d(a,b,y)\}} \varphi_b(\chi_{a,b}(V_a(y)) + \gamma_d(|w|)).
\end{aligned}$$

Define $\theta(s) := \varphi_b^{-1}(e^{T-\max\{0, \xi+d(a,b,y)\}} \varphi_b(s))$. We have $\bar{W}(g) = \varphi_b^{-1}(\bar{V}(g)) \leq \theta(\chi_{a,b}(V_a(y)) + \gamma_d(|w|))$. Hence, it follows from Lemma A.2 that

$$\begin{aligned}
\bar{W}(g) &\leq \theta(\chi_{a,b}(V_a(y))) + e^{\frac{\epsilon_b}{\kappa_b} (T-\max\{0, \xi+d(a,b,y)\})} \gamma_d(|w|) \\
&\leq \theta(\chi_{a,b}(V_a(y))) + e^{\kappa T} \gamma_d(|w|). \quad (62)
\end{aligned}$$

Meanwhile, it follows from (52) that

$$\begin{aligned}
\varphi_b(\theta(\chi_{a,b}(V_a(y)))) &= e^{T-\max\{0, \xi+d(a,b,y)\}} \varphi_b(\chi_{a,b}(V_a(y))) \\
&\leq e^{T-\xi-d(a,b,y)} \varphi_b(\chi_{a,b}(V_a(y)))
\end{aligned}$$

$$\begin{aligned}
&= e^{T-\xi} \varphi_a(V_a(y)) e^{-d(a,b,y)} \frac{\varphi_b(\chi_{a,b}(V_a(y)))}{\varphi_a(V_a(y))} \\
&= \bar{V}(z) e^{-d(a,b,y) + \int_1^{\chi_{a,b}(V_a(y))} \frac{\kappa_b}{a_b(r)} dr - \int_1^{V_a(y)} \frac{\kappa_a}{a_a(r)} dr} \\
&\leq e^{-\bar{\rho}(|y|)} \bar{V}(z) = e^{-\bar{\rho}(|z|_{\mathcal{A}^*})} \varphi_b(\bar{W}(z)).
\end{aligned}$$

Plug it into (62), we conclude that $\bar{W}(g) - \bar{W}(z) \leq -H(z) + \gamma_d(|w|)$, where $H(z) := \bar{W}(z) - \max_{i \in \mathcal{I}} \varphi_i^{-1}(e^{-\bar{\rho}(|z|_{\mathcal{A}^*})} \varphi_i(\bar{W}(z)))$, $\gamma_d(s) := e^{\kappa T} \gamma_d(s)$. Clearly $\gamma_d \in \mathcal{X}$. Similar to the proof of Theorem 4.2, because $H(z)$ is continuous in z , $H(z) = 0$ for all $z \in \mathcal{A}^*$ and $H(z) > 0$ for all $z \notin \mathcal{A}^*$, there exists $\alpha_d \in \mathcal{P}\mathcal{D}$ such that $H(z) \geq \alpha_d(|z|_{\mathcal{A}^*})$ and hence, for all $(z, w) \in D, g \in G(z, w)$,

$$\bar{W}(g) - \bar{W}(z) \leq -\alpha_d(|z|_{\mathcal{A}^*}) + \gamma_d(|w|). \quad (63)$$

In addition, the condition (53) ensures that $\bar{\rho}(|z|_{\mathcal{A}^*}) \geq \epsilon$ for all $z \in \mathcal{X}$. Thus, it is guaranteed by Lemma A.3 that $\alpha_d \in \mathcal{X}_\infty$ when $\alpha_i \in \mathcal{X}_\infty$ for all $i \in \mathcal{I}$.

Finally, depending on whether $\alpha_c, \alpha_d \in \mathcal{X}_\infty$ or $\mathcal{P}\mathcal{D}$, the conditions (56), (61), (63) imply that the hybrid system (18) is ISS by [3, Theorem 3.1] or iISS by [26, Theorem 1], respectively. In other words, there exist $\beta^* \in \mathcal{X}\mathcal{L}\mathcal{L}$, $\gamma_c^*, \gamma_d^* \in \mathcal{X}$ such that

$$\begin{aligned}
|z(t, j; \xi_0, u)|_{\mathcal{A}^*} &\leq \beta^*(|z_0|_{\mathcal{A}^*}, t, j) + \gamma_c^* \left(\text{ess sup}_{\hat{t}+\hat{j} \leq t+j} |u(\hat{t}, \hat{j})| \right) \\
&\quad + \gamma_d^* \sup_{\substack{(\hat{t}, \hat{j}) \in \text{dom } u \text{ s.t.} \\ (\hat{t}, \hat{j}+1) \in \text{dom } u, \hat{t}+\hat{j} \leq t+j}} |u(\hat{t}, \hat{j})|
\end{aligned}$$

holds for all $(t, j) \in \text{dom } z$, or additionally there exists $\alpha^* \in \mathcal{X}_\infty$ such that

$$\begin{aligned}
\alpha^*(|z(t, j; \xi_0, u)|_{\mathcal{A}^*}) &\leq \beta^*(|z_0|_{\mathcal{A}^*}, t, j) + \int_0^t \gamma_c^*(|u(s, j_s)|) ds \\
&\quad + \sum_{\substack{(\hat{t}, \hat{j}) \in \text{dom } u \text{ s.t.} \\ (\hat{t}, \hat{j}+1) \in \text{dom } u, \hat{t}+\hat{j} \leq t+j}} \gamma_d^*(|u(\hat{t}, \hat{j})|)
\end{aligned}$$

holds for all $(t, j) \in \text{dom } z$. It then follows from the relation between the solutions of the impulsive switched system (2) and the solutions of the hybrid system (18) given by Proposition 3.4 that by picking $\beta(s, t) = \beta^*(s, t, 0)$, $\gamma_1 = \gamma_c^*$, $\gamma_2 = \gamma_d^*$ and $\alpha_0 = \alpha^*$, we conclude the inequalities (4), (5), respectively. \square

Similar to the corollaries in Section 4.1, the formula for the function d in Theorem 5.2 can also be simplified when α_i are comparable or linear. These are summarized by the following results.

Corollary 5.3. *Consider the impulsive switched system with an input (2) and assume that Assumption 3 holds on the subsystems. Further assume that there exists $\alpha^* \in \mathcal{P}\mathcal{D}$ such that $\alpha_i = \kappa_i \alpha^*$ for some $\kappa_i \in \mathbb{R}$ and all $i \in \mathcal{I}$. Define functions*

$c : \mathcal{S} \rightarrow \mathbb{R}$ satisfying (25), (26) and $d : \mathcal{S}^2 \times \mathbb{R}^n \rightarrow \mathbb{R}$ with $d(i, j, x)$ continuous in x , satisfying

$$d(i, j, x) > \int_{V_i(x)}^{\chi_{i,j}(V_i(x))} \frac{1}{\alpha^*(r)} dr, \quad \forall i, j \in \mathcal{S}, x \in \mathbb{R}^n. \quad (64)$$

Then, for any $T > 0$, the system (2) is iISS uniformly over $\Sigma(c, d, T)$. Moreover, if $\alpha^* \in \mathcal{K}_\infty$, and instead of (52), there exists $\epsilon > 0$ such that

$$d(i, j, x) \geq \int_{V_i(x)}^{\chi_{i,j}(V_i(x))} \frac{1}{\alpha^*(r)} dr + \epsilon, \quad \forall i, j \in \mathcal{S}, x \in \mathbb{R}^n, \quad (65)$$

then, for any $T > 0$, the system (2) is ISS uniformly over $\Sigma(c, d, T)$.

Corollary 5.4. Consider the impulsive switched system with an input (2) and assume that Assumption 3 holds on the subsystems. Further assume that $\alpha_i(s) = \kappa_i s$ for some $\kappa_i \in \mathbb{R}$ and all $s \geq 0, i \in \mathcal{S}$. Define functions $c : \mathcal{S} \rightarrow \mathbb{R}$ satisfying (25), (26) and $d : \mathcal{S}^2 \times \mathbb{R}^n \rightarrow \mathbb{R}$ with $d(i, j, x)$ continuous in x , such that for some $\epsilon > 0$, it holds that

$$d(i, j, x) \geq \ln \chi_{i,j}(V_i(x)) - \ln V_i(x) + \epsilon, \quad \forall i, j \in \mathcal{S}, x \in \mathbb{R}^n. \quad (66)$$

Then, for any $T > 0$, the system (2) is ISS uniformly over $\Sigma(c, d, T)$.

5.2. Case studies for systems with inputs

We now illustrate through two case studies on how our theory developed in this work can be applied for finding a class of switching signals, uniformly over which a switched system with input is ISS or iISS. In the first example, a class of switched systems with inputs and nonlinear subsystems is studied. We show that although each subsystem is ISS, this system might not be ISS uniformly over any ADT switching signals. Nevertheless, our results still find a class of switching signals which preserves uniform ISS. These switching signals have the feature that the smaller magnitude of the state, the longer time it is required for the next switch to occur. Analogously, another class of switched systems with inputs and nonlinear subsystems is studied which might not be iISS uniformly over any ADT switching signals. Our results again find a class of switching signals which preserves uniform iISS. Unlike the previous example, this time, these switching signals have the feature that the larger magnitude of the state, the longer time it is required for the next switch to occur.

ISS switched system with nonlinear subsystems

Consider a switched system with nonlinear subsystems

$$\dot{x} = (x^\top P_i x)^k A_i x + B_i w \quad (67)$$

where $k > 0$ and assume that there are no jumps of state at switching instants. For simplicity, we assume that all A_i matrices are Hurwitz and P_i are positive definite matrices such that (41a) holds for some $\lambda_i > 0$. Since

$$\lim_{|x| \rightarrow 0} \frac{|(x^\top P_i x)^k A_i x|}{|x|} = 0,$$

it is implied by [7, Theorem 1] that there exist certain A_i matrices such that for any $\tau_d > 0$, there exists a switching signal σ satisfying ADT condition with ADT parameter τ_d , but the solution of the unforced switched system does not converge to the origin. Hence the switched system made up with subsystems (67) is not ISS uniformly for any ADT switching signals.

We now show that this switched system can be ensured to be ISS under stronger conditions on the switching signals. To this end, pick Lyapunov functions¹

$$V_i(x) := \sqrt{x^\top P_i x + 1} - 1. \quad (68)$$

We conclude that

$$\begin{aligned} \dot{V}_i(x(t)) &= \frac{1}{2(V_i(x) + 1)} \left((x^\top P_i x)^k x^\top (A_i^\top P_i + P_i A_i) x + 2x^\top P_i B_i u \right) \\ &\leq \frac{1}{2(V_i(x) + 1)} \left(-2\lambda_i (x^\top P_i x)^k x^\top P_i x + 2\sqrt{x^\top P_i x} \|P_i^{\frac{1}{2}} B_i\| |u| \right) \\ &\leq \frac{1}{2(V_i(x) + 1)} \left(-\lambda_i ((V_i(x) + 1)^2 - 1)^{k+1} + 2(V_i(x) + 1) \|P_i^{\frac{1}{2}} B_i\| |u| \right) \\ &= -\lambda_i \frac{(V_i(x) + 1)^2 - 1}{2(V_i(x) + 1)} + \|P_i^{\frac{1}{2}} B_i\| |u|. \end{aligned}$$

Hence we conclude (50a) with $\alpha_i(s) := \lambda_i \frac{(s+1)^2 - 1}{2(s+1)} \in \mathcal{K}_\infty$, and $\gamma_c(s) := \max_{i \in \mathcal{S}} \|P_i^{\frac{1}{2}} B_i\| s \in \mathcal{K}$. On the other hand, if we assume that there exist $\mu_{i,j} > 0$ such that $P_j \leq \mu_{i,j} P_i$, then

$$(V_j(x) + 1)^2 - 1 = x^\top P_j x \leq \mu_{i,j} x^\top P_i x = \mu_{i,j} ((V_i(x) + 1)^2 - 1).$$

Because it is assumed that the state does not jump at switching instants, we conclude that the condition (51) holds with $\chi_{i,j}(s) := \sqrt{\mu_{i,j}(s+1)^2 - \mu_{i,j} + 1} - 1$ and any $\gamma_d \in \mathcal{K}$.

It is observed that α_i are comparable with $\alpha^*(s) = \frac{(s+1)^2 - 1}{s+1}$, $\kappa_i = \frac{\lambda_i}{2}$. Hence we appeal to Corollary 5.3 and let $c(i) := \frac{\lambda_i}{2} + \epsilon$ for some $\epsilon > 0$. Meanwhile, define

$$\begin{aligned} d(i, j, x) &:= \int_{V_i(x)}^{\chi_{i,j}(V_i(x))} \frac{r+1}{((r+1)^2 - 1)^{k+1}} dr + \epsilon \\ &= \frac{1}{2k} \left(\frac{1}{(x^\top P_i x)^k} - \frac{1}{\mu_{i,j}^k (x^\top P_i x)^k} \right) + \epsilon \\ &= \frac{1 - \mu_{i,j}^{-k}}{2k(x^\top P_i x)^k} + \epsilon. \end{aligned}$$

We therefore conclude that if $\mu_{i,j} > 1$, then $d(i, j, x) \rightarrow \infty$ when $x \rightarrow 0$. In other words, when the state is closer to the origin, longer time between two consecutive switches is needed in order for the switched system to be ISS. We also observe that when a switch occurs at a state x that is sufficiently close to the origin, then the left-hand side of (6) evaluated over a sufficiently short time interval only containing that

¹We can also pick the Lyapunov functions $V_i(x) := \sqrt{x^\top P_i x} = |P_i^{\frac{1}{2}} x|$ for simpler computation despite the fact that these functions are not smooth at $x = 0$. Non-smoothness at the origin is not critical for our analysis.

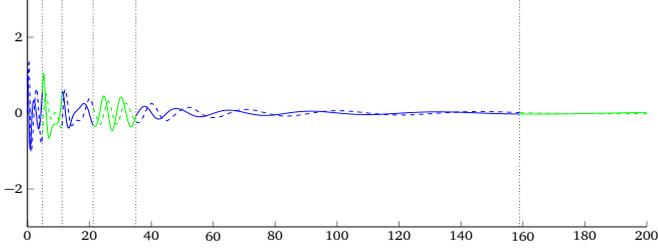


Figure 2: Plot of a solution of the system (67) with the given numerical parameters. The solid curve is $x_1(t)$ and dashed curve is $x_2(t)$. Blue denotes mode 1 and green is for mode 2. The vertical dotted lines indicate the switching instants.

switch will be larger than any finite T . Because ISS implies GAS when the system is unforced, we conclude that when the system (67) is unforced, switching signals in $\Sigma(c, d, T)$ which make it GAS must be eventually constant.

As a numerical demonstration, consider the case $\mathcal{S} = \{1, 2\}$, with A_i, P_i matrices given by (46), (48). We assume that the input matrices are $B_1 = B_2 = \begin{pmatrix} 0 & 1 \end{pmatrix}^\top$ and the input signal is $w(t) = e^{-0.1t} \sin(t)$. It is computed that $\mu_{1,2} = \mu_{2,1} = 2$. With $\epsilon = 0.1, k = 0.5, T = 100$ and initial state $x_0 = \begin{pmatrix} 1 & 0 \end{pmatrix}^\top$, The solution trajectory of the switched system (67) subjected to a switching signal $(\sigma, \mathcal{T}) \in \Sigma(c, d, T)$ is shown in Fig. 2. Indeed, convergence of solution is observed when the input is also convergent, which is guaranteed by the ISS property. It is also seen that the closer the state is to the origin, the longer dwell-time is needed for the next switch, which is consistent with the previous discussion.

iISS switched system with nonlinear subsystems

Now let us look at a different switched system with nonlinear subsystems

$$\dot{x} = (x^\top P_i x + 1)^{-k} A_i x + B_i w \quad (69)$$

where $k > 0$. Again assume that there are no jumps of state at switching instants, and P_i are positive definite matrices such that there exist $\lambda_i > 0$ for all $i \in \mathcal{S}$ and (41a) holds. This time because

$$\lim_{|x| \rightarrow \infty} \frac{|(x^\top P_i x + 1)^{-k} A_i x|}{|x|} = 0,$$

it is again implied by [7, Theorem 2] that there exist certain A_i matrices such that for any $\tau_d > 0$, there exists a switching signal σ satisfying ADT condition with ADT parameter τ_d , but the solution of the unforced switched system does not converge to the origin. Hence the switched system made up with subsystems (69) is not iISS uniformly for any ADT switching signals.

To show that this switched system can be ensured to be iISS under stronger conditions on the switching signals, again pick the Lyapunov functions (68). We conclude that

$$V_i(x) = \frac{1}{2(V_i(x) + 1)} \left((x^\top P_i x + 1)^{-k} x^\top (A_i^\top P_i + P_i A_i) x + 2x^\top P_i B_i u \right)$$

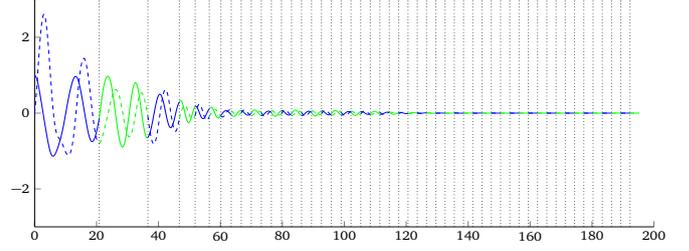


Figure 3: Plot of a solution of the system (69) with the given numerical parameters. The solid curve is $x_1(t)$ and dashed curve is $x_2(t)$. Blue denotes mode 1 and green is for mode 2. The vertical dotted lines indicate the switching instants.

$$\begin{aligned} &\leq \frac{1}{2(V_i(x) + 1)} \left(-\lambda_i (x^\top P_i x + 1)^{-k} x^\top P_i x + 2\sqrt{x^\top P_i x} \|P_i^{\frac{1}{2}} B_i\| |u| \right) \\ &\leq \frac{1}{2(V_i(x) + 1)} \left(-\lambda_i (V_i(x) + 1)^{-2k} ((V_i(x) + 1)^2 - 1) \right. \\ &\quad \left. + 2(V_i(x) + 1) \|P_i^{\frac{1}{2}} B_i\| |u| \right) \\ &= -\frac{\lambda_i}{2} \left((V_i(x) + 1)^{-2k+1} - (V_i(x) + 1)^{-2k-1} \right) + \|P_i^{\frac{1}{2}} B_i\| |u|. \end{aligned}$$

Hence we conclude (23a) with $\alpha_i(s) := \frac{\lambda_i}{2} ((s+1)^{-2k+1} - (s+1)^{-2k-1}) \in \mathcal{P}\mathcal{D}$, and the same function γ_c as for the previous example. Again these α_i functions are comparable with $\alpha^*(s) = (s+1)^{-2k+1} - (s+1)^{-2k-1}$, $\kappa_i = \frac{\lambda_i}{2}$. Hence we appeal to Corollary 5.3 and let $c(i) := \frac{\lambda_i}{2} + \epsilon$ for some $\epsilon > 0$. Meanwhile,

$$\begin{aligned} d(i, j, x) &:= \int_{V_i(x)}^{x_{i,j}(V_i(x))} \frac{1}{(r+1)^{-2k+1} - (r+1)^{-2k-1}} dr + \epsilon \\ &\geq \int_{V_i(x)}^{x_{i,j}(V_i(x))} (r+1)^{2k-1} dr + \epsilon \\ &= \frac{1}{2k} \left((\mu_{i,j} x^\top P_i x + 1)^k - (x^\top P_i x + 1)^k \right) + \epsilon. \end{aligned}$$

We also conclude that if $\mu_{i,j} > 1$, then $d(i, j, x) \rightarrow \infty$ when $|x| \rightarrow \infty$. In other words, when the state is further away from the origin, the cost of the switch from mode i to mode j approaches to infinity and it takes longer time between two consecutive switches in order for the switched system to be iISS.

We also perform a numerical demonstration for this system (69). With the same values of $\mathcal{S}, A_i, B_i, P_i$, input signal $w(t)$, initial state x_0 and parameters ϵ, k, T as for the previous numerical demonstration on system (67), the solution trajectory of the switched system (69) subjected to a switching signal $(\sigma, \mathcal{T}) \in \Sigma(c, d, T)$ is shown in Fig. 3. Indeed, convergence of solution is observed when the input has bounded integration, which is guaranteed by the iISS property. It is also seen that the further away the state is from the origin, the longer dwell-time is needed for the next switch, which is consistent with the previous discussion.

6. Conclusion

We have studied the stability of impulsive switched systems with nonlinear supply functions in this work. A broad class

of time/mode/state-dependent switching signals, which is shown to be a superset of many known sets of switching signals by choosing the appropriate functional parameters, are defined in this work. We have also shown that these switching signals can be generated online via the use of an auxiliary timer while the system is running. By choosing appropriate functional parameters according to the Lyapunov characterizations of the subsystems, the main results of this work conclude UGAS, ISS or iISS of the impulsive switched systems uniformly over our proposed classes of switching signals.

To build upon the results proposed in this paper, one may take several routes. One such research direction is to numerically compute the class of switching signals described via (6) directly from the system data while constructing the functions V_i stipulated in Assumption 1 or Assumption 3. This has been done recently for ADT constrained switching signals in [20, 10] and one could investigate the potential of those techniques for the switching signals proposed in this paper. Secondly, it would be interesting to investigate the utility of proposed class of switching signals in the design of control related applications.

A. Supporting lemmas for the main results

Modified from the results in [22], we have the following observations for the function φ_i defined in (31):

Lemma A.1. *Let $\kappa > 0$ and $s \geq t > 0$ such that $\varphi_i(s) \leq \kappa \varphi_i(t)$. It follows that $\alpha_i(s) \leq \kappa^{\frac{\tilde{c}_i}{\alpha_i(t)}} \alpha_i(t)$.*

Proof. Fix $s \geq t > 0$. Then, the inequality $\varphi_i(s) \leq \kappa \varphi_i(t)$ implies that

$$\exp\left(\int_1^s \frac{\tilde{c}(i)}{\alpha_i(r)} dr\right) \leq \kappa \exp\left(\int_1^t \frac{\tilde{c}(i)}{\alpha_i(r)} dr\right).$$

Take logarithm on both sides and subtract the integral on the right from the left, to get

$$\int_t^s \frac{\tilde{c}(i)}{\alpha_i(r)} dr \leq \ln(\kappa). \quad (\text{A.1})$$

Define $\tilde{\alpha}_i : [t, \infty) \rightarrow \mathbb{R}_+$ by $\tilde{\alpha}_i(r) := \alpha_i(t) + \tilde{c}_i(r-t)$. It follows from Assumption 2 that $\tilde{\alpha}_i(r) \geq \alpha_i(r)$ for all $r \geq t$. Hence, continuing from (A.1), we have

$$\begin{aligned} \ln(\kappa) &\geq \int_t^s \frac{\tilde{c}(i)}{\tilde{\alpha}_i(r)} dr = \int_t^s \frac{\tilde{c}(i)}{\alpha_i(t) + \tilde{c}_i(r-t)} dr \\ &= \frac{\tilde{c}(i)}{\tilde{c}_i} \ln \frac{\tilde{\alpha}_i(s)}{\alpha_i(t)} \geq \frac{\tilde{c}(i)}{\tilde{c}_i} \ln \frac{\alpha_i(s)}{\alpha_i(t)}, \end{aligned}$$

which results in $\alpha_i(s) \leq \kappa^{\frac{\tilde{c}_i}{\alpha_i(t)}} \alpha_i(t)$. \square

Lemma A.2. *The function $\theta : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ defined by $\theta(s) := \varphi_i^{-1}(\kappa \varphi_i(s))$ with any $\kappa \geq 1$ is globally one-sided Lipschitz with constant $\kappa^{\frac{\tilde{c}_i}{\alpha_i(t)}}$.*

Proof. To show θ is globally one-sided Lipschitz with constant $\kappa^{\frac{\tilde{c}_i}{\alpha_i(t)}}$, we need to show that $\theta(t_2) - \theta(t_1) \leq \kappa^{\frac{\tilde{c}_i}{\alpha_i(t)}}(t_2 - t_1)$ for all $t_2 \geq t_1 \geq 0$. In other words, we need to show that $\theta(t) - \kappa^{\frac{\tilde{c}_i}{\alpha_i(t)}} t$ is non-increasing in t over $\mathbb{R}_{\geq 0}$. To this end, we recall the derivatives (34), (59), with which we conclude that for any $t > 0$,

$$\begin{aligned} \frac{d}{dt} \theta(t) &= \frac{d}{dt} (\varphi_i^{-1}(\kappa \varphi_i(t))) \\ &= \frac{\alpha_i(\varphi_i^{-1}(\kappa \varphi_i(t))) \kappa \tilde{c}(i) \varphi_i(t)}{\kappa \tilde{c}(i) \varphi_i(t) \alpha_i(t)} = \frac{\alpha_i(\theta(t))}{\alpha_i(t)}. \end{aligned}$$

Let $s := \theta(t)$, so that $\varphi_i(s) = \kappa \varphi_i(t)$. Since $\kappa \geq 1$, $s \geq t > 0$ and Lemma A.1 yields $\alpha_i(s) \leq \kappa^{\frac{\tilde{c}_i}{\alpha_i(t)}} \alpha_i(t)$. Therefore,

$$\frac{d}{dt} (\theta(t) - \kappa^{\frac{\tilde{c}_i}{\alpha_i(t)}} t) = \frac{\alpha_i(s)}{\alpha_i(t)} - \kappa^{\frac{\tilde{c}_i}{\alpha_i(t)}} \leq 0.$$

Hence $\theta(t) - \kappa^{\frac{\tilde{c}_i}{\alpha_i(t)}} t$ is non-increasing and this completes the proof. \square

Lemma A.3. *For some $\epsilon > 0$, define a function*

$$\gamma(s) := s - \max_{i \in \mathcal{I}} \varphi_i^{-1}(e^{-\epsilon} \varphi_i(s)). \quad (\text{A.2})$$

Then it holds that $\gamma \in \mathcal{X}_\infty$ when $\alpha_i \in \mathcal{X}_\infty$ for all $i \in \mathcal{I}$.

Proof. For each $i \in \mathcal{I}$, let $\gamma_i(s) := s - \varphi_i^{-1}(e^{-\epsilon} \varphi_i(s))$. Then clearly $\gamma \in \mathcal{X}_\infty$ if $\gamma_i \in \mathcal{X}_\infty$ for all $i \in \mathcal{I}$. To show the latter, it follows from the definition of γ_i that $\varphi_i((s - \gamma_i(s))) = e^{-\epsilon} \varphi_i(s)$ and, after plugging (31) in and taking logarithm, gives

$$\int_1^{s-\gamma_i(s)} \frac{\tilde{c}(i)}{\alpha_i(r)} dr = -\epsilon + \int_1^s \frac{\tilde{c}(i)}{\alpha_i(r)} dr.$$

Hence, we have

$$\int_{s-\gamma_i(s)}^s \frac{1}{\alpha_i(r)} dr = \epsilon > 0. \quad (\text{A.3})$$

When $\alpha_i \in \mathcal{X}_\infty$, $\frac{1}{\alpha_i(r)}$ monotonically decreases to 0 as r increases to infinity; therefore, (A.3) holds for all $s > 0$ only when the interval of integration grows to infinity with respect to s . In other words, we conclude that $\alpha_i \in \mathcal{X}_\infty$ and this completes the proof. \square

References

- [1] D. Angeli, E. D. Sontag, and Y. Wang. A characterization of integral input-to-state stability. *IEEE Transactions on Automatic Control*, 45(6):1082–1097, June 2000.
- [2] D. D. Bainov and P. S. Simeonov. *Systems with Impulse Effects: Stability, Theory and Applications*. New York: Academic Press, 1989.
- [3] C. Cai and A. R. Teel. Characterizations of input-to-state stability for hybrid systems. *Systems & Control Letters*, 58(1):47–53, 2009.
- [4] S. Dashkovskiy and A. Mironchenko. Input-to-state stability of nonlinear impulsive systems. *SIAM Journal on Control and Optimization*, 51(3):1962–1987, 2013.
- [5] W. P. Dayawansa and C. F. Martin. A converse Lyapunov theorem for a class of dynamical systems which undergo switching. *IEEE Transactions on Automatic Control*, 44(4):751–760, 1999.

- [6] R.A. Decarlo, M.S. Branicky, S. Pettersson, and B. Lennartson. Perspectives and results on the stability and stabilizability of hybrid systems. *Proceedings of the IEEE*, 88(7):1069–1082, 2000.
- [7] M. Della Rossa and A. Tanwani. Instability of dwell-time constrained switched nonlinear systems. *Systems & Control Letters*, 162:105164, 2022.
- [8] R. Goebel, R. G. Sanfelice, and A. R. Teel. *Hybrid Dynamical Systems: Modeling, Stability, and Robustness*. Princeton University Press, 2012.
- [9] W. M. Haddad, V. S. Chellaboina, and S. G. Nersisov. *Impulsive and Hybrid Dynamical Systems: Stability, Dissipativity, and Control*. Princeton University Press, 2006.
- [10] S. Hafstein and A. Tanwani. Linear programming based lower bounds on average dwell-time via multiple Lyapunov functions. *European Journal of Control*, 74:100838, 2023.
- [11] H. Haimovich and J. L. Mancilla-Aguilar. A characterization of integral ISS for switched and time-varying systems. *IEEE Transactions on Automatic Control*, 63(2):578–585, Feb 2018.
- [12] H. Haimovich and J. L. Mancilla-Aguilar. Strong ISS implies strong iISS for time-varying impulsive systems. *Automatica*, 122:109224, 2020.
- [13] J. K. Hale. *Ordinary Differential Equations*. Krieger Publishing Company, Malabar, Florida, 2nd edition, 1980.
- [14] J. P. Hespanha, D. Liberzon, and A. R. Teel. Lyapunov conditions for input-to-state stability of impulsive systems. *Automatica*, 44(11):2735 – 2744, 2008.
- [15] J. P. Hespanha and A. S. Morse. Stability of switched systems with average dwell-time. In *Proceedings of the 38th IEEE Conf. on Decision and Control*, volume 3, pages 2655–2660, Dec 1999.
- [16] Ö. Karabacak. Dwell time and average dwell time methods based on the cycle ratio of the switching graph. *Systems & Control Letters*, 62(11):1032–1037, 2013.
- [17] H. Khalil. *Nonlinear Systems, 3rd ed.* Prentice Hall, Englewood Cliffs, NJ, 2002.
- [18] A. Kundu, D. Chatterjee, and D. Liberzon. Generalized switching signals for input-to-state stability of switched systems. *Automatica*, 64:270–277, 2016.
- [19] D. Liberzon. *Switching in Systems and Control*. Birkhäuser, Boston, MA, 2003.
- [20] S. Liu, S. Martinez, and J. Cortes. Average dwell-time minimization of switched systems via sequential convex programming. *IEEE Control Systems Letters*, 6:1076–1081, 2022.
- [21] S. Liu, A. Russo, D. Liberzon, and A. Cavallo. Integral-input-to-state stability of switched nonlinear systems under slow switching. *IEEE Transactions on Automatic Control*, 67(11):5841–5855, 2021.
- [22] S. Liu, A. Tanwani, and D. Liberzon. ISS and integral-ISS of switched systems with nonlinear supply functions. *Mathematics of Control, Signals and Systems*, 34:297–327, 2022.
- [23] J. L. Mancilla-Aguilar, H. Haimovich, and P. Feketa. Uniform stability of nonlinear time-varying impulsive systems with eventually uniformly bounded impulse frequency. *Nonlinear Analysis: Hybrid Systems*, 38:100933, 2020.
- [24] A. S. Morse. Dwell-time switching. In *Proceedings of the 2nd European Control Conference*, pages 176–181, 1993.
- [25] M. A. Müller and D. Liberzon. Input/output-to-state stability and state-norm estimators for switched nonlinear systems. *Automatica*, 48(9):2029 – 2039, 2012.
- [26] N. Noroozi, A. Khayatian, and R. Geiselhart. A characterization of integral input-to-state stability for hybrid systems. *Mathematics of Control, Signals and Systems*, 29(3):13, Jun 2017.
- [27] L. Praly and Y. Wang. Stabilization in spite of matched unmodeled dynamics and an equivalent definition of input-to-state stability. *Mathematics of Control, Signals and Systems*, 9(1):1–33, Mar 1996.
- [28] A. Russo and S. Liu. Do 0-GAS-guaranteeing impulse sequences preserve ISS or iISS properties? Not always. In *Proceedings of the 62nd IEEE Conf. on Decision and Control*, pages 7649–7654, 2023.
- [29] A. M. Samoilenko and N. A. Perestyuk. *Impulsive Differential Equations*. World Scientific Publishing, 1995.
- [30] R. Sanfelice, D. Copp, and P. Nanez. A toolbox for simulation of hybrid systems in matlab/simulink: hybrid equations (HyEQ) toolbox. In *Proceedings of the 16th international conference on Hybrid systems: computation and control*, pages 101–106, 2013.
- [31] R. G. Sanfelice. *Hybrid Feedback Control*. Princeton University Press, 2021.
- [32] E. D. Sontag. Smooth stabilization implies coprime factorization. *IEEE Transactions on Automatic Control*, 34(4):435–443, 1989.
- [33] Z. Sun and S. S. Ge. *Stability Theory of Switched Dynamical Systems*. Springer London, 2011.
- [34] L. Vu, D. Chatterjee, and D. Liberzon. Input-to-state stability of switched systems and switching adaptive control. *Automatica*, 43(4):639 – 646, 2007.
- [35] W. Xiang and J. Xiao. Stabilization of switched continuous-time systems with all modes unstable via dwell time switching. *Automatica*, 50(3):940–945, 2014.
- [36] H. Yang, B. Jiang, V. Cocquempot, and H. Zhang. Stabilization of switched nonlinear systems with all unstable modes: Application to multi-agent systems. *IEEE Transactions on Automatic Control*, 56(9):2230–2235, 2011.
- [37] G. Zhai, B. Hu, K. Yasuda, and A. N. Michel. Stability analysis of switched systems with stable and unstable subsystems: An average dwell time approach. *International Journal of Systems Science*, 32(8):1055–1061, 2001.
- [38] G. Zhang and A. Tanwani. ISS Lyapunov functions for cascade switched systems and sampled-data control. *Automatica*, 105:216 – 227, 2019.