

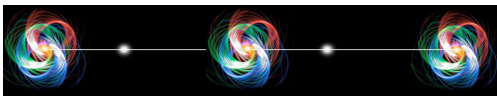
State polynomials for nonlinear Bell inequalities

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Joint work with Igor Klep, Jurij Volčič & Jie Wang

MEGA 2024, MPI Leipzig

30 July 2024



Motivation: Bell inequalities

State polynomials

NPA hierarchy for state polynomials

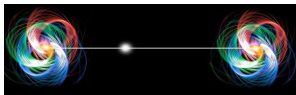
Back to Bell inequalities

Linear Bell inequalities

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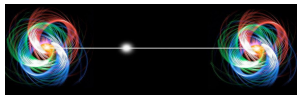
Alice \leftrightarrow Source \rightsquigarrow Bob

Correlations = conditional joint probabilities:

$$P(ab|st) = P(\text{Alice, Bob answer } a, b \mid \text{Alice, Bob are asked } s, t)$$

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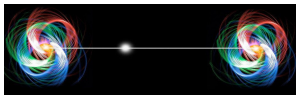
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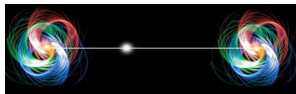
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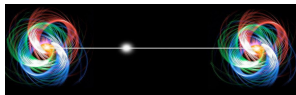
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classical correlations = convex comb. of deterministic correlations

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$$-P_a(1|0) - P_b(1|0) + P(11|00) + P(11|01) + P(11|10) - P(11|11) \leq 0$$

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Inequality on eigenvalues of noncommutative polynomials

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- If \mathcal{H}_A is finite-dim then a quantum state $\lambda \in \mathcal{S}(\mathcal{H}_A) = \text{SDP}$ matrix with unit trace

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- entangled states cannot be written as mixtures of product states

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↔ noncommutative analogue of [Lasserre 01] hierarchy for classical polynomial optimization

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Covariance in the quantum (commuting) model

$$\text{cov}(A, B) = \lambda(AB) - \lambda(A)\lambda(B) \text{ where } \lambda \text{ is a state}$$

\Leftrightarrow max over all states $\lambda =$ **quantum violation**

Covariance Bell inequalities [Pozsgay et al. 17]

Classical model = A_i, B_j commute & satisfy a ball constraint

↔ classical moment problem

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The state λ is given by the integration w.r.t. a proba μ built from E
[Schmüdgen 12]

$$\lambda(f) = \int_{\mathbb{R}^n} f \, d\mu$$

Covariance Bell inequalities [Pozsgay et al. 17]

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$(A_1 A_2 A_3 / B_1 B_2 B_3)$: strategy where Alice and Bob deterministically output A_x and B_y for inputs x and y

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💡 Concrete λ yields $\text{cov}_{3322} = 5$

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$(A_1 A_2 A_3 / B_1 B_2 B_3)$: strategy where Alice and Bob deterministically output A_x and B_y for inputs x and y

💡 Concrete λ yields $\text{cov}_{3322} = 5$

What are the classical bound and the maximal quantum violation?

Bilocal Bell inequality [Tavakoli et al. 21-22]



Alice \leftrightarrow Source 1 \rightsquigarrow Bob \leftrightarrow Source 2 \rightsquigarrow Charlie

Observers hold particles from different sources and therefore a priori share no correlations

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A party that holds multiple shares originating from different sources can perform entangled measurements to a posteriori distribute entanglement between $[\cdot \cdot \cdot]$ systems in the network

Bilocal Bell inequality [Tavakoli et al. 21-22]

Binary random variables A_i, B_j, C_k

$$\frac{1}{3} \sum_{i \in \{1,2,3\}} \left(E(B_i C_i) - E(A_i B_i) \right) - \sum_{\{i,j,k\} = \{1,2,3\}} E(A_i B_j C_k)$$

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What is the quantum violation?

Motivation: Bell inequalities

State polynomials

NPA hierarchy for state polynomials

Back to Bell inequalities

Moment polynomials

Elements of $\mathcal{M}[x]$

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real vars $x = (x_1, \dots, x_n)$

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at a proba μ on \mathbb{R}^2 with fourth order moments and a pair

$X = (X_1, X_2) \in \mathbb{R}^2$, f evaluates as

$$f(\mu, X) = X_1 X_2 \int x_1 x_2^3 d\mu - X_2^2 \left(\int x_1^2 d\mu \right)^3 + X_2 - 2$$

State polynomials

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at a state $\lambda : \mathcal{B}(\mathcal{H}) \rightarrow \mathbb{R}$ and a pair $X = (X_1, X_2) \in \mathcal{B}(\mathcal{H})^2$, f evaluates as

$$f(\lambda, X) = X_1 X_2 \lambda(X_1 X_2^3) - X_2^2 \left(\lambda(X_1^2) \right)^3 + X_2 - 2$$

Related business

State polynomials

$$\tau(x_1^2)\tau(x_2) + \tau(x_2) \text{ with } \tau(w) = \tau(w^*)$$

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💡 Proba is a state \Rightarrow **moment** polynomials are **state** polynomials

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Objective function $f \in \mathcal{M}[x]$

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Two types of constraints:

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💡 NPA hierarchy to optimize over $\mathcal{M}[x]$

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💡 NPA hierarchy to optimize over $\mathcal{S}\langle x \rangle$

Hierarchies for polynomial optimization

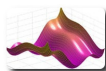
NP-hard NON CONVEX problem $f_{\min} = \inf f(X)$

Theory

(Primal)

$$\inf \int f d\mu$$

with μ proba \Rightarrow



INFINITE LP

(Dual)

$$\sup \lambda$$

\Leftarrow with $f - \lambda \geq 0$

Hierarchies for polynomial optimization

NP-hard NON CONVEX problem $f_{\min} = \inf f(X)$

Practice

(Primal **Relaxation**)

$$\text{moments } \int X^\alpha d\mu$$

finite number \Rightarrow



SDP

(Dual **Strengthening**)

$$f - \lambda = \text{sum of squares}$$

\Leftarrow fixed degree

LASSERRE'S HIERARCHY of **CONVEX PROBLEMS** $\uparrow f_{\min}$

[Lasserre '01]

degree r & n vars $\implies \binom{n+2r}{n}$ **SDP VARIABLES**



A simple example

$$f_{\min} = \min f(X) \text{ over } K(S)$$

Semialgebraic set $K(S) = \{X \in \mathbb{R}^n : s(X) \geq 0, \quad s \in S\}$

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$$\overbrace{X_1 X_2}^f = -\frac{1}{8} + \frac{1}{2} \overbrace{\left(X_1 + X_2 - \frac{1}{2}\right)^2}^{\sigma_0} + \frac{1}{2} \overbrace{X_1(1 - X_1)}^{\sigma_1} + \frac{1}{2} \overbrace{X_2(1 - X_2)}^{\sigma_2}$$

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Sums of squares (SOS) σ_j

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Sums of squares (SOS) σ_j

$$\text{Quadratic module: } \text{QM}(S)_r = \left\{ \sigma_0 + \sum_j \sigma_j s_j, \text{ deg } \sigma_j s_j \leq 2r \right\}$$

Hierarchies for polynomial optimization

$$f_{\min} = \min_{X \in K(S)} f(X)$$

- $\mathcal{P}(K(S))$: proba on $K(S)$
- quadratic module $\text{QM}(S) = \left\{ \sigma_0 + \sum_j \sigma_j s_j, \text{ with } \sigma_j \text{ SOS} \right\}$

Infinite-dimensional linear programs (LP)

(Primal)	=	(Dual)
$\inf \int_{K(S)} f d\mu$		$\sup \lambda$
s.t. $\mu \in \mathcal{P}(K(S))$		s.t. $\lambda \in \mathbb{R}$
		$f - \lambda \in \text{QM}(S)$

Hierarchies for polynomial optimization

$$f_{\min} = \min_{X \in K(S)} f(X)$$

- Pseudo-moment sequences \mathbf{y} up to order r
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Finite-dimensional semidefinite programs (SDP)

(Moment)	=	(SOS)
$f_r = \inf \sum_{\alpha} f_{\alpha} y_{\alpha}$		$\sup \lambda$
s.t. $\mathbf{M}_{r-r_j}(s_j \mathbf{y}) \succcurlyeq 0$		s.t. $\lambda \in \mathbb{R}$
$y_0 = 1$		$f - \lambda \in \text{QM}(S)_r$

Hierarchies for polynomial optimization

💡 Moment matrices are indexed by monomials

$$\mathbf{M}_1(\mathbf{y}) = \begin{array}{c} 1 \\ x_1 \\ x_2 \end{array} \left(\begin{array}{c|cc} 1 & & \\ \hline & y_{10} & y_{01} \\ \hline & y_{10} & y_{11} \\ & y_{01} & y_{02} \end{array} \right)$$

Hierarchies for polynomial optimization

Theorem [Putinar 93, Lasserre 01]: positive polynomials

For $f \in \mathbb{R}[x]$, $S \subseteq \mathbb{R}[x]$, if $\underbrace{N}_{>0} - \sum_i x_i^2 \in \text{QM}(S)$ then

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Proof sketch

NPA hierarchy for moment polynomials

Objective function $f \in \mathcal{M}[x]$

$$\text{for Bell } f = \frac{1}{3} \sum_{i \in \{1,2,3\}} \left(m(b_i c_i) - m(a_i b_i) \right) - \sum_{\{i,j,k\}=\{1,2,3\}} m(a_i b_j c_k) \in \mathcal{M}$$

Two types of constraints:

- $s_1(X) \geq 0$ with $s_1 \in \mathbb{R}[x]$

$$X \in K(S_1) \\ \text{for Bell } a_i^2 = b_j^2 = c_k^2 = 1$$

- $s_2(\mu) \geq 0$ with $s_2 \in \mathcal{M}$ μ proba on $K(S_1)$

$$\mu \in \mathcal{K}(S_1, S_2) \\ \text{for Bell } m(a_1 a_2 a_3 c_1 c_2 c_3) = m(a_1 a_2 a_3) m(c_1 c_2 c_3)$$

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 moment matrices & quadratic modules

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Moment matrices are (slightly) more complicated than in $\mathbb{R}[x]$

$$\mathbf{M}_1(\mathbf{y}) = \begin{array}{c} 1 \\ x_1 \\ x_2 \\ m_{10} \\ m_{01} \end{array} \left(\begin{array}{c|cccc} 1 & & x_1 & x_2 & m_{10} & m_{01} \\ \hline 1 & | & y_{1000} & y_{0100} & y_{0010} & y_{0001} \\ \hline y_{1000} & | & y_{2000} & y_{1100} & y_{1010} & y_{1001} \\ y_{0100} & | & y_{1100} & y_{0200} & y_{0110} & y_{0101} \\ y_{0010} & | & y_{1010} & y_{0110} & y_{0020} & y_{0011} \\ y_{0001} & | & y_{1001} & y_{0101} & y_{0011} & y_{0002} \end{array} \right)$$

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Quadratic module $\text{QM}(S_1, S_2)$ is also more complicated

$$\begin{aligned} \sum p^2_{\mathbf{m}}(q^2 s) : & \quad s \in \{1\} \cup S_1 \quad p, q \in \mathcal{M}[x] \\ \sum p^2 s : & \quad s \in S_1 \cup S_2 \quad p \in \mathcal{M}[x] \end{aligned}$$

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Theorem [Klep-M.-Volčič 23]: positive moment polynomials

For $f \in \mathcal{M}$, $S_1 \subseteq \mathbb{R}[x]$, $S_2 \subseteq \mathcal{M}$, if $\underbrace{N}_{>0} - \sum_i x_i^2 \in \text{QM}(S_1)$ then

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At order $r = 2$, x_1x_2 and x_2x_1 are needed

$\tau(x_1x_2) = \tau(x_2x_1)$ but $\tau(x_1^2x_2) \neq \tau(x_1x_2x_1)$ in general

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For $f \in \mathcal{S}$, $S \subseteq \mathcal{S}\langle x \rangle$, if $\underbrace{N}_{>0} - \sum_i x_i^2 \in \text{QM}(S \cap \mathbb{R}\langle \underline{x} \rangle)$ then

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More efficient NPA hierarchies

“SPARSE” cost f and constraints

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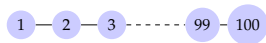
Correlative sparsity: few variable products in f

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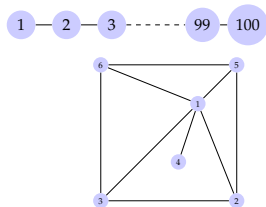


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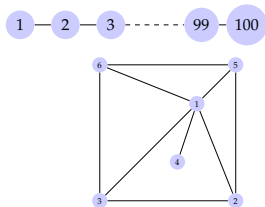
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Term sparsity: few terms in f



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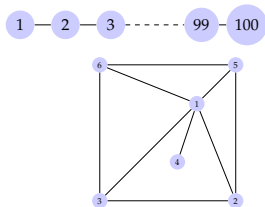
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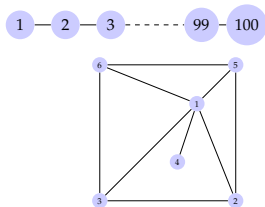
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Universal algebras of binary observables:



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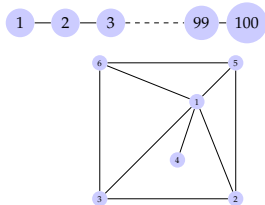
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Universal algebras of binary observables:

$$\rightsquigarrow \text{group } G \text{ of constraints } x_i^2 = 1 \quad x_ix_j = x_jx_i$$



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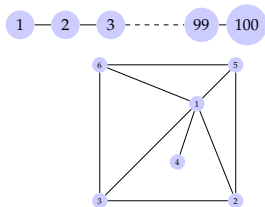
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💡 Index SDP matrices by $H \subseteq G$ generated by the support of f

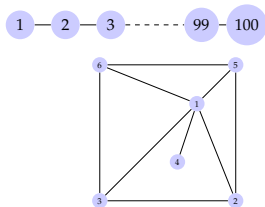


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$$\rightsquigarrow f = x_1x_2 + x_2x_3 + \cdots + x_{99}x_{100}$$



Term sparsity: few terms in f

$$\rightsquigarrow f = x_1^{99}x_2 + x_1x_2^{100}$$

Universal algebras of binary observables:

$$\rightsquigarrow \text{group } G \text{ of constraints } x_i^2 = 1 \quad x_i x_j = x_j x_i$$

💡 Index SDP matrices by $H \subseteq G$ generated by the support of f

PERFORMANCE



VS



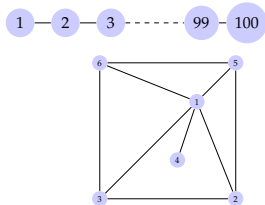
ACCURACY

More efficient NPA hierarchies

“SPARSE” cost f and constraints

Correlative sparsity: few variable products in f

$$\rightsquigarrow f = x_1x_2 + x_2x_3 + \cdots + x_{99}x_{100}$$



Term sparsity: few terms in f

$$\rightsquigarrow f = x_1^{99}x_2 + x_1x_2^{100}$$

Universal algebras of binary observables:

$$\rightsquigarrow \text{group } G \text{ of constraints } x_i^2 = 1 \quad x_ix_j = x_jx_i$$

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PERFORMANCE

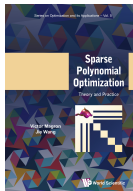


VS



ACCURACY

Tons of applications: computer arithmetic, deep learning, entanglement, optimal power-flow, analysis of dynamical systems, matrix ranks



Back to Bell inequalities

Binary A_i, B_j

$$\begin{aligned}\text{cov}_{3322} &= \text{cov}(A_1, B_1) + \text{cov}(A_1, B_2) + \text{cov}(A_1, B_3) \\ &\quad + \text{cov}(A_2, B_1) + \text{cov}(A_2, B_2) - \text{cov}(A_2, B_3) \\ &\quad + \text{cov}(A_3, B_1) - \text{cov}(A_3, B_2)\end{aligned}$$

Back to Bell inequalities

Binary A_i, B_j

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NPA hierarchy for \mathcal{M} and $r = 2$: SDP with 4 146 variables

$$f_2 = 4.5$$

Back to Bell inequalities

Binary A_i, B_j

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NPA hierarchy for \mathcal{M} and $r = 2$: SDP with 4146 variables $f_2 = 4.5$
same local bound as [Pozsgay et al. 17] 💡 classical bound = $f_{\max} = 4.5$

Back to Bell inequalities

Binary A_i, B_j

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NPA hierarchy for \mathcal{M} and $r = 2$: SDP with 4146 variables $f_2 = 4.5$
same local bound as [Pozsgay et al. 17] 💡 classical bound = $f_{\max} = 4.5$


NPA hierarchy for \mathcal{S} and $r = 2$: $f_2 = 5$

Back to Bell inequalities

Binary A_i, B_j

$$\begin{aligned}\text{cov}_{3322} &= \text{cov}(A_1, B_1) + \text{cov}(A_1, B_2) + \text{cov}(A_1, B_3) \\ &\quad + \text{cov}(A_2, B_1) + \text{cov}(A_2, B_2) - \text{cov}(A_2, B_3) \\ &\quad + \text{cov}(A_3, B_1) - \text{cov}(A_3, B_2)\end{aligned}$$

NPA hierarchy for \mathcal{M} and $r = 2$: SDP with 4 146 variables $f_2 = 4.5$
same local bound as [Pozsgay et al. 17]  classical bound = $f_{\max} = 4.5$

NPA hierarchy for \mathcal{S} and $r = 2$: $f_2 = 5$
same local bound as [Pozsgay et al. 17]  quantum bound = $f_{\max} = 5$

Back to Bell inequalities

Binary A_i, B_j, C_k

$$\frac{1}{3} \sum_{i \in \{1,2,3\}} \left(E(B_i C_i) - E(A_i B_i) \right) - \sum_{\{i,j,k\}=\{1,2,3\}} E(A_i B_j C_k)$$

Back to Bell inequalities

Binary A_i, B_j, C_k

$$\frac{1}{3} \sum_{i \in \{1,2,3\}} \left(E(B_i C_i) - E(A_i B_i) \right) - \sum_{\{i,j,k\}=\{1,2,3\}} E(A_i B_j C_k)$$

satisfying bilocality constraints

$$E(A_1 A_2 A_3 C_1 C_2 C_3) = E(A_1 A_2 A_3) E(C_1 C_2 C_3)$$

+ similar factorization constraints

Back to Bell inequalities

Binary A_i, B_j, C_k

$$\frac{1}{3} \sum_{i \in \{1,2,3\}} \left(E(B_i C_i) - E(A_i B_i) \right) - \sum_{\{i,j,k\}=\{1,2,3\}} E(A_i B_j C_k)$$

satisfying bilocality constraints

$$E(A_1 A_2 A_3 C_1 C_2 C_3) = E(A_1 A_2 A_3) E(C_1 C_2 C_3)$$

+ similar factorization constraints & vanishing constraints

$$E(A_i) = E(B_i) = E(C_i) = 0 \quad \text{for } i \in \{1,2,3\}$$

$$E(A_i B_j) = E(B_i C_j) = 0 \quad \text{for } i \neq j$$

$$E(A_i B_j C_k) = 0 \quad \text{for } |\{i,j,k\}| \leq 2$$

Back to Bell inequalities

[Tavakoli et al. 21-22] local classical bound of 3

Back to Bell inequalities

[Tavakoli et al. 21-22] local classical bound of 3

$$\sup \frac{1}{3} \sum_{i \in \{1,2,3\}} \left(m(b_i c_i) - m(a_i b_i) \right) - \sum_{\{i,j,k\}=\{1,2,3\}} m(a_i b_j c_k)$$

s.t.

$$m(a_1 a_2 a_3 \ c_1 c_2 c_3) = m(a_1 a_2 a_3) \ m(c_1 c_2 c_3)$$

$$a_i^2 = b_j^2 = c_k^2 = 1 \text{ and } m(a_i) = m(b_j) = m(c_k) = 0$$

$$m(a_i b_j) = m(b_j c_k) = 0$$

$$m(a_i b_j c_k) = 0 \text{ for } |\{i, j, k\}| \leq 2$$

Back to Bell inequalities

[Tavakoli et al. 21-22] local classical bound of 3

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$r = 3$: SDP with 31 017 variables

$f_3 = 4$

Back to Bell inequalities

[Tavakoli et al. 21-22] local classical bound of 3

$$\sup \frac{1}{3} \sum_{i \in \{1,2,3\}} \left(m(b_i c_i) - m(a_i b_i) \right) - \sum_{\{i,j,k\} = \{1,2,3\}} m(a_i b_j c_k)$$

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$$m(a_i b_j c_k) = 0 \text{ for } |\{i, j, k\}| \leq 2$$

$r = 3$: SDP with 31 017 variables

$$f_3 = 4$$

We extracted a local classical bound of 4  classical bound = $f_{\max} = 4$

Back to Bell inequalities

[Tavakoli et al. 21-22] local quantum bound of 4

Back to Bell inequalities

[Tavakoli et al. 21-22] local quantum bound of 4

$$\sup \frac{1}{3} \sum_{i \in \{1,2,3\}} \left(\tau(b_i c_i) - \tau(a_i b_i) \right) - \sum_{\{i,j,k\}=\{1,2,3\}} \tau(a_i b_j c_k)$$

s.t.

$$\tau(a_1 a_2 a_3 c_1 c_2 c_3) = \tau(a_1 a_2 a_3) m(c_1 c_2 c_3)$$

$$a_i^2 = b_j^2 = c_k^2 = 1 \text{ and } \tau(a_i) = \tau(b_j) = \tau(c_k) = 0$$

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Back to Bell inequalities

[Tavakoli et al. 21-22] local quantum bound of 4

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$r = 3$: SDP with 3018 constraints (few seconds)

$f_3 = 4.46$

Back to Bell inequalities

[Tavakoli et al. 21-22] local quantum bound of 4

$$\sup \frac{1}{3} \sum_{i \in \{1,2,3\}} \left(\tau(b_i c_i) - \tau(a_i b_i) \right) - \sum_{\{i,j,k\}=\{1,2,3\}} \tau(a_i b_j c_k)$$

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$r = 3$: SDP with 3018 constraints (few seconds)

$$f_3 = 4.46$$

$r = 4$: SDP with 64878 constraints (few hours)

$$f_4 = 4.38$$

Back to Bell inequalities

[Tavakoli et al. 21-22] local quantum bound of 4

$$\sup \frac{1}{3} \sum_{i \in \{1,2,3\}} \left(\tau(b_i c_i) - \tau(a_i b_i) \right) - \sum_{\{i,j,k\}=\{1,2,3\}} \tau(a_i b_j c_k)$$

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$r = 3$: SDP with 3 018 constraints (few seconds)

$$f_3 = 4.46$$

$r = 4$: SDP with 64 878 constraints (few hours)

$$f_4 = 4.38$$

$r = 5$: SDP with 1 352 093 constraints (one week)

$$f_5 = 4.37$$

Back to Bell inequalities

[Tavakoli et al. 21-22] local quantum bound of 4

$$\sup \frac{1}{3} \sum_{i \in \{1,2,3\}} \left(\tau(b_i c_i) - \tau(a_i b_i) \right) - \sum_{\{i,j,k\}=\{1,2,3\}} \tau(a_i b_j c_k)$$

s.t.

$$\tau(a_1 a_2 a_3 c_1 c_2 c_3) = \tau(a_1 a_2 a_3) m(c_1 c_2 c_3)$$

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$$\tau(a_i b_j) = \tau(b_j c_k) = 0$$

$$\tau(a_i b_j c_k) = 0 \text{ for } |\{i, j, k\}| \leq 2$$

$r = 3$: SDP with 3 018 constraints (few seconds)

$$f_3 = 4.46$$

$r = 4$: SDP with 64 878 constraints (few hours)

$$f_4 = 4.38$$

$r = 5$: SDP with 1 352 093 constraints (one week)

$$f_5 = 4.37$$

We still don't know the quantum bound f_{\max} !

Conclusion

Conclusion

Positivity certificates for moment and state polynomials under compact polynomial inequality constraints

Conclusion

Positivity certificates for moment and state polynomials under compact polynomial inequality constraints



NPA hierarchies to certify classical and quantum bounds of nonlinear Bell inequalities

Conclusion

Positivity certificates for moment and state polynomials under compact polynomial inequality constraints



NPA hierarchies to certify classical and quantum bounds of nonlinear Bell inequalities

Hilbert-Artin analogues (theoretical results not explained in this talk):

Conclusion

Positivity certificates for moment and state polynomials under compact polynomial inequality constraints



NPA hierarchies to certify classical and quantum bounds of nonlinear Bell inequalities

Hilbert-Artin analogues (theoretical results not explained in this talk):

💡 State polynomials, positive over all matrices and matricial states, are sums of squares with denominators

Conclusion

Positivity certificates for moment and state polynomials under compact polynomial inequality constraints



NPA hierarchies to certify classical and quantum bounds of nonlinear Bell inequalities

Hilbert-Artin analogues (theoretical results not explained in this talk):

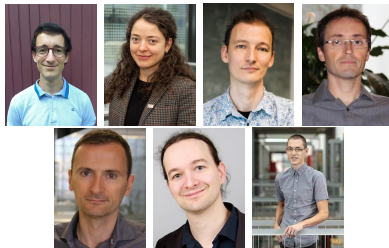
💡 State polynomials, positive over all matrices and matricial states, are sums of squares with denominators

💡 Moment polynomials positive on measures are sums of squares, up to arbitrarily small perturbation (generalization of [Lasserre 06])

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COMPUTE

nonCommutative polynOMial oPtimisation for qUanTum nEtworks











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Thank you for listening!

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