Exploiting sparsity in polynomial optimization

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LAAS CNRS

https://homepages.laas.fr/vmagron/SparsePOPJNCF23.pdf

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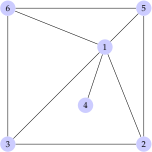

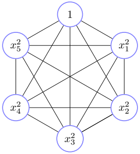

What is sparse polynomial optimization?

Looks like a regular polynomial optimization problem (POP):

$$\begin{aligned} \inf \quad & f(\mathbf{x}) \\ \text{s.t.} \quad & \mathbf{x} \in \mathbf{X} = \{\mathbf{x} \in \mathbb{R}^n : g_j(\mathbf{x}) \geqslant 0\} \end{aligned}$$

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PERFORMANCE  VS  ACCURACY

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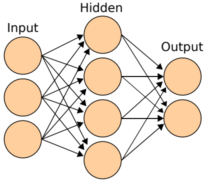
Deep learning

$\rightsquigarrow$ robustness, computer vision



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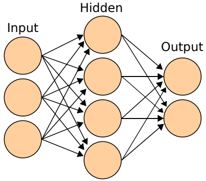
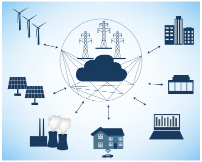


Power systems
$\rightsquigarrow$ AC optimal power-flow, stability



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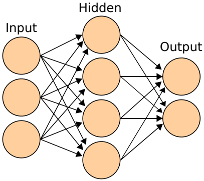

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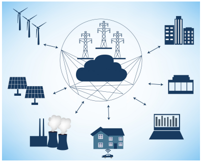
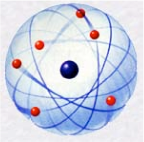


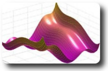
Quantum Systems
$\rightsquigarrow$ condensed matter, entanglement



The Moment-SOS Hierarchy for POP

NP-hard NON CONVEX Problem $f_{\min} = \inf f(\mathbf{x})$

Theory

(Primal)		(Dual)
$\inf \displaystyle\int f \, d\mu$		$\sup \quad \lambda$
with μ proba $\Rightarrow$	INFINITE LP	$\Leftarrow$ with $f - \lambda \geqslant 0$

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Practice

(Primal **Relaxation**)

moments $\int \mathbf{x}^\alpha \, d\mu$

(Dual **Strengthening**)

$f - \lambda =$ **sum of squares**

finite number $\Rightarrow$ **SDP** $\Leftarrow$ **fixed** degree

LASSERRE'S HIERARCHY of **CONVEX PROBLEMS** $\uparrow f_{\min}$
[Lasserre '01]

degree r & n vars $\Longrightarrow \binom{n+2r}{n}$ **SDP** VARIABLES

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HOW TO OVERCOME THIS NO-FREE LUNCH RULE?

An example

NP hard General Problem: $f_{\min} := \min_{\mathbf{x} \in \mathbf{X}} f(\mathbf{x})$

Semialgebraic set $\mathbf{X} = \{\mathbf{x} \in \mathbb{R}^n : g_j(\mathbf{x}) \geqslant 0\}$

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$$\overbrace{x_1 x_2}^{f} =$$

$$-\frac{1}{8} + \overbrace{\frac{1}{2}\left(x_1 + x_2 - \frac{1}{2}\right)^2}^{\sigma_0} + \overbrace{\frac{1}{2}}^{\sigma_1} \overbrace{x_1(1 - x_1)}^{g_1} + \overbrace{\frac{1}{2}}^{\sigma_2} \overbrace{x_2(1 - x_2)}^{g_2}$$

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Sums of squares (SOS) σ_j

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Sums of squares (SOS) σ_j

Quadratic module: $\mathcal{M}(\mathbf{X})_r = \left\{ \sigma_0 + \sum_j \sigma_j g_j, \ \deg \sigma_j \, g_j \leqslant 2r \right\}$

The Moment-SOS Hierarchy for POP

NP-hard NON CONVEX Problem $f_{\min} = \inf_{\mathbf{x} \in \mathbf{X}} f(\mathbf{x})$

- space $\mathcal{M}_+(\mathbf{X})$ of probability measures supported on $\mathbf{X}$
- quadratic module $\mathcal{Q}(\mathbf{X}) = \left\{ \sigma_0 + \sum_j \sigma_j g_j, \text{ with } \sigma_j \text{ SOS} \right\}$

Infinite-dimensional linear programs (LP)

(Primal) (Dual)

$$\begin{array}{llcll} \inf & \int_{\mathbf{X}} f \, d\mu & = & \sup & \lambda \\ \text{s.t.} & \mu \in \mathcal{M}_+(\mathbf{X}) & & \text{s.t.} & \lambda \in \mathbb{R} \\ & & & & f - \lambda \in \mathcal{Q}(\mathbf{X}) \end{array}$$

The Moment-SOS Hierarchy for POP

NP-hard NON CONVEX Problem $f_{\min} = \inf_{\mathbf{x} \in \mathbf{X}} f(\mathbf{x})$

- Pseudo-moment sequences $\mathbf{y}$ up to order r
- Truncated quadratic module $\mathcal{Q}(\mathbf{X})_r$

Finite-dimensional semidefinite programs (SDP)

(Moment)			(SOS)	
inf	$\sum_{\alpha} f_\alpha y_\alpha$	$=$	sup	λ
s.t.	$\mathbf{M}_{r-r_j}(g_j \, \mathbf{y}) \succcurlyeq 0$		s.t.	$\lambda \in \mathbb{R}$
	$y_0 = 1$			$f - \lambda \in \mathcal{Q}(\mathbf{X})_r$

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\end{array}
$$

Primal-dual "SPARSE" variants?

Sparse SDP

Correlative sparsity

Term sparsity

Ideal sparsity

Tutorial session

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Sparse matrices

Symmetric matrices indexed by graph vertices

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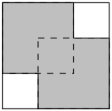
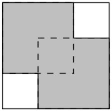


💡 no edge between 1 and 3 $\iff$ 0 entry in the $(1, 3)$ entry

Sparse matrices

Symmetric matrices indexed by graph vertices



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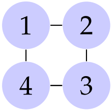

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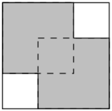

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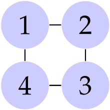

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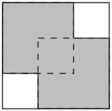

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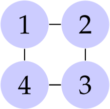

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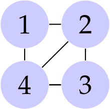

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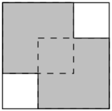

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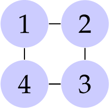

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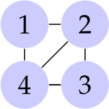
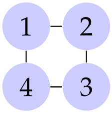


clique = a fully connected subset of vertices

Chordal extensions



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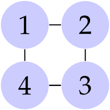

Fact

Any non-chordal graph can always be extended to a chordal graph, by adding appropriate edges

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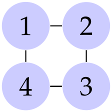

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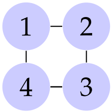

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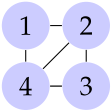
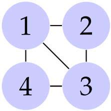
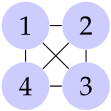




approximately minimal



maximal

Theorem [Gavril ’72, Vandenberghe & Andersen ’15]

The maximal cliques of a chordal graph can be enumerated in linear time in the number of nodes and edges.

Running intersection property (RIP)

RIP Theorem for chordal graphs [Blair & Peyton '93]

For a chordal graph with maximal cliques $I_1, \ldots, I_p$:

$$\text{(RIP)} \qquad \boxed{\forall k < p \quad I_{k+1} \cap \bigcup_{j \leqslant k} I_j \subseteq I_i \quad \text{for some } i \leqslant k}$$

(possibly after reordering)

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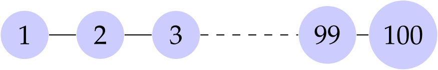
- RIP holds for chains



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- RIP always holds for $p = 2$
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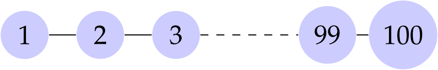
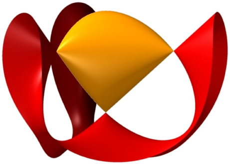


- RIP holds for numerous applications!

Semidefinite Programming (SDP)

$$\min_{\mathbf{y}} \quad \mathbf{c}^\top \mathbf{y}$$
$$\text{s.t.} \quad \sum_i \mathbf{F}_i \, y_i \succcurlyeq \mathbf{F}_0$$



Spectrahedron

- Linear cost $\mathbf{c}$
- Symmetric matrices $\mathbf{F}_0, \mathbf{F}_i$
- Linear matrix inequalities "$\mathbf{F} \succcurlyeq 0$" ($\mathbf{F}$ has nonnegative eigenvalues)

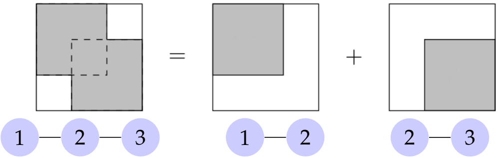
Sparse SDP matrices

Theorem [Griewank Toint '84, Agler et al. '88]

Chordal graph G with n vertices & maximal cliques I_1, I_2

$Q_G \succcurlyeq 0$ with nonzero entries corresponding to edges of G

$\implies Q_G = P_1{}^T Q_1 P_1 + P_2{}^T Q_2 P_2$ with $Q_k \succcurlyeq 0$ indexed by I_k



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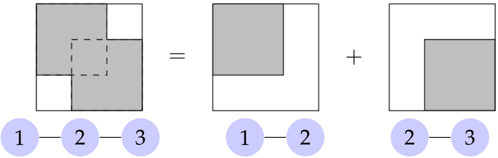

What are P_1, P_2?

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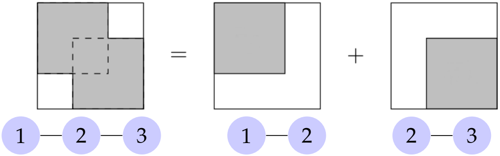

What are P_1, P_2? $P_1 \in \mathbb{R}^{|I_1| \times n}$

$$P(i,j) = \begin{cases} 1 & \text{if } I(i) = j \\ 0 & \text{otherwise} \end{cases}$$

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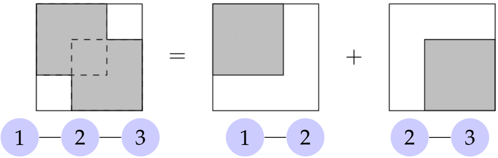

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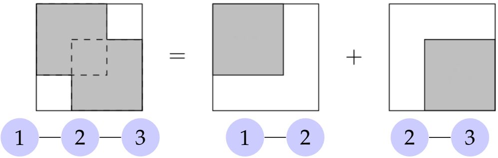

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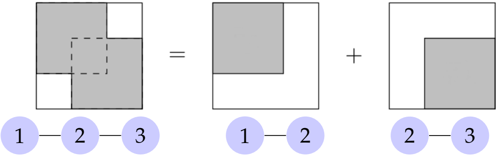

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${P_1}^T Q_1 P_1$ inflates a $|I_1| \times |I_1|$ matrix Q_1 into a sparse $n \times n$ matrix

Sparse SDP

Correlative sparsity

Term sparsity

Ideal sparsity

Tutorial session

What is correlative sparsity?

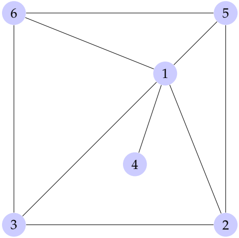
Exploit few links between **variables** [Lasserre, Waki et al. '06]

$$f(\mathbf{x}) = x_2 x_5 + x_3 x_6 - x_2 x_3 - x_5 x_6 + x_1(-x_1 + x_2 + x_3 - x_4 + x_5 + x_6)$$

Correlative sparsity pattern (csp) graph G

Vertices = $\{1, \ldots, n\}$

$(i, j) \in$ Edges $\Longleftrightarrow x_i x_j$ appears in f



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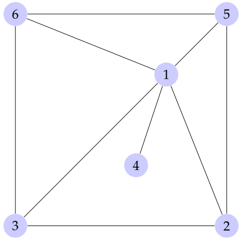

Similar construction with constraints $\mathbf{X} = \{\mathbf{x} \in \mathbb{R}^n : g_j(\mathbf{x}) \geqslant 0\}$

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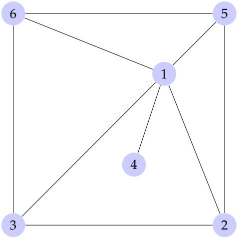
Chordal graph after adding edge $(3, 5)$



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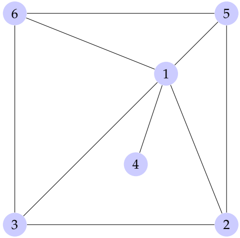

maximal cliques $I_1 = \{1, 4\}$ $I_2 = \{1, 2, 3, 5\}$ $I_3 = \{1, 3, 5, 6\}$

$f = f_1 + f_2 + f_3$ where f_k involves **only** variables in I_k

Let us index moment matrices and SOS with the cliques I_k

A sparse variant of Putinar's Positivstellensatz

Convergence of the Moment-SOS hierarchy is based on:

Theorem [Putinar '93] Positivstellensatz

If $\mathbf{X}$ contains a ball constraint $N - \sum_i x_i^2 \geqslant 0$ then

$$f > 0 \text{ on } \mathbf{X} = \{\mathbf{x} : g_j(\mathbf{x}) \geqslant 0\} \implies f = \sigma_0 + \sum_j \sigma_j g_j \text{ with } \sigma_j \text{ SOS}$$

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Theorem: Sparse Putinar's representation [Lasserre '06]

$f = \sum_k f_k$, f_k depends on $\mathbf{x}(I_k)$

$f > 0$ on $\mathbf{X}$

Each g_j depends on some I_k

RIP holds for (I_k)

ball constraints for each $\mathbf{x}(I_k)$

$\implies$

$$\boxed{f = \sum_k (\sigma_{0k} + \sum_{j \in J_k} \sigma_j g_j)}$$

SOS σ_{0k} "sees" vars in I_k

σ_j "sees" vars from g_j

A first key message

$\boxed{\text{SUMS OF SQUARES PRESERVE SPARSITY}}$

A proof of sparse Putinar's Positivstellensatz

Let $\mathbf{X} = \{\mathbf{x} : g_j(\mathbf{x}) \geqslant 0\}$ be compact and $f = \sum_k f_k$, with f_k depends on $\mathbf{x}(I_k)$, and $f > 0$ on $\mathbf{X}$

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Prove this lemma by induction on the number of subsets I_k

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If RIP holds for (I_k) then
$f = \sum_k h_k$, with h_k depends on $\mathbf{x}(I_k)$, and $h_k > 0$ on $\mathbf{X}_k$

- Prove this lemma by induction on the number of subsets I_k

- Then apply Putinar to each h_k

Sparse moment matrices

For each subset I_k, submatrix of $\mathbf{M}_r(\mathbf{y})$ corresponding of rows & columns indexed by monomials in $\mathbf{x}(I_k)$

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$I_1 = \{1,4\} \implies$ monomials in x_1, x_4

$$\mathbf{M}_1(\mathbf{y}, I_1) = \begin{pmatrix} 1 & | & y_{1,0,0,0,0,0} & y_{0,0,0,1,0,0} \\ & - & - & - \\ y_{1,0,0,0,0,0} & | & y_{2,0,0,0,0,0} & y_{1,0,0,1,0,0} \\ y_{0,0,0,1,0,0} & | & y_{1,0,0,1,0,0} & y_{0,0,0,2,0,0} \end{pmatrix}$$

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💡 same for each localizing matrix $\mathbf{M}_r(g_j\mathbf{y})$

Sparse primal-dual Moment-SOS hierarchy

$f_{\min} = \inf_{\mathbf{x} \in \mathbf{X}} f(\mathbf{x})$ with $\mathbf{X} = \{\mathbf{x} : g_j(\mathbf{x}) \geqslant 0\}$

Dense Moment-SOS hierarchy

(Moment) (SOS)

$$\begin{aligned} \inf \quad & \sum_\alpha f_\alpha y_\alpha & = \quad & \sup \quad \lambda \\ \text{s.t.} \quad & \mathbf{M}_r(\mathbf{y}) \succcurlyeq 0 & & \text{s.t.} \quad \lambda \in \mathbb{R} \\ & \mathbf{M}_{r-r_j}(g_j\,\mathbf{y}) \succcurlyeq 0 & & \qquad f - \lambda = \sigma_0 + \sum_j \sigma_j g_j \\ & y_0 = 1 & & \end{aligned}$$

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(Moment) (SOS)

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\inf \quad & \sum_\alpha f_\alpha \, y_\alpha & = \quad \sup \quad & \lambda \\
\text{s.t.} \quad & \mathbf{M}_r(\mathbf{y}, I_k) \succcurlyeq 0 & \text{s.t.} \quad & \lambda \in \mathbb{R} \\
& \mathbf{M}_{r-r_j}(g_j \, \mathbf{y}, I_k) \succcurlyeq 0 \, , j \in J_k, \forall k & & f - \lambda = \sum_k (\sigma_{k0} + \sum_{j \in J_k} \sigma_j g_j) \\
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\end{array}
$$

RIP holds for (I_k) + ball constraints for each $\mathbf{x}(I_k)$ $\implies$ Primal and dual optimal value converge to $f_{\min}$ by sparse Putinar

Computational cost

$f_{\min} = \inf_{\mathbf{x} \in \mathbf{X}} f(\mathbf{x})$ with $\mathbf{X} = \{\mathbf{x} : g_j(\mathbf{x}) \geqslant 0, j \leqslant m\}$

$\tau = \max\{|I_1|, \ldots, |I_p|\}$

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$$\begin{array}{llcll} & \text{(Moment)} & & & \text{(SOS)} \\ \inf & \sum_{\alpha} f_\alpha y_\alpha & = & \sup & \lambda \\ \text{s.t.} & \mathbf{M}_r(\mathbf{y}, I_k) \succcurlyeq 0 & & \text{s.t.} & \lambda \in \mathbb{R} \\ & \mathbf{M}_{r-r_j}(g_j\, \mathbf{y}, I_k) \succcurlyeq 0\,, j \in J_k, \forall k & & & f - \lambda = \sum_k (\sigma_{k0} + \sum_{j \in J_k} \sigma_j g_j) \\ & y_0 = 1 & & & \end{array}$$

$(m+p)$ SOS in at most τ vars of degree $\leqslant 2r$

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Sparse Moment-SOS hierarchy

(Moment) (SOS)

$$\inf \sum_\alpha f_\alpha \, y_\alpha \quad = \quad \sup \lambda$$

s.t. $\mathbf{M}_r(\mathbf{y}, I_k) \succcurlyeq 0$ s.t. $\lambda \in \mathbb{R}$

$\mathbf{M}_{r-r_j}(g_j \, \mathbf{y}, I_k) \succcurlyeq 0 \, , j \in J_k, \forall k$ $f - \lambda = \sum_k (\sigma_{k0} + \sum_{j \in J_k} \sigma_j g_j)$

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💡 $(m+p) \, \mathcal{O}(r^\tau)$ SDP vars

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Sparse Moment-SOS hierarchy

(Moment)		(SOS)
$\inf \sum_\alpha f_\alpha y_\alpha$	$=$	$\sup \lambda$
s.t. $\mathbf{M}_r(\mathbf{y}, I_k) \succcurlyeq 0$		s.t. $\lambda \in \mathbb{R}$
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$(m+p)$ SOS in at most τ vars of degree $\leqslant 2r$

$(m+p)\,\mathcal{O}(r^\tau)$ SDP vars vs $(m+1)\,\mathcal{O}(r^n)$ in the dense SDP

Sparse linear program over measures

In the dense setting:

$$\begin{aligned} f_{\min} = \inf_{\mu} \quad & \int_{\mathbf{X}} f \, d\mu \\ \text{s.t.} \quad & \mu \in \mathcal{M}_+(\mathbf{X}) \end{aligned}$$

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$\mathbf{X}_{kj}$ = only "sees" variables indexed by $I_k \cap I_j$

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In the sparse setting:

$\mathbf{X}_k = \{\mathbf{x}(I_k) : g_j(\mathbf{x}) \geqslant 0 : j \in J_k\}$ = the subspace of $\mathbf{X}$ which only "sees" variables indexed by I_k

$\mathbf{X}_{kj}$ = only "sees" variables indexed by $I_k \cap I_j$

💡 one measure μ_k for each I_k $\to$ marginals $\pi_{kj}\mu_k$ on $\mathcal{M}_+(\mathbf{X}_{kj})$

Sparse linear program over measures

In the dense setting:

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Sparse moment SDPs relax the sparse LP over measures:

$$f_{\text{cs}} = \inf_{\mu_k} \quad \sum_k \int_{\mathbf{X}_k} f_k \, d\mu_k \qquad \text{s.t.} \quad \pi_{jk}\mu_j = \pi_{kj}\mu_k \,, \quad \mu_k \in \mathcal{M}_+(\mathbf{X}_k)$$

The dual of sparse Putinar's Positivstellensatz

Theorem [Lasserre '06]

$$\text{RIP holds for } (I_k) \implies f_{\min} = f_{\text{cs}} = \inf_{\mu_k} \sum_k \int_{\mathbf{X}_k} f_k \, d\mu_k$$

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$$\begin{aligned} \text{s.t. } & \pi_{jk}\mu_j = \pi_{kj}\mu_k \\ & \mu_k \in \mathcal{M}_+(\mathbf{X}_k) \end{aligned}$$

Proof: there exists $\mu \in \mathcal{M}_+(\mathbf{X})$ with marginal μ_k on $\mathbf{X}_k$

$$\begin{tikzcd} & \mathcal{M}_+(\mathbf{X}) \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ \mathcal{M}_+(\mathbf{X}_1) \arrow[dr, "\pi_{21}"'] & & \mathcal{M}_+(\mathbf{X}_2) \arrow[dl, "\pi_{12}"] \\ & \mathcal{M}_+(\mathbf{X}_{12}) & \end{tikzcd}$$

A first (dual) key message

$\boxed{\text{THE MEASURE LP PRESERVES SPARSITY}}$

Extracting minimizers: the dense case

Let $r_{\min}$ be the minimal relaxation order.

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Theorem: dense extraction [Lasserre & Henrion '05]

Assume that the moment SDP has an optimal solution $\mathbf{y}$ with cost f^r and

$$\operatorname{rank} \mathbf{M}_{r'}(\mathbf{y}) = \operatorname{rank} \mathbf{M}_{r'-r_{\min}}(\mathbf{y}) \text{ for some } r' \leqslant r\,.$$

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Then $f^r = f_{\min}$ and the LP over measures has an optimal solution $\mu \in \mathcal{M}_+(\mathbf{X})$ supported on $t = \operatorname{rank} \mathbf{M}_{r'}(\mathbf{y})$ points of $\mathbf{X}$.

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Extraction possible with the `Gloptipoly` software

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Then $f_{\text{cs}}^r = f_{\min} = f_{\text{cs}}$ & sparse measure LP has optimal solution $\mu_k \in \mathcal{M}_+(\mathbf{X}_k)$ supported on $t_k = \operatorname{rank} \mathbf{M}_r(\mathbf{y}, I_k)$ points of $\mathbf{X}_k$.

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- RIP is not required!
- Extract $\mathbf{x}(k)$ from $\mathbf{M}_r(\mathbf{y}, I_k) \implies$ minimizer $\mathbf{x}$ with $(x_i)_{i \in I_k} = \mathbf{x}(k)$

Application to rational functions

$$f_{\min} = \inf_{\mathbf{x} \in \mathbf{X}} \sum_i \frac{p_i(\mathbf{x})}{q_i(\mathbf{x})} \,, \quad q_i > 0 \text{ on } \mathbf{X} \,, \quad p_i, q_i \text{ depends only on } I_i$$

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Application to roundoff errors

[Magron Constantinides Donaldson '17]

Exact $f(\mathbf{x}) = x_1 x_2 + x_3 x_4$

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3: Bound $\ell(\mathbf{x}, \mathbf{e})$ with **SPARSE SUMS OF SQUARES**

$I_k \to \{\mathbf{x}, e_k\} \implies \boxed{m\, r^{n+1} \text{ instead of } r^{n+m}}$ SDP vars

Application to roundoff errors

$$f = x_2x_5 + x_3x_6 - x_2x_3 - x_5x_6 + x_1(-x_1 + x_2 + x_3 - x_4 + x_5 + x_6)$$

$$\mathbf{x} \in [4.00, 6.36]^6, \quad \mathbf{e} \in [-\epsilon, \epsilon]^{15}, \quad \epsilon = 2^{-53}$$

Dense SDP: $\binom{6+15+4}{6+15} = 12650$ variables $\rightsquigarrow$ **Out of memory**

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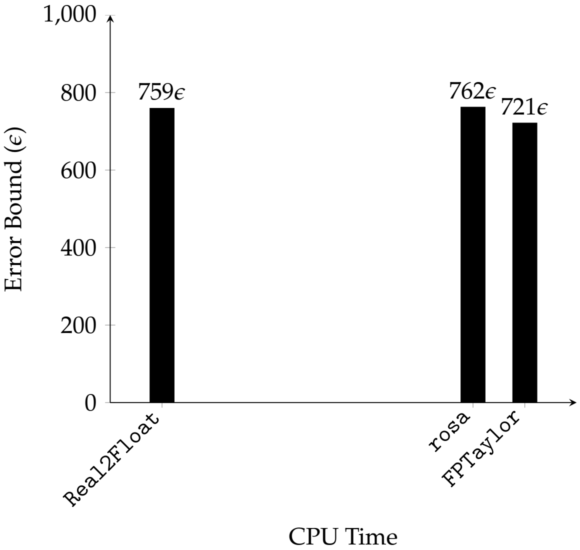
SMT-based `rosa` tool: 762ϵ (19 $\times$ more CPU)

Application to roundoff errors



Extension to noncommutative optimization

Self-adjoint noncommutative variables $a_i, b_j \in \mathcal{B}(\mathcal{H})$

$$f = a_1(b_1 + b_2 + b_3) + a_2(b_1 + b_2 - b_3) + a_3(b_1 - b_2) - b_1 - 2b_1 - b_2$$

with $a_1 a_2 \neq a_2 a_1$, **involution** $(a_1 b_3)^\star = b_3 a_1$

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MINIMAL EIGENVALUE OPTIMIZATION

$$\lambda_{\min} = \inf\{\langle f(a, b)\mathbf{v}, \mathbf{v}\rangle : (a, b) \in \mathbf{X}, \|\mathbf{v}\| = 1\}$$

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$$\begin{aligned} \lambda_{\min} &= \inf \{ \langle f(a,b)\mathbf{v}, \mathbf{v} \rangle : (a,b) \in \mathbf{X}, \|\mathbf{v}\| = 1 \} \\ &= \sup \quad \lambda \\ &\quad \text{s.t.} \quad f(a,b) - \lambda \mathbf{I} \succcurlyeq 0\,, \quad \forall (a,b) \in \mathbf{X} \end{aligned}$$

Extension to noncommutative optimization

Ball constraint $N - \sum_i x_i^2 \succcurlyeq 0$ in $\mathbf{X}$

Theorem: NC Putinar's representation [Helton & McCullough '02]

$f \succ 0$ on $\mathbf{X}$ $\implies$ $\boxed{f = \sum_i s_i^\star s_i + \sum_j \sum_i t_{ji}^\star g_j t_{ji}}$ with s_i, $t_{ji} \in \mathbb{R}\langle \underline{x} \rangle$

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NC variant of Lasserre's Hierarchy for $\lambda_{\min}$:

replace "$f - \lambda \mathbf{I} \succcurlyeq 0$ on $\mathbf{X}$" by $f - \lambda \mathbf{I} = \sum_i s_i^\star s_i + \sum_j \sum_i t_{ji}^\star g_j t_{ji}$
with s_i, t_{ji} of **bounded** degrees = SDP

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Self-adjoint noncommutative (NC) variables $x = (x_1, \ldots, x_n)$

Theorem [Helton & McCullough '02]

$f \succcurlyeq 0 \Leftrightarrow f$ SOS (all positive polynomials are sums of squares)

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Theorem: Sparse NC Positivstellensatz [Klep Magron Povh '21]

$f = \sum_k f_k$, f_k depends on $x(I_k)$
$f > 0$ on $\{\mathbf{x} : g_j(\mathbf{x}) \geqslant 0\}$
chordal graph with cliques I_k
ball constraints for each $\mathbf{x}(I_k)$

$\Longrightarrow$

$$\boxed{f = \sum_{k,i} (s_{ki}^\star s_{ki} + \sum_{j \in J_k} t_{ji}^\star g_j t_{ji})}$$

s_{ki} "sees" vars in I_k
t_{ji} "sees" vars from g_j

Application to violation of Bell inequalities

$\mathbf{I}_{3322}$ Bell inequality (entanglement in quantum information)

$$f = a_1(b_1 + b_2 + b_3) + a_2(b_1 + b_2 - b_3) + a_3(b_1 - b_2) - a_1 - 2b_1 - b_2$$

Maximal violation levels $\to$ **upper bounds** on $\lambda_{\max}$ of f on $\{a, b : a_i^2 = a_i \quad b_i^2 = b_i \quad a_i b_j = b_j a_i\}$

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level	sparse	dense [Pál & Vértesi '18]
2	0.2550008	0.2509397

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Maximal violation levels $\to$ **upper bounds** on $\lambda_{\max}$ of f on

$\{a, b : a_i^2 = a_i \quad b_i^2 = b_i \quad a_i b_j = b_j a_i\}$

$I_k \to \{a_k, b_1, b_2, b_3\}$

level	sparse	dense [Pál & Vértesi '18]
2	0.2550008	0.2509397
3	0.2511592	0.2508756

Application to violation of Bell inequalities

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6	0.2508753977180	(1 hour)

PERFORMANCE  VS  ACCURACY

Application to SOS of bounded degrees

Theorem: sparse BSOS representation [Weisser et al. '18]

If $0 \leqslant g_j \leqslant 1$ on $\mathbf{X}$, $f > 0$ on $\mathbf{X}$ & RIP holds for (I_k) then

$$f = \sum_k \left(\sigma_k + \sum_{\alpha,\beta} c_{k,\alpha\beta} \prod_{j \in J_k} {g_j}^{\alpha_j} (1 - g_j)^{\beta_j} \right) ,$$

with σ_k SOS of degree $\leqslant 2r$

Application to sparse positive definite forms

Theorem: [Reznick '95] Positivstellensatz

pd form $f \implies \boxed{f = \dfrac{\sigma}{\|\mathbf{x}\|_2^{2r}}}$ with σ SOS, $r \in \mathbb{N}$

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Sparse $f \sum_k f_k$, with f_k only depends on I_k

RUNNING INTERSECTION PROPERTY (RIP)

$$\boxed{\forall k \quad \underbrace{I_k \cap \bigcup_{j<k} I_j}_{\hat{I}_k} \subseteq I_{s_k} \quad \text{for some } s_k < k}$$

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Theorem: sparse Reznick [Mai Lasserre Magron '20]

RIP $\implies \boxed{f = \sum_k \frac{\sigma_k}{{H_k}^r}}$ with σ_k SOS only depends on I_k

Uniform H_k involve products $\|\mathbf{x}(I)\|_2^2$ for $I \in \{I_k, \hat{I}_k, \hat{I}_i : s_i = k\}$

More and more applications!

Robust Geometric Perception [Yang & Carlone '20]

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Volume computation [Tacchi et al. '21]

Robustness of implicit deep networks [Chen et al. '21]

Sparse SDP

Correlative sparsity

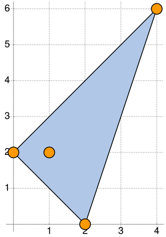
Term sparsity

Ideal sparsity

Tutorial session

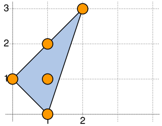
Term sparsity via Newton polytope

$f = 4x_1^4x_2^6 + x_1^2 - x_1x_2^2 + x_2^2$
spt(f) = $\{(4,6),(2,0),(1,2),(0,2)\}$



Newton polytope $\mathscr{B} = \operatorname{conv}$ (spt(f))

Squares in SOS decomposition $\subseteq \frac{\mathscr{B}}{2} \cap \mathbb{N}^n$
[Reznick '78]



$$f = \begin{pmatrix} x_1 & x_2 & x_1x_2 & x_1x_2^2 & x_1^2x_2^3 \end{pmatrix} \underbrace{Q}_{\succcurlyeq 0} \begin{pmatrix} x_1 \\ x_2 \\ x_1x_2 \\ x_1x_2^2 \\ x_1^2x_2^3 \end{pmatrix}$$

Term sparsity: the unconstrained case

$$f = x_1^2 - 2x_1x_2 + 3x_2^2 - 2x_1^2x_2 + 2x_1^2x_2^2 - 2x_2x_3 + 6x_3^2 + 18x_2^2x_3 - 54x_2x_3^2 + 142x_2^2x_3^2$$

[Reznick '78] $\rightarrow f = \begin{pmatrix} 1 & x_1 & x_2 & x_3 & x_1x_2 & x_2x_3 \end{pmatrix} \underbrace{Q}_{\succcurlyeq 0} \begin{pmatrix} 1 \\ x_1 \\ x_2 \\ x_3 \\ x_1x_2 \\ x_2x_3 \end{pmatrix}$

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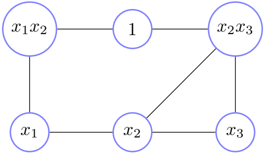
💡 **Term** sparsity pattern graph G



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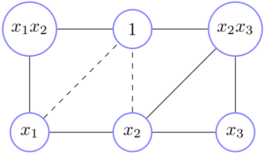
+ chordal extension G'



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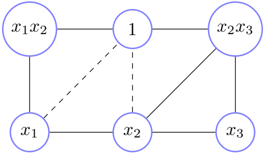

Replace Q by $Q_{G'}$ with nonzero entries at edges of G'
$\rightsquigarrow$ 6 + 9 = 15 "unknown" entries in $Q_{G'}$

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At step r of the hierarchy, tsp graph G has

Nodes V = monomials of degree $\leqslant r$

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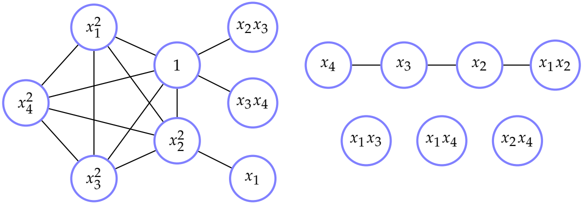
An example with $r = 2$

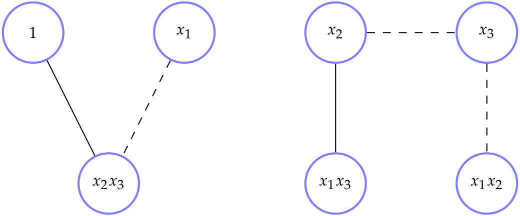
$f = x_1^4 + x_1 x_2^2 + x_2 x_3 + x_3^2 x_4^2$

$g_1 = 1 - x_1^2 - x_2^2 - x_3^2 \quad g_2 = 1 - x_3 x_4$



Term sparsity: support extension

$\alpha' + \beta' = \alpha + \beta$ and $(\alpha, \beta) \in E \Rightarrow (\alpha', \beta') \in E$



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$\rightsquigarrow$ support extension

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By iteratively performing support extension & chordal extension

$$G^{(1)} = G' \subseteq \cdots \subseteq G^{(s)} \subseteq G^{(s+1)} \subseteq \cdots$$

💡 Two-level hierarchy of lower bounds for $f_{\min}$, indexed by sparse order s and relaxation order r

Term sparsity: primal moment relaxations

Let G' be a chordal extension of G with maximal cliques (C_i)

$$C_i \longmapsto \mathbf{M}_{C_i}(\mathbf{y})$$

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Each constraint $G_j \rightsquigarrow G_j^{(s)} \rightsquigarrow$ cliques $C_{j,i}^{(s)}$

Term sparsity: primal moment relaxations

Let $C_{j,i}^{(s)}$ be the maximal cliques of $G_j^{(s)}$. For each $s \geq 1$

$$
\begin{aligned}
f_{\mathrm{ts}}^{r,s} = \quad \inf \quad & \textstyle\sum_\alpha f_\alpha y_\alpha \\
\text{s.t.} \quad & \mathbf{M}_{C_{0,i}^{(s)}}(\mathbf{y}) \succcurlyeq 0 \\
& \mathbf{M}_{C_{j,i}^{(s)}}(g_j\, \mathbf{y}) \succcurlyeq 0 \\
& y_0 = 1
\end{aligned}
$$

💡 dual yields the TSSOS hierarchy

A two-level hierarchy of lower bounds

$$
\begin{array}{cccc}
f_{\mathrm{ts}}^{r_{\min},1} \leq & f_{\mathrm{ts}}^{r_{\min},2} \leq & \cdots \leq & f^{r_{\min}} \\
\wedge| & \wedge| & & \wedge| \\
f_{\mathrm{ts}}^{r_{\min}+1,1} \leq & f_{\mathrm{ts}}^{r_{\min}+1,2} \leq & \cdots \leq & f^{r_{\min}+1} \\
\wedge| & \wedge| & & \wedge| \\
\vdots & \vdots & \vdots & \vdots \\
\wedge| & \wedge| & & \wedge| \\
f_{\mathrm{ts}}^{r,1} \leq & f_{\mathrm{ts}}^{r,2} \leq & \cdots \leq & f^{r} \\
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\vdots & \vdots & \vdots & \vdots
\end{array}
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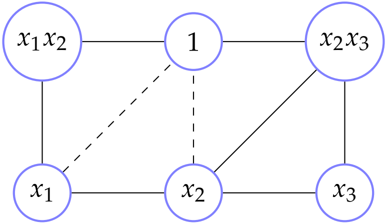
Different choices of chordal extensions



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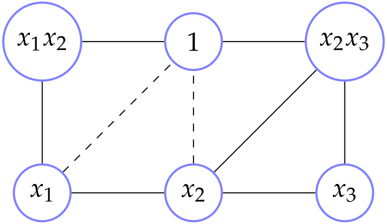

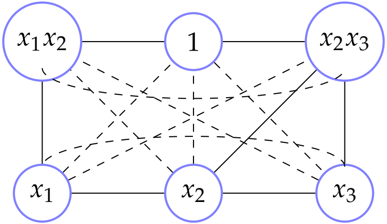

Term sparsity: convergence guarantees

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Newton polytope $\rightsquigarrow \mathscr{B} = (1 \quad x_1 x_2 \quad x_1 x_2^2 \quad x_1^2 x_2 \quad x_1^2 x_2^2)$

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$x_2 \mapsto -x_2$

Sign-symmetries blocks $(1 \quad x_1 x_2^2 \quad x_1^2 x_2^2) \quad (x_1 x_2 \quad x_1^2 x_2)$

TSSOS blocks $(1 \quad x_1 x_2^2 \quad x_1^2 x_2^2) \quad (x_1 x_2) \quad (x_1^2 x_2)$

A second key message

$\boxed{\textbf{TSSOS preserves the block structure related to sign-symmetries}}$

Comparison with (S)DSOS

Let f be a nonnegative polynomial of degree $2d$
f is SOS $\Leftrightarrow f = \mathbf{v}^T \mathbf{Q} \mathbf{v}$ with $\mathbf{Q} \succcurlyeq 0 \rightsquigarrow$ semidefinite program
where $\mathbf{v}$ contains $1, x_1, \dots, x_n, x_1^2, \dots, x_n^d$

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To reduce the number of “unknown” entries in $\mathbf{Q}$, one can force: [Ahmadi & Majumdar ’14]

1. $\mathbf{Q}$ diagonally dominant: $\mathbf{Q}_{ii} \geqslant \sum_{j \neq i} \mathbf{Q}_{ij} \rightsquigarrow$ linear program
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Theorem [Lasserre Magron Wang '21]

The first TSSOS relaxation is always more accurate than the SDSOS relaxation

Combining correlative & term sparsity

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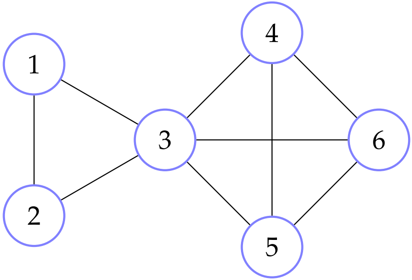
💡 a two-level CS-TSSOS hierarchy of lower bounds for $f_{\min}$

Combining correlative & term sparsity

$$f = 1 + \sum_{i=1}^{6} x_i^4 + x_1x_2x_3 + x_3x_4x_5 + x_3x_4x_6 + x_3x_5x_6 + x_4x_5x_6$$

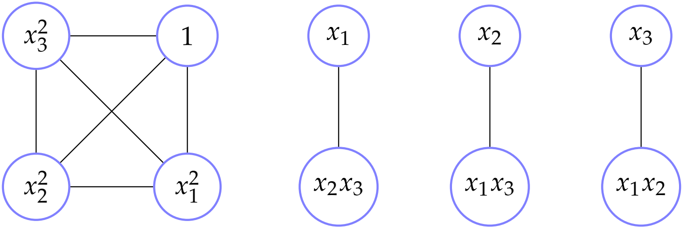
csp graph



Combining correlative & term sparsity

$$f = 1 + \sum_{i=1}^{6} x_i^4 + x_1x_2x_3 + x_3x_4x_5 + x_3x_4x_6 + x_3x_5x_6 + x_4x_5x_6$$

tsp graph for the first clique



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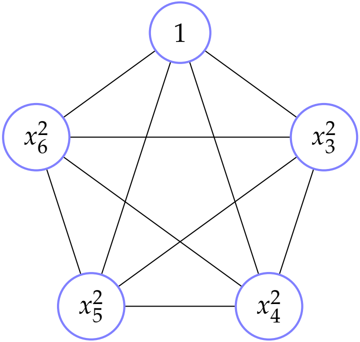

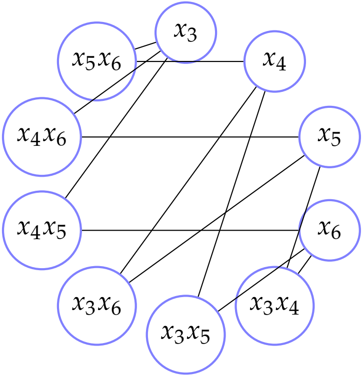

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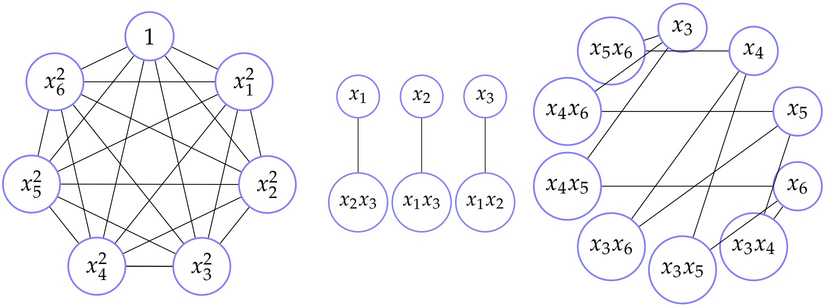
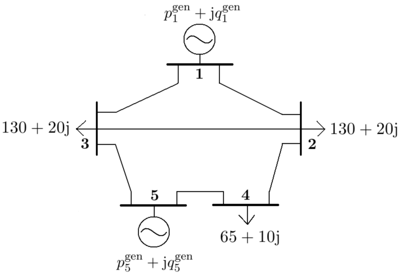
tsp graph without correlative sparsity



Application to optimal power-flow



Optimal Powerflow $n \simeq \mathbf{10^3}$

[Josz et al. '18]

$$
\begin{cases}
\inf\limits_{V_i, S_s^g, S_{ij}} & \sum_{s \in G} \left(\mathbf{c}_{2s} (\Re(S_s^g))^2 + \mathbf{c}_{1s} \Re(S_s^g) + \mathbf{c}_{0s} \right) \\
\text{s.t.} & \angle V_{\text{ref}} = 0, \\
& \mathbf{S}_s^{gl} \leq S_s^g \leq \mathbf{S}_s^{gu} \ \forall s \in G, \quad \boldsymbol{v}_i^l \leq |V_i| \leq \boldsymbol{v}_i^u \ \forall i \in N \\
& \sum_{s \in G_i} S_s^g - \mathbf{S}_i^d - \mathbf{Y}_i^s |V_i|^2 = \sum_{(i,j) \in E_i \cup E_i^R} S_{ij}, \quad \forall i \in N \\
& S_{ij} = \left(\mathbf{Y}_{ij}^* - \mathbf{i} \frac{\mathbf{b}_{ij}^c}{2} \right) \frac{|V_i|^2}{|\mathbf{T}_{ij}|^2} - \mathbf{Y}_{ij}^* \frac{V_i V_j^*}{\mathbf{T}_{ij}}, \quad S_{ji} = \cdots \\
& |S_{ij}| \leq \mathbf{s}_{ij}^u, \ \boldsymbol{\theta}_{ij}^{\Delta l} \leq \angle (V_i V_j^*) \leq \boldsymbol{\theta}_{ij}^{\Delta u}, \quad \forall (i,j) \in E
\end{cases}
$$

Application to optimal power-flow

mb = the maximal size of blocks

m = number of constraints

n	m	CS ($r = 2$)			CS+TS ($r = 2, s = 1$)		
		mb	time (s)	gap	mb	time (s)	gap
114	315	66	5.59	0.39%	31	2.01	0.73%
348	1809	253	–	–	34	278	0.05%
766	3322	153	585	0.68%	44	33.9	0.77%
1112	4613	496	–	–	31	410	0.25%
4356	18257	378	–	–	27	934	0.51%
6698	29283	1326	–	–	76	1886	0.47%

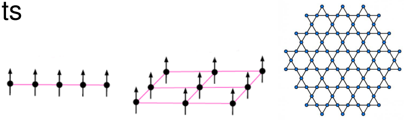
Application to noncommutative optimization

Ground-state energy $\Leftrightarrow$ minimal eigenvalue of an Hamiltonian

$$H = \sum_{\langle i,j \rangle} \left(x_i \, x_j + y_i \, y_j + z_i \, z_j \right)$$

spin states (x_i, y_i, z_i), constraints

Lattices: 1D 2D Kagome



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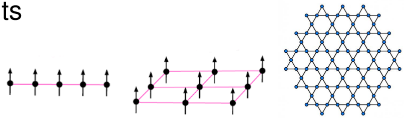

First neighbors interactions : $H = \sum_{i=1}^{n} x_i x_{i+1} + y_i y_{i+1} + z_i z_{i+1}$

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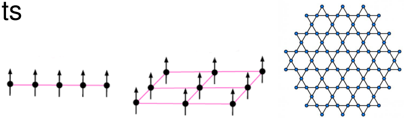

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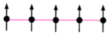

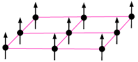

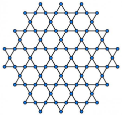

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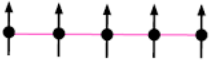
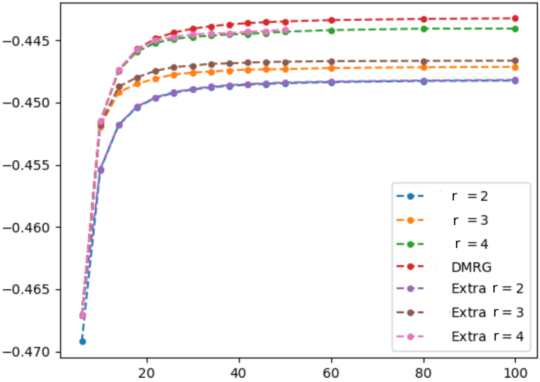
Existing $\pm$ efficient techniques: quantum Monte Carlo & variational algorithms $\Rightarrow$ **upper bounds** on minimal energy

Application to noncommutative optimization

Lower bounds of the energy 1D lattice





Dense $r = 4$, $n = 10^2 \Rightarrow 10^{11}$ variables (solvers handle $\simeq 10^4$)

Application to noncommutative optimization

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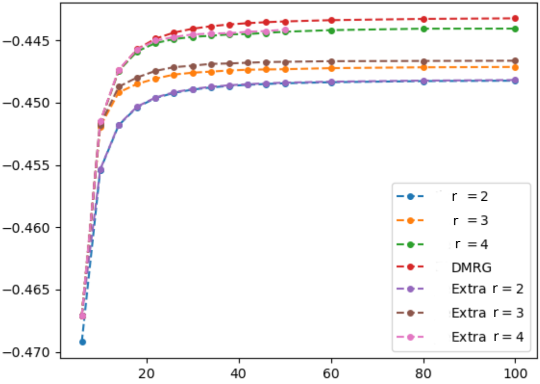

Dense $r = 4$, $n = 10^2 \Rightarrow 10^{11}$ variables (solvers handle $\simeq 10^4$)

Sparse solved within 1 hour on PFCALCUL at LAAS

Application to noncommutative optimization

CLASSICAL WORLD

$$\psi^*(A_1 \otimes B_1 + A_1 \otimes B_2 + A_2 \otimes B_1 - A_2 \otimes B_2)\psi \leqslant 2$$

for separable states $\psi \in \mathbb{C}^k \otimes \mathbb{C}^k$ and matrices A_j, B_j satisfying $A_j^* = A_j$, $A_j^2 = I$, $B_j^* = B_j$, $B_j^2 = I$

Application to noncommutative optimization

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TSIRELSON'S BOUND: $2 \to 2\sqrt{2}$ for maximally entangled states $\psi^*(A \otimes B)\psi = \operatorname{tr}(AB)$

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TSIRELSON'S BOUND: $2 \to 2\sqrt{2}$ for maximally entangled states $\psi^*(A \otimes B)\psi = \mathrm{tr}(AB)$

$$2\sqrt{2} = \mathrm{tr}_{\max}\{a_1 b_1 + a_1 b_2 + a_2 b_1 - a_2 b_2 : a_j^2 = b_j^2 = 1\}$$

Application to noncommutative optimization

COVARIANCES OF QUANTUM CORRELATIONS

$$\mathrm{cov}_{\psi}(A, B) = \psi^*(A \otimes B)\psi - \psi^*(A \otimes I)\psi \cdot \psi^*(I \otimes B)\psi$$

Application to noncommutative optimization

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for separable states but ...

Application to noncommutative optimization

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for separable states but . . . 5 for **one** maximally entangled state

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💡 2nd dense SDP relaxation of the corresponding trace problem outputs 5

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for separable states but . . . 5 for **one** maximally entangled state

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2nd sparse SDP gives also 5 . . . 10 times faster

Application to networked systems stability

Lyapunov function

$$f = \sum_{i=1}^{N} a_i (x_i^2 + x_i^4) - \sum_{i,k=1}^{N} b_{ik} x_i^2 x_k^2 \quad a_i \in [1, 2] \quad b_{ik} \in [\frac{0.5}{N}, \frac{1.5}{N}]$$

$\rightsquigarrow \binom{N+2}{2}(\binom{N+2}{2} + 1)/2$ “unknown” entries in $Q = 231$ for $N = 5$

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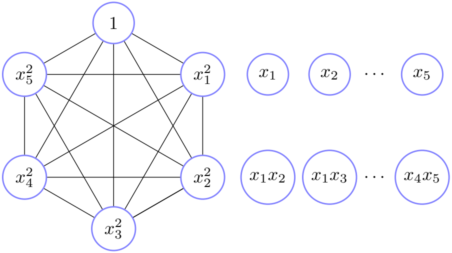

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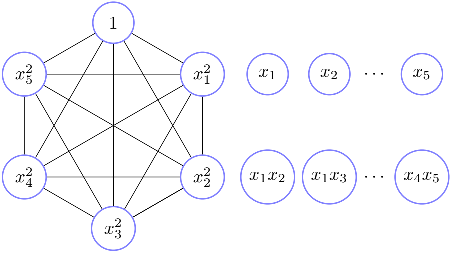

$\rightsquigarrow (N+1)^2$ "unknown" entries in $Q_G = 36$ for $N = 5$

Proof that $f \geqslant 0$ for $N = 80$ in ~ 10 seconds!

Application to networked systems stability

Duffing oscillator Hamiltonian $V = \sum_{i=1}^{N} a_i \left(\frac{x_i^2}{2} - \frac{x_i^4}{4}\right) + \frac{1}{8} \sum_{i,k=1}^{N} b_{ik} (x_i - x_k)^4$

On which domain $V > 0$?

$$f = V - \sum_{i=1}^{N} \underbrace{\lambda_i}_{>0} x_i^2 (g - x_i^2) \geqslant 0$$

$\implies V > 0$ when $x_i^2 < g$

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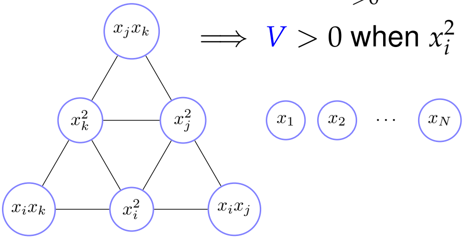

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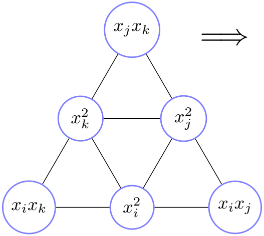

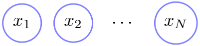

tsp graph G

$\rightsquigarrow$ $\frac{N(N+1)}{2} + 6\binom{N}{2} + N$ "unknown" entries in $Q_G = 80$ for $N = 5$

Proof that $f \geqslant 0$ for $N = 50$ in ~ 1 second!

Application to joint spectral radius (JSR)

Given $\mathcal{A} = \{A_1, \ldots, A_m\} \subseteq \mathbb{R}^{n \times n}$, the JSR is

$$\rho(\mathcal{A}) := \lim_{k \to \infty} \max_{\sigma \in \{1,\ldots,m\}^k} ||A_{\sigma_1} A_{\sigma_2} \cdots A_{\sigma_k}||^{\frac{1}{k}}$$

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Tons of applications:

- stability of switched linear systems
- continuity of wavelet functions
- trackability of graphs
- $\cdots$

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... NP-hard to compute/approximate

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Theorem [Parrilo & Jadbabaie '08]

Given $\mathcal{A} = \{A_1, \ldots, A_m\} \subseteq \mathbb{R}^{n \times n}$, if a positive definite form f of degree $2r$ satisfies

$$f(A_i \mathbf{x}) \leqslant \gamma^{2r} f(\mathbf{x}) \quad \forall i, \mathbf{x}$$

Then $\rho(\mathcal{A}) \leqslant \gamma$

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Bisection on γ + SDP

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- At fixed r, replace f by a "term sparse" f

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Theorem: Sparse JSR [Maggio Magron Wang '21]

$$\rho(\mathcal{A}) \leqslant \rho^r(\mathcal{A}) \leqslant \rho^{r,s}(\mathcal{A}) = \inf_{f \in \mathbb{R}[\mathscr{A}^{(s)}], \gamma} \gamma$$

$$\text{s.t.} \begin{cases} f(\mathbf{x}) - ||\mathbf{x}||_2^{2r} \text{ SOS}(\mathscr{A}^{(s)}) \\ \gamma^{2r} f(\mathbf{x}) - f(A_i \mathbf{x}) \text{ SOS}(\mathscr{A}_i^{(s)}) \end{cases}$$

Application to joint spectral radius (JSR)

Closed-loop system evolves according to either a completed or a missed computation (A_H or A_M): $\mathcal{A} = \{A_H {A_M}^i \mid i < m\}$

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💡 takes less than 10 seconds with the Sparse JSR!

Ideal sparsity

$$f_{\min} = \inf\{f(x_1, x_2) : x_1 x_2 = 0\}$$

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Theorem [Korda-Laurent-Magron-Steenkamp '22]

Ideal-sparse hierarchies provide better bounds than the dense ones

ACCURACY

Application to matrix ranks

Given a symmetric nonnegative matrix A, find the smallest r_+ s.t.

$$A = \sum_{\ell=1}^{r_+} a_\ell {a_\ell}^T \qquad \text{for } a_\ell \geqslant 0$$

r_+ is called the completely positive rank

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✓ Relax/convexify with a linear program over measures

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$$K_A = \{ \mathbf{x} : \sqrt{A_{ii}} x_i - x_i \geqslant 0 \,, \quad A_{ij} - x_i x_j \geqslant 0 \ (i,j) \in E_A \,, \\ x_i x_j = 0 \ (i,j) \in \overline{E}_A \,, \quad A - \mathbf{x}\mathbf{x}^T \succcurlyeq 0 \}$$

Application to matrix ranks

Random instances, order 2

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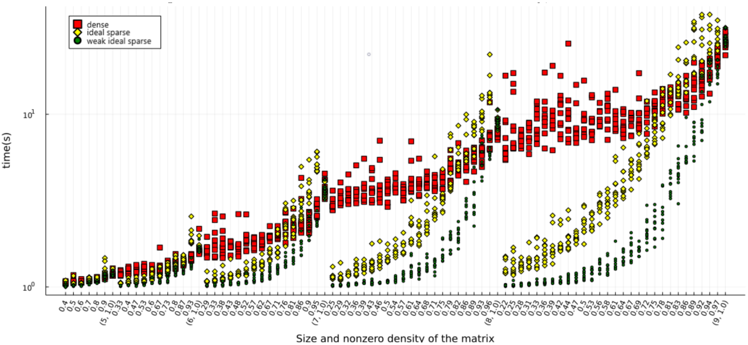

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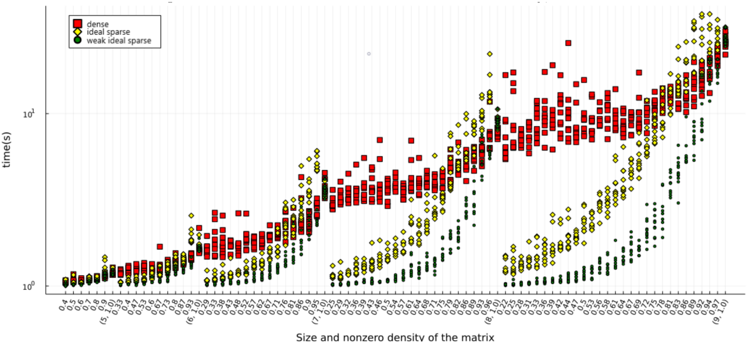

PERFORMANCE  AND  ACCURACY

Conclusion

SPARSITY EXPLOITING CONVERGING HIERARCHIES to minimize polynomials, eigenvalue/trace, joint spectral radii

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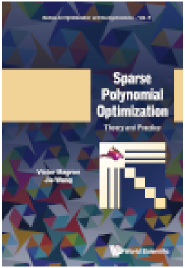
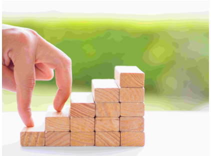


Further topics

Correlative sparsity: Convergence rate?



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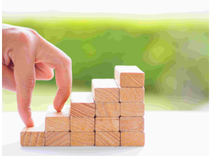

Correlative sparsity: Convergence rate?

Term sparsity: Smart solution extraction?

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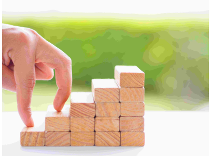

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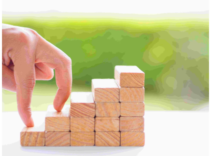

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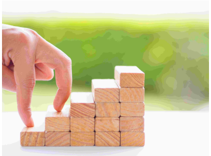

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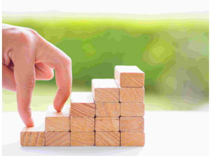

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Tons of applications!

Thank you for your attention!

`https://homepages.laas.fr/vmagron`

GITHUB:TSSOS

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Sparse SDP

Correlative sparsity

Term sparsity

Ideal sparsity

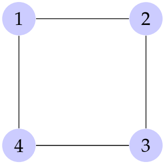
Tutorial session

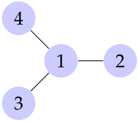
Newton polytope

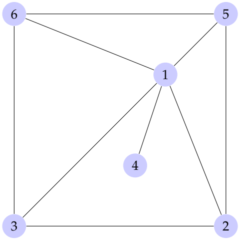
Motzkin $f = x_1^4 x_2^2 + x_1^2 x_2^4 - 3x_1^2 x_2^2 + 1$

1. Compute the Newton polytope of f
2. Show that f is not SOS

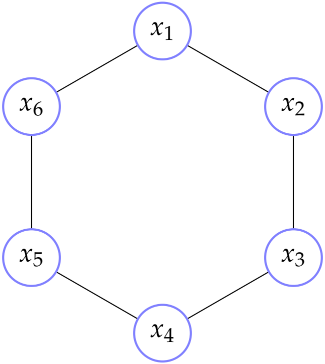
Chordal or not chordal?



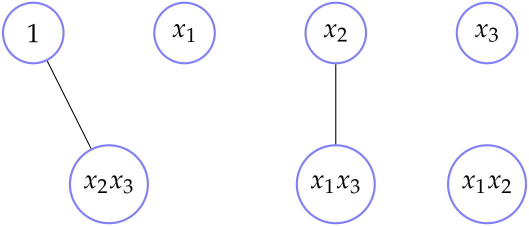




Chordal extension



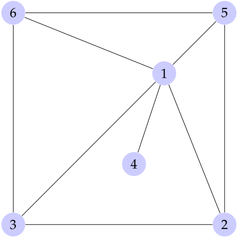
Support extension



How big is CS?

$$f(\mathbf{x}) = x_2x_5 + x_3x_6 - x_2x_3 - x_5x_6 + x_1(-x_1 + x_2 + x_3 - x_4 + x_5 + x_6)$$

Chordal graph after adding edge $(3, 5)$



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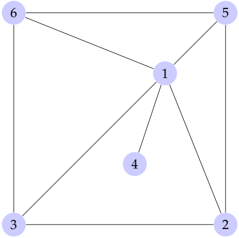

Chordal graph after adding edge $(3, 5)$

How many SDP variables in the dense and sparse relaxation at order $r = 1, 2, 3$?

Moment matrix

Write the first (correlative) sparse moment relaxation of

$$\begin{aligned} \inf_{\mathbf{x}} \quad & x_1 x_2 + x_1 x_3 + x_1 x_4 \\ \text{s.t.} \quad & x_1^2 + x_2^2 \leqslant 1 \\ & x_1^2 + x_3^2 \leqslant 1 \\ & x_1^2 + x_4^2 \leqslant 1 \end{aligned}$$

Measure LP preserves sparsity

$f = f_1 + f_2$, f_k depends on I_k, $\mathbf{X}$ compact & each g_j depends either on I_1 or I_2.

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$f = f_1 + f_2$, f_k depends on I_k, $\mathbf{X}$ compact & each g_j depends either on I_1 or I_2. Prove that

$$f_{\min} = \inf_{\mu \in \mathcal{M}_+(\mathbf{X})} \int_{\mathbf{X}} f \, d\mu = f_{\text{cs}} = \begin{array}{rl} \displaystyle\inf_{\mu_1, \mu_2} & \int_{\mathbf{X}_1} f_1 \, d\mu_1 + \int_{\mathbf{X}_2} f_2 \, d\mu_2 \\ \text{s.t.} & \pi_{12}\mu_1 = \pi_{21}\mu_2 \\ & \mu_1 \in \mathcal{M}_+(\mathbf{X}_1) \,, \quad \mu_2 \in \mathcal{M}_+(\mathbf{X}_2) \end{array}$$

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$f = f_1 + f_2$, f_k depends on I_k, $\mathbf{X}$ compact & each g_j depends either on I_1 or I_2. Prove that

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$$\text{s.t.} \quad \pi_{12}\mu_1 = \pi_{21}\mu_2$$

$$\mu_1 \in \mathcal{M}_+(\mathbf{X}_1)\,, \quad \mu_2 \in \mathcal{M}_+(\mathbf{X}_2)$$

(μ_k) feasible for $f_{\mathrm{cs}} \implies \exists \mu \in \mathcal{M}_+(\mathbf{X})$ with marginal μ_k on $\mathbf{X}_k$

$$\begin{tikzcd}
 & \mathcal{M}_+(\mathbf{X}) \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\
\mathcal{M}_+(\mathbf{X}_1) \arrow[dr, "\pi_{21}"'] & & \mathcal{M}_+(\mathbf{X}_2) \arrow[dl, "\pi_{12}"] \\
 & \mathcal{M}_+(\mathbf{X}_{12}) &
\end{tikzcd}$$

How big is TSSOS? (1/2)

$$f = \sum_{i=1}^{N} (x_i^2 + x_i^4) - \sum_{i,k=1}^{N} x_i^2 x_k^2$$

How many entries in the dense & sparse SOS/moment matrices?

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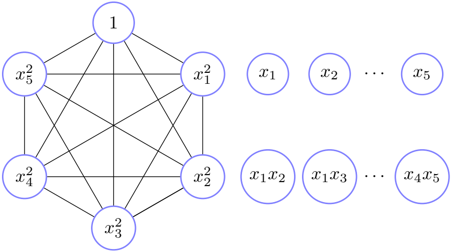
tsp graph G



How big is TSSOS? (2/2)

$$f = \sum_{i=1}^{N} \left(\frac{x_i^2}{2} - \frac{x_i^4}{4}\right) + \sum_{i,k=1}^{N} (x_i - x_k)^4$$

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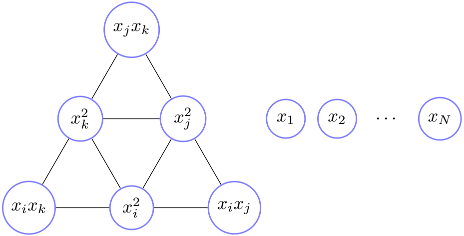

SOS + sparse + RIP $\not\Rightarrow$ sparse SOS (1/2)

$f_1 = x_1^4 + (x_1 x_2 - 1)^2 \quad f_2 = x_2^2 x_3^2 + (x_3^2 - 1)^2 \quad f = f_1 + f_2$

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Compute the dense relaxation f^2
Compare with the correlative sparse relaxation f^2_{cs}
Compare with the term sparse relaxation $f^{2,s}_{\text{ts}}$ for $s = 1, 2, \ldots$

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Install and run TSSOS:

```
] add https://github.com/wangjie212/TSSOS
using TSSOS, DynamicPolynomials
```

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] add https://github.com/wangjie212/TSSOS
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@polyvar x1 x2 x3; x=[x1;x2;x3];
f1 = x1^4+(x1*x2-1)^2; f2 = x2^2*x3^2+(x3^2-1)^2;
f = f1+f2
```

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Install and run TSSOS:

```
] add https://github.com/wangjie212/TSSOS
using TSSOS, DynamicPolynomials
@polyvar x1 x2 x3; x=[x1;x2;x3];
f1 = x1^4+(x1*x2-1)^2; f2 = x2^2*x3^2+(x3^2-1)^2;
f = f1+f2
dense2,sol,data=cs_tssos_first([f], x, 2,
CS=false,TS=false);
```

SOS + sparse + RIP $\not\Rightarrow$ sparse SOS (2/2)

Download from `https://homepages.laas.fr/vmagron/ncball`:

$f = f_1 + f_2 \qquad \mathbb{B}_{\mathsf{nc}} = \{x : 1 - x_1^2 - x_2^2 - x_3^2 \succcurlyeq 0, 1 - x_2^2 - x_3^2 - x_4^2 \succcurlyeq 0\}$

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Compute $\lambda_{\min}(f)$ on $\mathbb{B}_{\mathsf{nc}}$ with 2nd dense relaxation

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Compute $\lambda_{\min}(f)$ on $\mathbb{B}_{\text{nc}}$ with 2nd dense relaxation

```
cs_nctssos_first([f;ncball],x,2,CS=false, TS=false,
obj="eigen");
```

SOS + sparse + RIP $\not\Rightarrow$ sparse SOS (2/2)

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$f = f_1 + f_2 \qquad \mathbb{B}_{\text{nc}} = \{x : 1 - x_1^2 - x_2^2 - x_3^2 \succcurlyeq 0, 1 - x_2^2 - x_3^2 - x_4^2 \succcurlyeq 0\}$

Compute $\lambda_{\min}(f)$ on $\mathbb{B}_{\text{nc}}$ with 2nd dense relaxation

```
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obj="eigen");
```

Compare with the correlative and term sparse relaxations