# Trace polynomial optimization with applications in quantum information


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Joint work with Felix Huber, Igor Klep and Jurij Volčič


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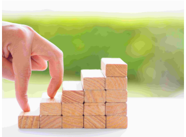

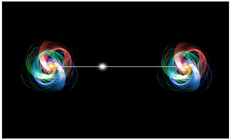

What is noncommutative optimization?

Eigenvalue optimization

Trace optimization

SDP hierarchies

Polynomial Bell inequalities

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$S \subset \operatorname{Sym} \mathbb{T}$ $\quad x_j$ operators from finite von Neumann algebra

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$\mathcal{A} = L^\infty(\mathcal{X}, \mu)$ is a vNa in $L^2(\mathcal{X}, \mu)$

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- More complicated ones!

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- Univ case [Klep-Pascoe-Volcic 21]: $f \succcurlyeq 0 \Rightarrow f =$ SOHS/SOHS
- Multilinear case [Huber 21]

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classical correlations = convex combinations of deterministic correlations

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$$-P_a(1|0) - P_b(1|0) + P(1,1|0,0) + P(1,1|0,1) + P(1,1|1,0) - P(1,1|1,1) \leqslant 0$$

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$X_s^a, Y_t^b$ are bounded operators on separable Hilbert spaces s.t.:

$$X_s^a Y_t^b = Y_t^b X_s^a, \quad X_s^a X_s^a = X_s^a, \qquad Y_t^b Y_t^b = Y_t^b$$

$$X_s^a X_s^{a'} = Y_t^b Y_t^{b'} = 0, \quad \sum_a X_s^a = \sum_b Y_t^b = I$$

“Mathematically”: inequality on eigenvalues of noncommutative polynomials

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[Pozas et al 19] extension $\rightarrow$ identify correlations not attainable in entanglement-swapping scenario (quantum teleportation)

Quantum physics operators $x_i, y_j$ satisfy causal constraints:

$$\operatorname{tr}(x_1 x_2 y_1 y_2) - \operatorname{tr}(x_1 x_2) \operatorname{tr}(y_1 y_2) = 0.$$

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$$\lambda_{\min} = \inf_{\mathbf{v},\mathcal{H}} \{ \langle f(x)\mathbf{v}, \mathbf{v} \rangle : x \in \mathcal{D}_S, \|\mathbf{v}\| = 1 \,, \mathbf{v} \in \mathcal{H} \}$$

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$\mathcal{M}(S)$ **Archimedean quadratic module**: $N - \sum_i x_i^2 \succcurlyeq 0$

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$f - \lambda\mathbf{1} = \sum_i h_i^\star h_i + \sum_s \sum_i t_{si}^\star s t_{si}$ with $h_i$, $t_{si}$ of **bounded** degrees

# Sparse eigenvalue optimization

Self-adjoint noncommutative (NC) variables $x = (x_1, \ldots, x_n)$

**Theorem [Helton & McCullough '02]**

$f \succcurlyeq 0 \Leftrightarrow f$ SOS (all positive polynomials are sums of squares)

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**Theorem: Sparse NC Positivstellensatz [Klep Magron Povh '21]**

$f = \sum_k f_k$, $f_k$ depends on $x(I_k)$

$f > 0$ on $\mathcal{D}_S$

Each $g_j$ depends on some $I_k$

RIP holds for $(I_k)$

Ball constraints for each $x(I_k)$

$\Longrightarrow$

$$\boxed{f = \sum_{k,i} (s_{ki}^\star s_{ki} + \sum_{j \in J_k} t_{ji}{}^\star g_j t_{ji})}$$

$s_{ki}$ "sees" vars in $I_k$

$t_{ji}$ "sees" vars from $g_j$

# Sparse eigenvalue optimization

$\mathbf{I}_{3322}$ Bell inequality (entanglement in quantum information)

$f = x_1(x_4 + x_5 + x_6) + x_2(x_4 + x_5 - x_6) + x_3(x_4 - x_5) - x_1 - 2x_4 - x_5$

Maximal violation levels $\rightarrow$ **upper bounds** on $\lambda_{\max}$ of $f$ on $\{x : x_i^2 = x_i, x_i x_j = x_j x_i$ if $i \in \{1, 2, 3\}, j \in \{4, 5, 6\}\}$

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| 6 | 0.25087539771 80 | (1 hour) |

PERFORMANCE  VS  ACCURACY

What is noncommutative optimization?

Eigenvalue optimization

Trace optimization

SDP hierarchies

Polynomial Bell inequalities

# Trace optimization

$$\mathrm{tr}_{\min} = \inf\{\mathrm{tr}(f(x)) : x \in D_S\}$$

$$\begin{aligned} = \sup \quad & m \\ \text{s.t.} \quad & \mathrm{tr}(f(x) - m) \geqslant 0\,, \quad \forall x \in \mathcal{D}_S \end{aligned}$$

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Disproving Connes' embedding conjecture: $\mathrm{tr}_{\min}^{\mathrm{II}_1} < \mathrm{tr}_{\min}$

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**How to extend it to sums of trace products** T?

# Kadison-Dubois representation theorem

$$\chi_{\mathcal{M}} := \{\varphi : \mathsf{T} \to \mathbb{R} \mid \varphi \text{ homomorphism}, \varphi(\mathcal{M}) \subseteq \mathbb{R}_{\geqslant 0}, \varphi(1) = 1\}$$

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**Theorem: Kadison-Dubois [Marshall 08]**

Given an Archimedean quadratic module $\mathcal{M} \subseteq \mathsf{T}$ & $f \in \mathsf{T}$:

$$\forall \varphi \in \chi_{\mathcal{M}} \quad \varphi(f) \geqslant 0 \qquad \Leftrightarrow \qquad \forall \varepsilon > 0 \quad f + \varepsilon \in \mathcal{M}$$

# Representation of positive elements of T

For $S \subseteq \mathsf{T}$, "augment" $S$ with traces of hermitian squares:

$$S(N) = S \cup \{\operatorname{tr}(pp^\star) \mid p \in \mathbb{R}\langle \underline{x} \rangle\} \cup \{N^k - \operatorname{tr}(x_j^{2k}) \mid k \in \mathbb{N}\} \subseteq \mathsf{T}$$

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Elements of $\mathcal{M}(S(N))$ are

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**Lemma [Klep-M.-Volcic 20]**

$\mathcal{M}(S(N))$ is archimedean

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**Lemma [Klep-M.-Volcic 20]**

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**Proof**

By induction: $\forall$ word $w$, $m \pm \mathrm{tr}(w) \in \mathcal{M}(S(N))$ for some $m > 0$

$$w = x_j^{2k} \implies N^k + 1 + 2\,\mathrm{tr}(x_j^k) = (N^k - \mathrm{tr}(x_j^{2k})) + \mathrm{tr}\left((x_j^k + 1)^2\right)$$

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# Tracial words & moment matrices

$\mathbb{T}$-words = $\{\prod_i \operatorname{tr}(u_i) v : u_i, v$ words $\}$ and T-words

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Tracial degree = up to cyclic equivalence

$\mathbf{W}_r^{\mathbb{T}}$ = vector of $\mathbb{T}$-words of with tracial degree $\leqslant r$

$n = 1$: $\mathbf{W}_2^{\mathbb{T}}$ contains $1, x_1, x_1^2, \operatorname{tr}(x_1), \operatorname{tr}(x_1^2), \operatorname{tr}(x_1)x_1$

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Tracial moment matrix $\mathbf{M}_r^{\mathbb{T}}(L)$ for a linear functional $L$ : T $\to \mathbb{R}$:

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Trace localizing matrix $\mathbf{M}_r^{\mathbb{T}}(sL)$ for $s \in \mathbb{T}$:

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# SDP hierarchy for T

Reminder:

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There is no duality gap and $f_r \to f_{\min}^{\mathrm{II}_1}$ as $r \to \infty$

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$S \subset \operatorname{Sym} \mathbb{T}$

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What is noncommutative optimization?

Eigenvalue optimization

Trace optimization

SDP hierarchies

Polynomial Bell inequalities

# Polynomial Bell inequalities

## CLASSICAL WORLD

$$\psi^*(A_1 \otimes B_1 + A_1 \otimes B_2 + A_2 \otimes B_1 - A_2 \otimes B_2)\psi \leqslant 2$$

for separable states $\psi \in \mathbb{R}^k \otimes \mathbb{R}^k$ and matrices $A_j, B_j$ satisfying $A_j^* = A_j$, $A_j^2 = I$, $B_j^* = B_j$, $B_j^2 = I$

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💡 2nd sparse SDP gives also 5 . . . 10 times faster

# Conclusion and perspectives

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# Thank you for your attention!

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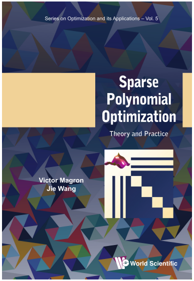

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