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1 MSO sets and MTES for dummies

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8 — Abstract —

9 Structural analysis-based diagnosis allows for the extraction of a wealth of information and properties
10 by studying a structural model that represents a physical system. This diagnosis approach is centered
11 on structurally overdetermined sets, which enable the generation of residuals for fault detection
12 and isolation. As the 'for Dummies' editorial collection, this article aims at taking on complex
13 concepts and making them easy to understand. It aims to clarify and compare key concepts in
14 structural analysis, focusing on Minimally Structurally Overdetermined (MSO) sets and Minimal
15 Test Equation Supports (MTES). Additionally, we explain and illustrate the Dulmage-Mendelsohn
16 decomposition, which helps identify structurally overdetermined parts of the system and plays
17 a important role in the structural analysis process. Through detailed exploration and practical
18 examples, we demonstrate the roles, applications, and interrelations of these sets, highlighting their
19 respective strengths and limitations. The paper provides an overview of the algorithms used to
20 identify and use these sets, including a theoretical and practical comparison of their computational
21 efficiency and diagnostic capabilities.

22 **2012 ACM Subject Classification** Computing methodologies → Knowledge representation and
23 reasoning

24 **Keywords and phrases** Structural analysis, MTES sets, MSO sets

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29 **1** Introduction

30 In the field of model-based fault diagnosis, the ability to detect and isolate faults within
31 complex systems is crucial for ensuring reliability and safety. By studying the model, it is
32 possible to identify faults, which are indicative of abnormal or faulty behavior within the
33 system.

34 Structural analysis consists in abstracting a model by keeping only the links between
35 equations and variables. The main advantages of structural analysis are that the approach
36 can be applied to large scale systems, linear or non linear, even under uncertainty. In
37 the field, the concept of Minimally Structurally Overdetermined (MSO) sets [6] has been
38 widely adopted due to its effectiveness in isolating faults by leveraging the redundancy in
39 system models. However, while MSO sets have proven valuable, they come with limitations,
40 particularly in terms of computational complexity when applied to large-scale systems.

41 To address these issues, more refined approaches have been introduced, such as Minimal
42 Test Equation Supports (MTES), presented in [5], or Fault Driven Minimal Structurally
43 Overdetermined (FMSO) sets introduced by [7]. These methods aim to enhance diagnosis
44 precision while aiming at reducing computational demands, yet their practical application
45 and comparative advantages remain areas of ongoing exploration.



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46 As the 'for Dummies' editorial collection, this article aims at taking on complex concepts
 47 and making them easy to understand. The objective is to clarify and compare various key
 48 concepts used in structural analysis model-based fault diagnosis, specifically focusing on
 49 MSO sets, MTES, or FMSO sets. By providing a detailed exploration of these concepts, this
 50 paper seeks to clarify their roles, applications, and interrelations. Through carefully chosen
 51 examples, we illustrate how each of these concepts can be applied to different scenarios,
 52 highlighting their respective strengths and limitations. This comparative analysis is intended
 53 to enhance the understanding of these diagnosis tools.

54 This paper is structured as follows. Section 1 gives some theoretical foundations on
 55 structural analysis. Then in Section 3 MSO sets are defined and explained. The algorithm
 56 for their identification is also presented. Next, Section 4 discuss the motivations behind
 57 the introduction of MTES and describe the algorithm for their computation. A comparison
 58 between the performance of MSO and MTES is then conducted in Section 5. To illustrate
 59 the practical application of these concepts, a realistic example is provided in Section 6,
 60 including structural modeling, algorithmic complexity analysis, and an evaluation of diagnostic
 61 capability. Section 7 concludes the paper.

62 **2** Background

63 One of the key concepts to understand the notion of MSO sets or even MTES is the Dulmage
 64 Mendelsohn decomposition of a bipartite graph [4].

65 Let the system description consist of a set of n_e equations involving a set of variables
 66 partitioned into a set Z of n_Z known (or measured) variables and a set X of n_X unknown
 67 (or unmeasured) variables. We refer to the vector of known variables as z and the vector
 68 of unknown variables as x . The system may be impacted by the presence of n_f faults that
 69 appear as parameters in the equations. The set of faults is denoted by F and we refer to the
 70 vector of faults as f .

71 **► Definition 1.** (*System*) A system, denoted $\Sigma(z, x, f)$ or Σ for short, is any set of equations
 72 relating z , x and f . The equations $e_i(z, x) \subseteq \Sigma(z, x, f)$, $i = 1, \dots, n_e$, are assumed to be
 73 differential or algebraic in z and x .

74 Consider the illustrative example given by system (1), from [8], named Σ_1 that will be
 75 taken as a running example in this section. It is composed of six equations e_1 to e_6 relating
 76 the known variables $Z = \{z_1, z_2\}$, the unknown variables $X = \{x_1, x_2, x_3, x_4, x_5\}$ and the set
 77 of system faults $F = \{f_1, f_2\}$. Besides a, b, c are constant parameters.

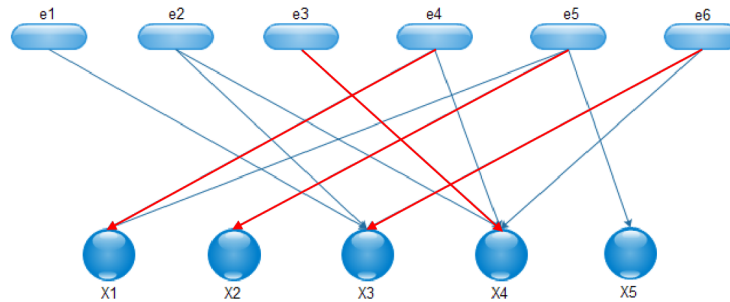
$$\begin{aligned}
 e_1 : \quad & \dot{x}_3 = e^{x_3} - a \\
 e_2 : \quad & x_3^2 = -b\dot{x}_4 + f_1 \\
 e_3 : \quad & z_1 = x_4 \\
 e_4 : \quad & z_2 = x_1 + b^2 + x_4 \\
 e_5 : \quad & \dot{x}_1 = e^{x_2} + x_5 \\
 e_6 : \quad & \dot{x}_3 = x_4 + c + f_2
 \end{aligned} \tag{1}$$

79 We can represent a system $\Sigma(z, x, f)$ by a biadjacency matrix, in which each row stands
 80 for an equation and each column an unknown variable. An X in position (i, j) means that
 81 equation i contains the unknown variable j . For example, Σ_1 can be represented by the
 82 biadjacency matrix of Table 1.

	x_1	x_2	x_3	x_4	x_5
e_1			×		
e_2			×	×	
e_3				×	
e_4	×			×	
e_5	×	×			×
e_6			×	×	

■ **Table 1** Biadjacency matrix of Σ_1

84 Equivalently, a system $\Sigma(z, x, f)$ can be represented by a bipartite graph $G(E, X)$. E
 85 and X are the sets of nodes, an edge between node E_i and node X_j means that equation
 86 i contains the unknown variable j . The bipartite graph associated with the illustrative
 87 example Σ_1 is shown in Figure 1 [8].
 88



■ **Figure 1** The bipartite graph of Σ_1 [8].

89 ► **Definition 2.** (*Maximal Matching*). A maximal matching is a set of edges such that these
 90 edges do not share any common nodes, and no more edges can be added to this set without
 91 losing this property.

92 In Table 1, as well as in Figure 1, a maximal matching is marked in red. It is the largest set
 93 of edges such that there are no common nodes (each equation and each variable appears at
 94 most once).

95 ► **Definition 3.** (*Free equation*). For a given maximal matching, an equation is free if it is
 96 not part of this maximal matching.

97 In the matching represented in Table 1, e_1 and e_2 are free.

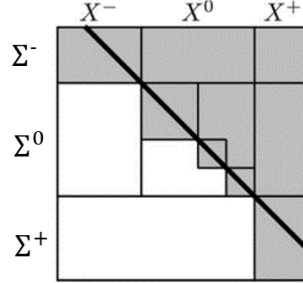
98 ► **Definition 4.** (*Free variable*). For a given maximal matching, a variable is free if it is not
 99 part of this maximal matching.

100 In the matching represented in Table 1, x_5 is free.

101 ► **Definition 5.** (*Alternating path*). Let M_{max} be a maximal matching. P , a path in the
 102 graph G , is an alternating path if it successively passes through edges that belong to M_{max}
 103 and edges that do not belong to M_{max} .

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104 The Dulmage-Mendelsohn decomposition allows separating a model Σ into 3 parts:
 105 an overdetermined part, denoted Σ^+ , an exactly determined part, denoted Σ^0 , and an
 106 underdetermined part, denoted Σ^- , as shown in Figure 2 .



■ **Figure 2** Dulmage-Mendelsohn decomposition of a model Σ

107 The overdetermined part Σ^+ of Σ is the set of equations $e \in E$ such that, for any maximal
 108 matching, there exists an alternating path between at least one free equation and e [6]. Σ^+
 109 can also be defined as the set of equations $e \in E$ such that there is at least one maximal
 110 matching in which e is free.

111 The part Σ^0 contains equations that are not in Σ^+ but are such that the set $\Sigma^0 \cup \Sigma^+$ is
 112 the maximal set of equations for which there exists a complete matching of the unknown
 113 variables in $\Sigma^0 \cup \Sigma^+$ into $\Sigma^0 \cup \Sigma^+$ [6]. We can also say that Σ^0 consists of the equations e
 114 that are matched in every maximal matching and do not contain any free variables.

115 The rest of the equations belongs to Σ^- . Σ^- only contains equations if the maximal
 116 matchings do not cover all the variables (if there exists at least one free variable in a maximal
 117 matching). Σ^- contains equations associated with the free variables.

118 *Running example:* Σ_1^+ contains all the equations that are free in at least one maximal
 119 matching. In the matching represented in Table 1, e_1 and e_2 are free so these two equations
 120 are in Σ_1^+ . But $(e_1x_3, e_2x_4, e_4x_1, e_5x_2)$ is also a maximal matching, and in this matching, e_3
 121 and e_6 are free. So e_3 and e_6 are also in Σ_1^+ . So, we have $\Sigma_1^+ = \{e_1, e_2, e_3, e_6\}$.

122 $\Sigma_1^0 = \{e_4\}$ because the set $\{e_1, e_2, e_3, e_4, e_6\}$ is the biggest set such that there exists a
 123 complete matching of the unknown variables in this set into this set. If we add e_5 in that
 124 set, it would not be a complete matching because either x_2 or x_5 would not be part of the
 125 matching. Another way to see this is that e_4 is necessary to every maximal matching, as it
 126 is the only equation that contains x_1 (e_5 also contains x_1 , but e_5 is used to cover x_2 or x_5).
 127 Thus, it is impossible to obtain a maximal matching without e_4 , and e_4 does not contain any
 128 free variable, so e_4 is part of Σ_1^0 .

129 The remaining equation, e_5 , is in Σ_1^- . That is because e_5 is the only equation that
 130 contains both x_2 and x_5 , so for every maximal matching, one of these two variables is free.

131 Note that, as shown above with Σ_1 , if a set of equations Σ has more equations than
 132 unknown variables, it does not necessarily mean that Σ^0 and Σ^- are empty.

133 ► **Definition 6.** (PSO set). A set Σ of equations is a PSO set (proper structurally overde-
 134 terminated set) if $\Sigma = \Sigma^+$. In other words, Σ is a PSO set if $\Sigma^0 = \Sigma^- = \emptyset$.

135 System represented Σ_1 as a whole is not a PSO set because Σ^0 and Σ^- are not empty. But
 136 the subset of equations $\{e_1, e_2, e_3\}$ is a PSO set: the three equations are free in at least one
 137 maximal matching, so $\Sigma = \Sigma^+$ (see Table 2).

	x_3	x_4
e_1	×	
e_2	×	×
e_3		×

■ **Table 2** The subset of equations $\{e_1, e_2, e_3\}$

138 ► **Definition 7.** (*Structural redundancy*). For a given set of equations Σ , the structural
 139 redundancy $\varphi(\Sigma)$ is defined by $\varphi(\Sigma) = |\Sigma^+| - |X^+|$, where $|\Sigma^+|$ is the number of equations
 140 in Σ^+ , and $|X^+|$ is the number of unknown variables involved in Σ^+ .

141 PSO sets have structural redundancy, which is key to diagnosis since it allows residual
 142 generation. We recall hereafter some important definitions [8].

143 ► **Definition 8** (ARR for $\Sigma(z, x, \mathbf{f})$). Let $\Sigma(z, x, \mathbf{f})$ be a system. Then, a relation of the form
 144 $\text{arr}(z, \dot{z}, \ddot{z}, \dots) = r$ is an Analytical Redundancy Relation (ARR) for $\Sigma(z, x, \mathbf{f})$ if for each z
 145 consistent with $\Sigma(z, x, \mathbf{f})$ the relation evaluation is $r = 0$, i.e., the ARR is satisfied.

146 ► **Definition 9** (Residual Generator for $\Sigma(z, x, \mathbf{f})$). A system taking a subset of the variables
 147 z as input, and generating a scalar signal r as output, is a residual generator for the model
 148 $\Sigma(z, x, \mathbf{f})$ if, for all z consistent with $\Sigma(z, x, \mathbf{f})$, it holds that $\lim_{t \rightarrow \infty} r(t) = 0$.

149 ARRs are residual generators and are used to verify the consistency of observations with
 150 the system model. The ARRs hold true if the observed behavior of the system meets the
 151 model constraints. They can be derived from the system model by eliminating unknown
 152 variables [7]. PSO sets can be used to generate ARRs as they identify the redundant parts
 153 of the system's model.

154 3 MSO sets

155 It is often more practical to focus on smaller, manageable subsets of the PSO sets that still
 156 retain the necessary redundancy for effective fault diagnosis.

157 ► **Definition 10.** (*MSO set*). A PSO set Σ is an MSO set (minimal structurally overde-
 158 termined) if no proper subset is a PSO set. MSO sets are PSO sets such that $\varphi(\Sigma) = 1$.

159 MSO sets are interesting because they are just overdetermined and exactly identify a set of
 160 equations from which a residual generator can be obtained. [6].

161 In our running example, the subset $\{e_1, e_2, e_3\}$ is an MSO set because it is a PSO set,
 162 and there is no subset which is also a PSO set. Furthermore, $\varphi(\Sigma) = 3 - 2 = 1$.

163 A deep first search algorithm¹ to find MSO sets is presented in [6]. The general idea is
 164 to start with a PSO set Σ , to remove an equation e , and to compute the overdetermined
 165 part of the remaining set $(\Sigma \setminus \{e\})^+$. When an equation e is removed from a PSO set, the
 166 structural redundancy decreases by one. So the previous process is done until the structural
 167 redundancy becomes one. If $\varphi((\Sigma \setminus \{e\})^+) = 1$, then an MSO set is found.

168 *Running example:* To illustrate that, let us apply the first steps of the algorithm to the
 169 following example Σ_2 (cf. Table 3).

¹ All the algorithms presented in this article are provided in the "Fault Diagnosis Toolbox" available here:
<https://faultdiagnostoolbox.github.io/>

	x_1	x_2	x_3
e_1	×		
e_2		×	×
e_3	×	×	
e_4			×
e_5	×		×

■ **Table 3** Illustrative example Σ_2

170 The set $\Sigma_2 = \{e_1, e_2, e_3, e_4, e_5\}$ is a PSO set of structural redundancy 2. First, e_1 is
 171 removed. By applying the Dulmage-Mendelsohn decomposition on the remaining set, we have
 172 $(\Sigma_2 \setminus \{e_1\})^+ = \{e_2, e_3, e_4, e_5\}$. $\varphi((\Sigma_2 \setminus \{e_1\})^+) = 1$, which means that $\{e_2, e_3, e_4, e_5\}$ is an
 173 MSO set. Then, e_2 is removed from Σ_2 . By applying the Dulmage-Mendelsohn decomposition
 174 on the remaining set, we have $(\Sigma_2 \setminus \{e_2\})^+ = \{e_1, e_4, e_5\}$. $\varphi((\Sigma_2 \setminus \{e_2\})^+) = 1$, which means
 175 that $\{e_1, e_4, e_5\}$ is also an MSO set. And so on, until covering the entire tree.

176 Using this algorithm, all the MSO sets are found. However, this method is not optimal
 177 in terms of computation time because some MSO sets can be found multiple times. For
 178 example, here, when removing e_2 , $(\Sigma_2 \setminus \{e_2\})^+ = (e_1, e_4, e_5)$ and when removing e_3 ,
 179 $(\Sigma_2 \setminus \{e_3\})^+ = (e_1, e_4, e_5)$. Therefore, the same MSO set is found twice. That is why the
 180 concept of equivalence class is introduced.

181 ► **Definition 11.** (*Equivalence class*). Let Σ be a set of equations and (e_i, e_j) , two equations
 182 in Σ . e_i and e_j are in the same equivalence class $[e] = \{e_i, e_j\}$ if $e_i \notin (\Sigma \setminus \{e_j\})^+$.

183 This enables grouping equations into the same class such that, if they are removed from Σ ,
 184 the same overdetermined part is obtained.

185 For the system Σ_2 , the equivalence classes are $\{e_1\}$, $\{e_2, e_3\}$, $\{e_4\}$, $\{e_5\}$ because $e_2 \notin$
 186 $(\Sigma_2 \setminus \{e_3\})^+$ (the inverse is also true because it is an equivalence relation).

187 Thus, in the algorithm, it is no longer one single equation that is removed at each step,
 188 but equivalence classes, which prevent the same overdetermined part from being computed
 189 twice. At each step, equations that belong to the same equivalence class are lumped, one
 190 equivalence class is removed, and the overdetermined part of the remaining equations is
 191 computed. Note that for Σ_2 , this optimization does not make a huge difference in terms of
 192 computation time, but it makes a big difference for huge models, with many equations and
 193 MSO sets.

194 There is another issue handled by the algorithm. The same MSO set can be obtained
 195 if the order of equations removal is permuted. Let us consider the following system Σ_3 (cf.
 196 Table 4).

	x_1	x_2	x_3
e_1		×	
e_2	×	×	×
e_3	×		
e_4		×	×
e_5	×		
e_6			×

■ **Table 4** Illustrative example Σ_3

197 At the beginning, e_1 is removed. The overdetermined part of the remaining equations
 198 $(\Sigma_3 \setminus \{e_1\})^+$ is $\{e_2, e_3, e_4, e_5, e_6\}$. Then, e_2 is removed, and the overdetermined part of the
 199 remaining equations $(\Sigma_3 \setminus \{e_1, e_2\})^+$ is $\{e_3, e_5\}$, which is an MSO set. But if e_2 is removed
 200 first, and then e_1 , the same MSO set is obtained. To handle this problem, the algorithm uses
 201 a new input set R which contains the equivalence classes that are allowed to be removed.

202 At each node traversed by the algorithm, a Dulmage-Mendelsohn decomposition is
 203 performed. This is the most time-consuming part of the computation. This algorithm
 204 traverses a search tree, where each node represents a potentially overdetermined set of
 205 equations. Each level of the tree corresponds to the removal of an equation from the system,
 206 and each branch represents a subset of equations being tested. When an equation is removed
 207 from Σ (i.e., when moving down a level in the search tree), the structural redundancy $\varphi(\Sigma)$
 208 decreases by 1 [6].

209 Consider the worst-case scenario, where each equation belongs to a different equivalence
 210 class (i.e., the equations are removed from Σ one by one). For a system with n_e equations
 211 and a redundancy degree of φ , the maximum number of nodes the algorithm might traverse
 212 is given by the sum of possible equation subset for each level k of the tree. A bound on the
 213 number of nodes traversed by the algorithm to find MSO sets is then given by Equation (2).

$$214 \quad \sum_{k=0}^{\varphi-1} \binom{n_e}{k} \quad (2)$$

215 MSO sets are used to construct ARRs, and thus residual generators, but it is important
 216 to note that MSO sets do not take faults into account. If we are interested in system
 217 performance and we want to construct residual generators for diagnosis purposes, only MSO
 218 sets that contain equations possibly disrupted by faults are important.

219 ► **Definition 12** (Detectable fault). *A fault f is structurally detectable in a set of equations*
 220 *Σ if f is in the overdetermined part Σ^+ .*

221 ► **Definition 13** (Fault support). *The fault support $F_{\Sigma'}$ of a set of equations $\Sigma' \subseteq \Sigma$ is defined*
 222 *as the set of faults that are involved in the equations of Σ' [8].*

223 A special class of MSO sets can be defined: FMSO sets (Fault-Driven Minimal Structurally
 224 Overdetermined) that are MSO sets with a non empty fault support [8].

225 ► **Definition 14** (FMSO set). *A subset of equations $\Sigma' \subseteq \Sigma(z, x, f)$ is an FMSO set of*
 226 *$\Sigma(z, x, f)$ if (1) $F_{\Sigma'} \neq \emptyset$ and $\varphi(\Sigma') = 1$ that means $|\Sigma'| = |X_{\Sigma'}| + 1$, (2) no proper subset of*
 227 *Σ' is overdetermined.*

228 Therefore, if there exists a FMSO set in Σ which contains the fault f , then f is detectable
 229 in Σ . Those special MSO sets are candidates as residual generators as they contain just the
 230 necessary redundancy and their fault support is not empty.

231 **4 MTES**

232 **4.1 Motivation**

233 MSO sets can be used to obtain residual generators, but are they all useful? Let us take the
 234 previous system Σ_3 and add faults of the set $F = \{f_1, f_2, f_3\}$ (cf. Table 5).

	x_1	x_2	x_3	
e_1		×		f_1
e_2	×	×	×	f_2
e_3	×			
e_4		×	×	
e_5	×			
e_6			×	f_3

■ **Table 5** System Σ_3 with faults

235 For this example, there are 8 MSO sets given in Table 6.

	MSO sets	Faults
MSO_1	$\{e_3, e_5\}$	$\{\emptyset\}$
MSO_2	$\{e_2, e_4, e_5, e_6\}$	$\{f_2, f_3\}$
MSO_3	$\{e_2, e_3, e_4, e_6\}$	$\{f_2, f_3\}$
MSO_4	$\{e_1, e_4, e_6\}$	$\{f_1, f_3\}$
MSO_5	$\{e_1, e_2, e_5, e_6\}$	$\{f_1, f_2, f_3\}$
MSO_6	$\{e_1, e_2, e_4, e_5\}$	$\{f_1, f_2\}$
MSO_7	$\{e_1, e_2, e_3, e_6\}$	$\{f_1, f_2, f_3\}$
MSO_8	$\{e_1, e_2, e_3, e_4\}$	$\{f_1, f_2\}$

■ **Table 6** MSO sets and their fault support for Σ_3 with faults

236 MSO_2 and MSO_3 as well as MSO_6 and MSO_8 have the same fault support $\{f_2, f_3\}$
 237 and $\{f_1, f_2\}$, respectively. This means that the generated residuals would be sensitive to
 238 the same faults. Note also that if the equations of MSO_2 and MSO_3 are put together, the
 239 fault support of the resulting subset of equations remains $\{f_2, f_3\}$. In addition, some MSO
 240 sets have a non minimal fault support like MSO_5 whose fault support is $\{f_1, f_2, f_3\}$ that
 241 includes the fault support of MSO_3 and MSO_4 . This indicates that it is not necessary to
 242 compute all the MSO sets to obtain a parsimonious set of residual generators guaranteeing
 243 maximal diagnosability.

244 The above remark tells us that we need to change the perspective and focus on the
 245 possible fault supports of MSO sets rather than the MSO sets themselves. Hence the
 246 following definitions from [5] focusing on fault supports.

247 ► **Definition 15** (Test Support (TS)). *Given a model Σ and a set of faults \mathcal{F} , a subset of*
 248 *faults $\zeta \subseteq \mathcal{F}$ is a TS if there exists a PSO set $\Sigma' \subseteq \Sigma$ such that $F_{\Sigma'} = \zeta$.*

249 ► **Definition 16** (Minimal Test Support (MTS)). *Given a model Σ , a test support is an MTS*
 250 *if no proper subset is a test support.*

251 With the two above definition, it is now necessary to identify the subset of equations that
 252 correspond to TSs and MTSSs.

253 ► **Definition 17** (Test Equation Support (TES)). *A PSO set Σ is a TES if (1) $F_{\Sigma} \neq \emptyset$ and*
 254 *(2) for any PSO set Σ' such that $\Sigma' \supseteq \Sigma$, it holds that $F_{\Sigma'} \supseteq F_{\Sigma}$.*

255 ► **Definition 18** (Minimal Test Equation Support (MTES)). *A TES Σ is an MTES if there is*
 256 *no subset of Σ that is also a TES.*

257 The idea of MTES is to identify the PSOs with maximal number of equations that have a
 258 given test support [5]. In other words, for a given test support ζ , it is important to identify
 259 the maximal set of equations $\Sigma' = \Sigma'^+$ such that $F_{\Sigma'} = \zeta$, thus avoiding that two MTES
 260 have the same TS.

261 It is also important to pay attention to the minimality property, i.e., to identify MTES
 262 instead of simply TES. Indeed minimality avoids obtaining redundant residual generators.
 263 This can be proved by showing that, considering the set of MTES for a given system, an ARR
 264 generated from a given MTES is never redundant with respect to the set of ARRs generated
 265 from the other MTESs. This statement holds true based on the following proposition, which
 266 is proved in [3], specifically due to the second condition.

267 ► **Proposition 19.** *A given ARR_i is redundant with respect to a set of ARR_s , $i \in I$, $j \notin I$,
 268 if and only if there exists $I' \subseteq I$ such that:*

- 269 1. $\forall z(\text{observations})$, if all ARR_{i_s} , $i \in I'$, are satisfied by z , then ARR_j is satisfied by z .
- 270 2. The fault support of ARR_j contains the fault support of each ARR_i , $i \in I'$:

$$271 \quad \bigcup_{i \in I'} F_{ARR_i} \subseteq F_{ARR_j}.$$

272 In summary, the fault supports of the MTESs are sufficient to guarantee maximal
 273 diagnosability because they correspond to all the possible MTS, i.e. to all the fault supports
 274 of possible MSOs. Every MTES has a different MTS and no MTES is redundant.

275 Interestingly, the set of MTES is much smaller than the set of MSO sets, which results in
 276 computation time saving.

277 4.2 MTES based diagnosis process

278 The following steps, applied on system Σ_3 illustrate the process of diagnosis based on MTES.

279

280 Step 1: Specification

281 The first step involves defining the isolability requirements for the faults in the system.
 282 The pair of faults that need to be isolated are specified, for example:

- 283 ■ I_1 : isolate f_1 from f_2 .
- 284 ■ I_2 : isolate f_1 from f_3 .
- 285 ■ I_3 : isolate f_2 from f_3 .

286 Step 2: Compute the set of MTES

287 MTESs are computed for the system Σ_3 :

- 288 ■ $MTES_1 = \{e_2, e_3, e_4, e_5, e_6\}$ with corresponding MTS $\{f_2, f_3\}$ and $\varphi(MTES_1) = 2$.
- 289 ■ $MTES_2 = \{e_1, e_3, e_4, e_5, e_6\}$ with corresponding MTS $\{f_1, f_3\}$ and $\varphi(MTES_1) = 2$.
- 290 ■ $MTES_3 = \{e_1, e_2, e_3, e_4, e_5\}$ with corresponding MTS $\{f_1, f_2\}$ and $\varphi(MTES_1) = 2$.

291 Step 3: Find the subset of MTES that satisfies all the isolability specifications

292 In this step, we check if there exists a Minimal Hitting Set that satisfies the isolability
 293 conditions defined in Step 1. We examine whether each specification can be satisfied by
 294 finding a set of equations where the faults can be isolated according to the conditions.

- 295 ■ For I_1 (isolate f_1 from f_2), an MTES whose corresponding MTS includes f_1 and not f_2
 296 is required. It could be $MTES_2$.

- 297 ■ For I_2 (isolate f_1 from f_3): an MTES whose corresponding MTS includes f_1 and not f_3
 298 is required. It could be $MTES_3$.
- 299 ■ For I_3 (isolate f_2 from f_3): an MTES whose corresponding MTS includes f_2 and not f_3
 300 is required. $MTES_3$ also works.

301 **Step 4: Find an FMISO within each selected MTES**

- 302 ■ $MTES_2$ includes only $FMISO_4$ that can be used as residual generator.
- 303 ■ $MTES_3$ includes $FMISO_6$ and $FMISO_8$, so any of those could be used as residual
 304 generator.

305 **4.3 Algorithm**

306 An algorithm for computing the set of MTES is presented in [5]. The concept behind this
 307 algorithm is similar to the one used to find MSO sets. In each step, an equation is removed
 308 from the model, and the overdetermined part of the remaining set is recalculated. The main
 309 difference is that instead of removing all equations successively (or equivalence classes in
 310 the improved version), only equations associated with faults are removed. By doing so, the
 311 fault support of the resulting set decreases at each iteration, aligning with our goal of finding
 312 minimal fault supports.

313 The algorithm begins with a TES (the largest PSO set, ensuring no larger PSO set has
 314 the same fault support), and remains within the class of TES by progressively removing
 315 the fault-carrying equations. Thus, as these equations are removed, it is ensured that the
 316 resulting set continues to satisfy the properties of a TES.

317 The stopping condition for this algorithm is as follows: a TES Σ is considered an MTES
 318 if all fault-carrying equations in Σ belong to the same equivalence class. Indeed, when
 319 this condition is met, if the process goes on by removing a fault-carrying equation and
 320 recalculating the overdetermined part, the resulting equation set will no longer contain any
 321 fault-carrying equations. This stopping condition is tested at each iteration, and when it is
 322 satisfied, the TES Σ is confirmed as an MTES.

323 Another stopping condition is given in [5]. A TES $\Sigma' \subseteq \Sigma$ is an MTES if and only if
 324 $\varphi(\Sigma') = \varphi((\Sigma \setminus \Sigma_f)^+) + 1$, with Σ_f the set of equations in Σ affected by faults. $(\Sigma \setminus \Sigma_f)^+$
 325 represents the equations that can be added without modifying the fault support, while
 326 remaining within the class of PSO sets. If we take any TES, it can be extended with the
 327 equations from $(\Sigma \setminus \Sigma_f)^+$. The consequence of this condition is that all MTES have the
 328 same structural redundancy (for Σ_3 , the 3 MTES have structural redundancy equal to 2).
 329 Thus, either all MTES are MSO sets, i.e., their structural redundancy is 1, or none of them
 330 is exactly an MSO set because their structural redundancy is more than 1, in which case
 331 MTES obviously include several MSO sets. If all the equation of a system are affected by
 332 faults, then MTES are exactly MSO sets. The advantage of using this condition is that it
 333 avoids performing a Dulmage-Mendelsohn decomposition at each step.

334 If n_f is the number of fault-carrying equations, the number of nodes traversed by the
 335 search algorithm is bounded by Equation (3).

$$336 \quad \sum_{k=0}^{\varphi-1} \binom{n_f}{k}. \quad (3)$$

337 *Running example:* We now apply the algorithm step by step on the illustrative example
 338 Σ_4 presented in Table 7.

	x_1	x_2	x_3	x_4	x_5	
e_1		×	×	×		f_1
e_2	×	×				
e_3		×				f_2
e_4			×			f_3
e_5				×	×	
e_6	×					
e_7		×				
e_8					×	
e_9			×			

■ **Table 7** Illustrative example Σ_3

339 First, we consider the entire set. It is a PSO set, so there is no need to compute the
340 overdetermined part. The equivalence classes are: $\{e_1, e_5, e_8\}$, $\{e_2, e_6\}$, $\{e_3\}$, $\{e_4\}$, $\{e_7\}$, $\{e_9\}$.
341 The stopping condition is not verified because all fault-carrying equations are not in the
342 same class. Thus, $\{e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_9\}$ is only a TES, but not a MTES. Then,
343 an equivalence class containing a fault (f_1) is removed. The set $\{e_1, e_5, e_8\}$ is eliminated
344 (equivalent to removing e_1 and then computing the overdetermined part). The remaining set
345 is $\{e_2, e_3, e_4, e_6, e_7, e_9\}$. The equivalence classes are as follow: $\{e_2, e_6\}$, $\{e_3\}$, $\{e_7\}$, $\{e_4, e_9\}$.
346 Since the stopping condition is still not met, we proceed by removing the equivalence class
347 containing f_2 . The resulting set is $\{e_2, e_4, e_6, e_7, e_9\}$. This time, the stopping condition is
348 satisfied (only one fault remaining), $MTES_1$ is identified. The same process is applied to
349 find all the MTES.

350 5 Summary

351 To sum up the previous sections, MSO sets are sets of equations in a model where the number
352 of equations exceeds the number of unknowns by one. This overdetermination is minimal,
353 meaning that if any equation was removed, the set would no longer be overdetermined.

354 MSO sets are particularly useful in fault detection as they provide the basis for producing
355 residual generators. They retain the necessary and sufficient redundancy. However, as seen
356 in the previous sections, relying on the generation of MSO sets to generate the residual
357 generators that guarantee maximal diagnosability has some drawbacks. The first point is
358 that some MSO sets may not be sensitive to faults, making those irrelevant for fault detection
359 and diagnosis. Second, even if they can be generated systematically, finding all MSO sets
360 can be computationally expensive for large systems. Last but not least, they do not provide
361 any information about fault isolation.

362 FMSO sets are MSO sets but they only retain MSO sets with non-empty fault support,
363 so they are all relevant for fault detection and isolation. They are less numerous than MSO
364 sets, which reduces significantly computation time. However, as shown in Section 4.1, several
365 FMSO sets may have the same fault support and some may be redundant.

366 By focusing on fault supports rather than specific sets of equations, TESs effectively
367 identify exactly those fault supports that correspond to all the possible MSO sets. MTESs
368 gather all the equations from which all the FMSO sets with the corresponding fault support
369 can be generated. As a result, MTESs inherently address the problem of test selection. In
370 addition, generating MTES is obviously less computationally demanding than generating

Aspect	MSO sets	FMSO sets	MTES
Diagnosis target	residual generation	fault driven residual generation	isolability driven residual generation
Scalability	too complex for large systems	too complex for large systems	good scalability

■ **Table 8** Summary comparison of MSO sets, FMSO sets, and MTES

371 MSO sets or FMSO sets. Note however that, once MTESs are obtained, one FMSO set must
372 be extracted for each of them to achieve maximal diagnosability.

373 One should note that for some problem, in particular test selection for distributed systems,
374 having all the MSO sets at hand is an advantage because subsystem interconnections express
375 in terms of equations so equation subsets are as important as their fault supports [1, 7].

376 Table 8 gives a summary of the comparison of MSO sets, FMSO sets, and MTES.

377 6 Application of the concepts on a real example: the ADCS

378 In this section, all the concepts discussed earlier are explored through a realistic example:
379 the Attitude Determination and Control System (ADCS) of a Low Earth Orbit satellite [2].

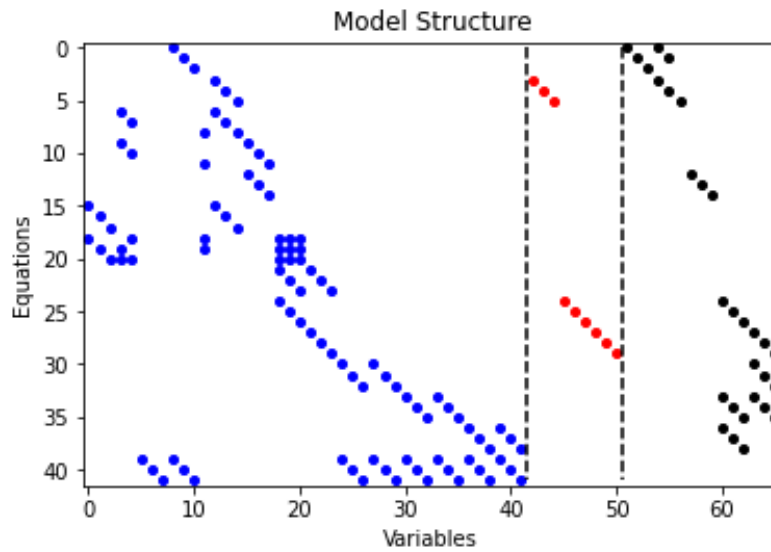
380 6.1 Structural model of the ADCS

381 In this example, the dynamics of a satellite's motion are modeled using equations that
382 describe the behavior of torque and angular velocity. The total torque acting on the satellite,
383 T , is the sum of the torques generated by the magnetorquers, reaction wheels, and external
384 disturbances. The rotational dynamics are influenced by the satellite's moment of inertia I
385 and angular velocity ω .

386 The global system is divided into 2 main parts: the Attitude Determination System (ADS)
387 and Attitude Control System (ACS). The ADS uses a combination of gyroscopes and vector
388 sensors (such as sun and star sensors) to estimate the satellite's attitude state, including
389 pitch, roll, and yaw angles, along with their corresponding rates. The resulting state vector
390 for the satellite is $X = [\psi, \theta, \phi, \dot{\psi}, \dot{\theta}, \dot{\phi}]$. These estimates are then fused together to provide
391 feedback to the ACS, which controls actuators like reaction wheels and magnetorquers to
392 adjust the satellite's attitude.

393 The structural model of the satellite's ADCS is enhanced by incorporating fault informa-
394 tion directly into the system's equations, with faults modeled as signals. The model considers
395 faults affecting the rate and vector sensors of the ADS (respectively frs and fvs), as well
396 as the reaction wheels of the ACS (frw), including various types such as hard, soft, and
397 intermittent faults. Each fault can affect all three axes of the system (x , y , and z).

398 The ADCS's structural model includes 42 equations including 42 unknowns variables, 15
399 known variables, and 9 faults that appear as parameters in the equations. The majority of
400 these equations are based on three behavioral equations that address the system dynamics
401 along the x , y , and z axes. The biadjacency matrix of the system is given by Figure 3. The
402 links between relations and unknown variables are represented in blue, those with faults in
403 red and those with known variables in black.



■ **Figure 3** Biadjacency matrix of the ADCS system

6.2 Results

Using the "Fault Diagnosis Toolbox"², 2448 MSO sets are found for this model, but only 9 MTES. There are also 2448 FMSO sets, as each MSO set has a non empty fault support.

In our example, we have $n_e = 42$, $n_f = 9$, and $\varphi = 9$. Applying Equation 3 results in a maximum of 511 nodes traversed during the search for MTES, whereas applying Equation 2 gives a maximum number of nodes exceeding 10^9 . These values clearly illustrate the difference in complexity between the two algorithms, with the MSO algorithm requiring the traversal of far more nodes. However, these are only upper bounds, as in practice, the equations can be grouped into equivalence classes, significantly reducing the actual number of nodes traversed by the algorithm. Table 9 presents the computation time required to compute both the MSO sets and the MTES. It shows that the computation time for MSO sets is significantly higher than for MTES in large systems.

Algorithm	Calculation time
MSO algorithm	0.534 s
MTES algorithm	0.239 s

■ **Table 9** Computation time of both algorithms

It is clear that the 2448 MSO sets for this model are not all useful for the generation of residuals. MSO sets with empty fault supports are not interesting, as they do not contribute to fault driven residual generation. This is why we focus on FMSO sets, which contain faults and are thus more relevant for diagnosis. However, in this case, all MSO sets contain faults, making the use of FMSO sets less critical. There must still be some MSO sets with identical

² <https://faultdiagnostictoolbox.github.io/>

13:14 MSO sets and MTES for dummies

421 fault supports or non-minimal ones. The MTES of this model, along with the associated
 422 fault supports, are given in Table 10.

	MTES	Fault support
$MTES_1$	$\{e_7, e_8, e_9, e_{10}, e_{11}, e_{12}, e_{13}, e_{14}, e_{15}, e_{16}, e_{17}, e_{18}, e_{19}, e_{20}, e_{21}, e_{24}, e_{30}\}$	$\{fvs_z\}$
$MTES_2$	$\{e_7, e_8, e_9, e_{10}, e_{11}, e_{12}, e_{13}, e_{14}, e_{15}, e_{16}, e_{17}, e_{18}, e_{19}, e_{20}, e_{21}, e_{23}, e_{29}\}$	$\{fvs_y\}$
$MTES_3$	$\{e_7, e_8, e_9, e_{10}, e_{11}, e_{12}, e_{13}, e_{14}, e_{15}, e_{16}, e_{17}, e_{18}, e_{19}, e_{20}, e_{21}, e_{22}, e_{28}\}$	$\{fvs_x\}$
$MTES_4$	$\{e_7, e_8, e_9, e_{10}, e_{11}, e_{12}, e_{13}, e_{14}, e_{15}, e_{16}, e_{17}, e_{18}, e_{19}, e_{20}, e_{21}, e_{27}\}$	$\{frs_z\}$
$MTES_5$	$\{e_7, e_8, e_9, e_{10}, e_{11}, e_{12}, e_{13}, e_{14}, e_{15}, e_{16}, e_{17}, e_{18}, e_{19}, e_{20}, e_{21}, e_{26}\}$	$\{frs_y\}$
$MTES_6$	$\{e_7, e_8, e_9, e_{10}, e_{11}, e_{12}, e_{13}, e_{14}, e_{15}, e_{16}, e_{17}, e_{18}, e_{19}, e_{20}, e_{21}, e_{25}\}$	$\{frs_x\}$
$MTES_7$	$\{e_6, e_9, e_{12}, e_{15}\}$	$\{frw_z\}$
$MTES_8$	$\{e_5, e_8, e_{11}, e_{14}\}$	$\{frw_y\}$
$MTES_9$	$\{e_4, e_7, e_{10}, e_{13}\}$	$\{frw_x\}$

■ **Table 10** MTES and their fault support for the ADCS example

423 For this example, all these MTES are MSO sets. These are the MSO sets with the
 424 minimal fault supports, containing the largest number of equations. The other MSO sets are
 425 not necessary as they have fault supports that are unions of these, or they can be expanded
 426 with other equations without altering the fault support while still remaining MSO sets. These
 427 9 MTES are necessary and sufficient to achieve maximal isolability as their fault support
 428 only includes one fault.

7 Conclusion

430 In this paper, a thorough examination and comparison of key concepts in model-based fault
 431 diagnosis is provided, specifically focusing on the structural analysis concepts named the
 432 Minimally Structurally Overdetermined (MSO) sets, Fault driven Minimally Structurally
 433 Overdetermined (FMSO) sets, and Minimal Test Equation Supports (MTES).

434 This article conducts a detailed analysis of the rules, applications and interrelationships
 435 of MSO sets, FMSO sets and MTESs, providing insights into their respective strengths
 436 and limitations. It then systematically examines the algorithms used to identify MSO and
 437 MTES sets, discussing their computational efficiency and impact on diagnosis capabilities.
 438 The final part of this work applies these concepts to a practical example, demonstrating
 439 their effectiveness in a real-world scenario. More specifically, the algorithms are applied to
 440 the Attitude Determination and Control System (ADCS) of a satellite in low Earth orbit,
 441 and the structural model, algorithmic complexity involved and the possibility of diagnosing
 442 faults are studied. This results highlight the practical usefulness of MTES in reducing the
 443 computational load while maintaining robust fault isolation capabilities.

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