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Steady State and Fluctuation Analysis for Kalman Filters with Poisson-Sampled Observations

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Abstract

We study optimal filtering for continuous-time linear stochastic systems with Poisson-sampled observation processes. For each realization of the sampled observation process, the posterior distribution is a Gaussian process whose mean and covariance are described by continuous-discrete process. We are particularly interested in analyzing the expectation of the first and second moment of the estimation error with respect to the sampling process. Using the system-theoretic properties like observability and controllability, our results provide tractable conditions on the mean sampling rate for convergence of the expected error covariance, its boundedness and convergence of expected estimation error to zero. Some comparisons are also drawn with the solution of Riccati differential equation associated with the continuous-observation process.

Keywords: Kalman filtering, Poisson-sampled observations, Error covariance bounds, Fixed-point analysis

1. Introduction

Filtering in stochastic dynamical systems is one of the fundamental research areas as it deals with computing an estimate of unknown variables using noisy observations. Apart from developing efficient algorithms, an important research direction is to study the filtering problem with constraints on the information available for computing the optimal distribution [1]. In particular, motivated by the idea of implementing filters subject to observations transmitted over networks through some communication protocols, it is natural to stipulate that the observations arrive at some random time instants [2, 3].

In discrete-time stochastic systems, the transmission of observations over a network may be subjected to dropouts, and the arrival of observations at a remote estimation site is modeled by a random process. In [4], the observations are subjected to a Bernoulli process. After computing the estimate, the performance of such estimators is analyzed by computing expectation of the error covariance matrix with respect to the Bernoulli process associated with the observations. In [5], a similar setup is concerned but the packet dropouts are modeled using a two state Markov chain. In certain cases, in addition to dropouts, the effect of communication channel is modeled via quantization

effect [6], where the authors study remote state estimation across a channel with an average power constraint and show that it is better to transmit an innovation term through the channel when data loss acknowledgment is available, else it is more efficient to transmit an estimate computed by the Kalman filter. In [7], the authors solve the problem of optimal estimation and control across multiple links with packet drops, where they separate out the control and estimation objective costs, and analyze the particular case of Kalman estimation forwarding over the links. Similar works involving stability of state estimation over discrete and continuous communication networks can be found in [8]. In these works, a common theme is to find conditions on channel parameters and system data that ensure state estimation with reasonable accuracy.

When dealing with continuous-time stochastic systems, and a discrete observation process, it is of interest to compute the conditional distribution of the state process conditioned upon this discrete observation process [9]. Such a setup for continuous-discrete filters is motivated by the implementation of filters over networks where the observation process is discretized by the presence of a digital channel between the plant and the filter. For certain technical reasons, and to better study the effect of mean sampling rate, the authors stipulate in their previous works that the sampling process is a Poisson counter. In particular, for a system class very close to the one studied in this paper, the authors in [10] propose a continuous-discrete filter and analyze the boundedness of error covariance as a function of the mean sampling rate. In related works, we note that stochastic optimal control problems under random sampling have been studied in [11]. Moreover, the

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authors of [12] study state estimation and optimal network resource allocation for continuous-time weakly-coupled linear systems with sensor measurements arriving at fixed discrete time intervals.

In this paper, we analyze Kalman filters for continuous-time stochastic systems when the observation process is Poisson-sampled. In particular, our focus is on studying the expectation of first and second moment of the error process. We provide condition using system-theoretic properties to show boundedness of the error covariance. In contrast to [10], we give explicit conditions in terms of controllability and observability Gramians, and the mean sampling rate. In addition, we show that similar conditions allow us to show the convergence of the expectation of first moment to zero asymptotically.

The remainder of the paper is organized as follows: In Section 2, we discuss the problem setup and provide an expression for expected covariance and estimation error. We provide the first main result on existence of fixed point for expected error covariance in Section 3. We then compute tractable analytical bounds on the error covariance in Section 4, and use them for fluctuation analysis in Section 5. The asymptotic convergence of first moment is studied in Section 6, and some concluding remarks appear in Section 7.

2. Preliminaries

In this section, we will describe the filtering problem that will be analyzed in this article. We first propose the system class and the associated measurement process. The optimal filter in our case is a continuous-discrete process for which we describe the dynamics for first moment of error and the error covariance. In the remainder of this paper, we analyze the expectations of these process.

2.1. System Class

We consider the class of stochastic linear systems of the form

$$dx_t = Ax_t dt + Gd\omega_t \quad (1)$$

where $(x_t)_{t \geq 0}$ is an \mathbb{R}^n -valued diffusion process describing the state. Let $(\Omega, \mathcal{F}, \mathbb{P})$ denote the underlying probability space. In (1), ω_t is an \mathbb{R}^m -valued standard Wiener process adapted to the filtration \mathcal{F} with $\mathbb{E}[d\omega_t d\omega_t^\top] = I_{m \times m} dt$ for all $t \geq 0$. The matrices $A \in \mathbb{R}^{n \times n}$ and $G \in \mathbb{R}^{n \times m}$ are constant.

We study the state estimation problem when the noisy output measurements are available only at random times. The motivation to work with randomly time-sampled measurements comes from several applications, such as, communication over networks which allow information packets to be sent at some discrete randomly distributed time instants. Thus, we consider a monotone nondecreasing sequence $(\tau_n)_{n \in \mathbb{N}}$ taking values in $\mathbb{R}_{\geq 0}$ which denote

the time instants at which the measurements are available for estimation. We introduce the process N_t defined as

$$N_t := \sup\{n \in \mathbb{N} \mid \tau_n \leq t\} \quad \text{for } t \in \mathbb{R}, \quad (2)$$

and it is assumed that $(N_t)_{t \geq 0}$ is a Poisson process of intensity $\lambda > 0$ and it is independent of the noise and the state processes. The Poisson process is among the most well-studied processes, and standard results (see, e.g., [13, §2.3]) show that it is memoryless and Markovian.

The discretized, and noisy, observation process is thus defined as

$$y_{\tau_{N_t}} = Cx(\tau_{N_t}) + \nu_{\tau_{N_t}}, \quad t \geq 0, \quad (3)$$

where $C \in \mathbb{R}^{p \times n}$ is a constant matrix, and ν_k is a sequence of i.i.d. Gaussian noise processes and $\nu_0 \sim \mathcal{N}(0, V)$. Equation (3) is motivated by the fact that a continuous observation process $dz_t = Cx_t dt + d\eta_t$ for a Wiener process $(\eta_t)_{t \geq 0}$ is formally equivalent to $y_t = Cx_t + \nu_t$, with the identifications $y_t \sim \frac{dz_t}{dt}$ and $\nu_t \sim \frac{d\eta_t}{dt}$, so that ν_t is a Gaussian process; see [9, Chapter 4] for further details. Our goal is to construct the estimate \hat{x}_t , which minimizes the mean square estimation error, using the observations $\mathcal{Y}_t := \{y_{\tau_k} \mid k \leq N_t\}$.

2.2. Optimal Filter

The basic problem in filter design is to find an estimate of the state process which minimizes the mean square estimation error. This optimal estimator is described by the expectation of the state process $(x_t)_{t \geq 0}$ conditioned upon the measurements observed over the interval $[0, t]$, that is, \mathcal{Y}_t . In particular, with the structure imposed on the system dynamics in this section, the conditional expectation is Gaussian and the two moments are simulated through ordinary differential equations with updates at times when a new measurement arrives. For an arbitrary strictly increasing real-valued sequence $(\tau_k)_{k \in \mathbb{N}^*}$, (where $\mathbb{N}^* := \mathbb{N} \cup \{0\}$) this procedure is also proposed in [9, Thm. 7.1]. If we specify a sequence $(\tau_k)_{k \in \mathbb{N}^*}$ so that it corresponds to the arrival times of a Poisson process, we simulate the mean of the conditional distribution as:

$$\dot{\hat{x}}_t = A\hat{x}_t dt, \quad t \in [\tau_{N_t}, \tau_{1+N_t}[\quad (4a)$$

$$\hat{x}_t^+ = \hat{x}_t + K_t(y_t - C\hat{x}_t), \quad t = \tau_{N_t} \quad (4b)$$

where the injection gain $K_t = P_t C^\top (C P_t C^\top + V)^{-1}$, and the process $(P_t)_{t \geq 0}$ is described as

$$\dot{P}_t = (AP_t + P_t A^\top + GG^\top), \quad t \in [\tau_{N_t}, \tau_{1+N_t}[\quad (5a)$$

$$\hat{P}_t^+ = P_t - P_t C^\top (C P_t C^\top + V)^{-1} C P_t, \quad t = \tau_{N_t}. \quad (5b)$$

We denote the estimation error $\chi_t := x_t - \hat{x}_t$ and the covariance of this error process is described by (5). From analysis perspective, we are also interested in studying the

first moment of the estimation error, which follows the dynamics

$$\begin{aligned} d\chi_t &= A\chi_t dt + Gd\omega_t, \\ \chi_{\tau_{N_t}^+} &= \chi_{\tau_{N_t}^-} - K_{\tau_{N_t}^-} \left(y_{\tau_{N_t}^-} - C\widehat{X}_{\tau_{N_t}^-} \right), \\ &= \left(I - K_{\tau_{N_t}^-} C \right) \chi_{\tau_{N_t}^-} - K_{\tau_{N_t}^-} \nu_{\tau_{N_t}^-}. \end{aligned}$$

2.3. Expected Covariance and Estimation Error

From analysis viewpoint, it is important to look at the expectation of the process $(P_t)_{t \geq 0}$ with respect to the sampling times $(\tau_{N_t})_{t \geq 0}$. From our previous works [10, Proposition 4.1], it can be shown that the expected covariance for the linear optimal filtering case, takes the form

$$\dot{P}_t = AP_t + P_t A^\top + GG^\top - \lambda P_t C^\top M_{P_t}^{-1} C P_t \quad (6)$$

where the matrix $M_{P_t} \in \mathbb{R}^{p \times p}$ is given by

$$M_{P_t} := C P_t C^\top + V. \quad (7)$$

Similarly, we denote by $\bar{\chi}_t := \mathbb{E}[\chi_t]$ the expected error with the expectation taken with respect to the sampling process N_t . Using [14, Theorem 1], the evolution of the expected value of the error are given by:

$$\dot{\bar{\chi}}_t = (A - \lambda \mathcal{K}_t C) \bar{\chi}_t. \quad (8)$$

Main Contribution.: The primary problem studied in this paper is to provide conditions in terms of the bounds on the mean sampling rate $\lambda > 0$ and the structural assumptions on controllability and observability of the pairs (A, G) and (A, C) that guarantee convergence to a steady state solution for \mathcal{P}_t . Using comparison with the Riccati differential equation associated with continuous-observation processes, we derive bounds on the expected error covariance. These results are then used for asymptotic analysis of the first moment of the error process $(\chi_t)_{t \geq 0}$, to provide conditions under which $\mathbb{E}[\chi_t]$ converges to zero asymptotically.

Reformulation.: For our purposes, it is also useful to introduce the notation $\mathcal{K}_t = P_t C^\top M_{P_t}^{-1}$, which results in the following equivalent expressions for \mathcal{P}_t :

$$\begin{aligned} \dot{\mathcal{P}}_t &= AP_t + P_t A^\top + GG^\top - \lambda (\mathcal{K}_t C P_t + P_t C^\top \mathcal{K}_t^\top) \\ &\quad + \lambda \mathcal{K}_t V \mathcal{K}_t^\top + \lambda \mathcal{K}_t C P_t C^\top \mathcal{K}_t^\top \\ &= AP_t + P_t A^\top + GG^\top + \lambda \mathcal{K}_t V \mathcal{K}_t^\top \\ &\quad + \lambda (I - \mathcal{K}_t C) P_t (I - \mathcal{K}_t C)^\top - \lambda P_t \\ &= AP_t + P_t A^\top + GG^\top + \lambda \mathcal{K}_t V \mathcal{K}_t^\top \\ &\quad + \lambda (I - \mathcal{K}_t C) P_t (I - \mathcal{K}_t C)^\top - \frac{\lambda}{2} P_t - \frac{\lambda}{2} P_t \\ &= \left(A - \frac{\lambda}{2} I \right) P_t + P_t \left(A - \frac{\lambda}{2} I \right)^\top + GG^\top \\ &\quad + \lambda \mathcal{K}_t V \mathcal{K}_t^\top + \lambda (I - \mathcal{K}_t C) P_t (I - \mathcal{K}_t C)^\top. \end{aligned}$$

Thus, by letting $A_\lambda := (A - \frac{\lambda}{2} I)$, (6) can now be equivalently expressed as

$$\begin{aligned} \dot{P}_t &= A_\lambda P_t + P_t A_\lambda^\top + GG^\top + \lambda \mathcal{K}_t V \mathcal{K}_t^\top \\ &\quad + \lambda (I - \mathcal{K}_t C) P_t (I - \mathcal{K}_t C)^\top. \quad (9) \end{aligned}$$

We denote the deterministic flow associated with (6) as $\phi_{s,t}(Q)$, with $s \leq t$ and $\phi_{s,s} := Q$, where $Q \in \mathbb{S}_+^{n \times n}$, where $\mathbb{S}_+^{n \times n}$ denotes the set of positive semi-definite matrices in $\mathbb{R}^{n \times n}$. Whenever $s = 0$, we denote $\phi_{0,t} := \phi_t$. When the focus is on specifying the solution of (9) with an initial condition $Q \in \mathbb{S}_+^{n \times n}$, then we equivalently write

$$\begin{aligned} \phi_t(Q) &= \mathcal{E}_t^\lambda Q \mathcal{E}_t^{\lambda \top} + \int_0^t \mathcal{E}_{t-s}^\lambda \left[\lambda (I - \mathcal{K}_s C) \phi_s(Q) (I - \mathcal{K}_s C)^\top \right. \\ &\quad \left. + \lambda \mathcal{K}_s V \mathcal{K}_s^\top + GG^\top \right] \mathcal{E}_{t-s}^{\lambda \top} ds. \quad (10) \end{aligned}$$

We also consider the following notation

$$\mathcal{E}_{s,t}^\lambda := \exp \left[\left(A - \frac{\lambda}{2} I \right) (t - s) \right]$$

and we let $\mathcal{E}_t^\lambda := \mathcal{E}_{0,t}^\lambda$. By a slight abuse of notation we use $\mathcal{K}_1 = \phi_t(Q_1) C^\top (C \phi_t(Q_1) C^\top + V)^{-1}$ and $\mathcal{K}_2 = \phi_t(Q_2) C^\top (C \phi_t(Q_2) C^\top + V)^{-1}$ to denote the time varying optimal gains corresponding to flows $\phi_t(Q_1)$ and $\phi_t(Q_2)$ when two positive semidefinite matrices Q_1 and Q_2 are involved in the same expression and when the time instant t is clear from the context. The notation $K_Q := Q C^\top (C Q C^\top + V)^{-1}$ is used to denote the gain when Q does not vary with time.

3. Asymptotics of Expected Covariance

We now develop the first main result which concerns the convergence of solution of equation (9) to its steady state. To do so, we need two basic assumptions on the system (1) and the observation process (3) which are stated below:

- (A1) The pair (A, C) is observable for each $\lambda > 0$, that is, there exists $\underline{\theta}_\lambda^C > 0$ such that,

$$\int_0^\infty \mathcal{E}_t^\lambda C^\top V^{-1} C \mathcal{E}_t^{\lambda \top} dt \geq \underline{\theta}_\lambda^C I.$$

- (A2) The pair (A, G) is controllable, for each $\lambda > 0$, that is, there exists $\underline{\theta}_\lambda^G > 0$ such that,

$$\int_0^\infty \mathcal{E}_t^\lambda G G^\top \mathcal{E}_t^{\lambda \top} dt \geq \underline{\theta}_\lambda^G I.$$

Also, we consider a constant β such that $CC^\top \leq \beta V$, and denote the logarithmic norm (associated with Euclidean induced matrix norm) of a matrix A by $\mu(A)$. For the Euclidean induced matrix norm, $\mu(A)$ can be computed as $\mu(A) = \lambda_{\max}(\frac{A+A^\top}{2})$ [15, Fact 11.15.7], where $\lambda_{\max}(\cdot)$ denotes the maximum eigenvalue of its matrix argument.

Theorem 1. Consider system (1) and the observation process (3) with N_t a Poisson process of intensity $\lambda > 0$, and suppose that Assumptions (A1) and (A2) are satisfied. If it holds that

$$\lambda(\lambda - 2\mu(A))\underline{\theta}_\lambda^C > 2\mu(A)\beta \quad (11)$$

then there exists a unique $\bar{\mathcal{P}}$ that solves the equation

$$0 = A\bar{\mathcal{P}} + \bar{\mathcal{P}}A^\top + GG^\top - \lambda\bar{\mathcal{P}}C^\top M_{\bar{\mathcal{P}}}^{-1}C\bar{\mathcal{P}}, \quad (12)$$

and for every $\mathcal{P}_0 \in \mathbb{S}_+^{n \times n}$, it holds that $\lim_{t \rightarrow \infty} \phi_t(\mathcal{P}_0) = \bar{\mathcal{P}}$.

Remark 1. Note that the condition (11) also implies that $\lambda > 2\mu(A)$. Assuming otherwise, if $\lambda \leq 2\mu(A)$, then the inequality holds only if $\mu(A)$ is negative and consequently $\lambda < 0$, which is not possible.

The proof of Theorem 1 involves several steps and in the process we can prove several intermediate results for the differential equation (6) of independent interest. These developments are carried out in Section 3.1 and Section 3.2, and used to complete the proof in Section 3.3.

3.1. Monotonicity and Recursive operator

We first prove two properties of the solutions of (6). The first one relates to the monotonicity with respect to initial conditions, and the second property relates to computing upper and lower bounds on the difference of solutions starting from different initial conditions.

Proposition 1. The error covariance flow $\phi_t(Q)$ is non-decreasing with respect to the Loewner partial order, that is, we have $\phi_t(Q_1) \geq \phi_t(Q_2)$, whenever $Q_1 \geq Q_2$.

Proof. From (10) it can be seen that

$$\begin{aligned} \phi_t(Q_1) - \phi_t(Q_2) &= \mathcal{E}_t^\lambda(Q_1 - Q_2)\mathcal{E}_t^{\lambda^\top} \\ &+ \lambda \int_0^t \mathcal{E}_{t-s}^\lambda \left[(I - \mathcal{K}_1 C)\phi_s(Q_1)(I - \mathcal{K}_1 C)^\top + \mathcal{K}_1 V \mathcal{K}_1^\top \right. \\ &\quad \left. - (I - \mathcal{K}_2 C)\phi_s(Q_2)(I - \mathcal{K}_2 C)^\top - \mathcal{K}_2 V \mathcal{K}_2^\top \right] \mathcal{E}_{t-s}^{\lambda^\top} ds. \end{aligned} \quad (13)$$

Using Lemma 4 in Appendix, and in particular equation (.2), we can write (13) equivalently as

$$\begin{aligned} \phi_t(Q_1) - \phi_t(Q_2) &= \mathcal{E}_t^\lambda(Q_1 - Q_2)\mathcal{E}_t^{\lambda^\top} \\ &+ \lambda \int_0^t \mathcal{E}_{t-s}^\lambda \left[(I - \mathcal{K}_1 C)\phi_s(Q_1) - (I - \mathcal{K}_2 C)\phi_s(Q_2) \right] \mathcal{E}_{t-s}^{\lambda^\top} ds. \end{aligned}$$

Recall the fact that given two positive definite matrices Q_1, Q_2 if $Q_1 \geq Q_2$, then, $Q_1^{-1} \leq Q_2^{-1}$. Note from Lemma 3 that, $(I - \mathcal{K}_s C)\phi_s(Q) = (\phi_s(Q)^{-1} + CV^{-1}C^\top)^{-1}$. If $\phi_s(Q_1) \geq \phi_s(Q_2) > 0$, we have

$$\begin{aligned} \phi_s(Q_2)^{-1} &\geq \phi_s(Q_1)^{-1} \\ \phi_s(Q_2)^{-1} + CV^{-1}C^\top &\geq \phi_s(Q_1)^{-1} + CV^{-1}C^\top \\ (\phi_s(Q_1)^{-1} + CV^{-1}C^\top)^{-1} &\geq (\phi_s(Q_2)^{-1} + CV^{-1}C^\top)^{-1} \\ (I - \mathcal{K}_1 C)\phi_s(Q_1) &\geq (I - \mathcal{K}_2 C)\phi_s(Q_2). \end{aligned}$$

Thus the integral on the right-hand side of (13) is nonnegative definite, whence the result follows. \square

Proposition 2. Let $Q_1, Q_2 \in \mathbb{S}_+^{n \times n}$. Consider the two flows $Q_1 \mapsto \phi_t(Q_1)$ and $Q_2 \mapsto \phi_t(Q_2)$ and their difference $\phi_t(Q_1) - \phi_t(Q_2)$ as in (13). Then, we have

$$\begin{aligned} \phi_t(Q_1) - \phi_t(Q_2) &\geq \mathcal{E}_t^\lambda(Q_1 - Q_2)\mathcal{E}_t^{\lambda^\top} \\ &+ \lambda \int_0^t \mathcal{E}_{t-s}^\lambda \left[(I - \mathcal{K}_1 C)(\phi_s(Q_1) - \phi_s(Q_2))(I - \mathcal{K}_1 C)^\top \right] \mathcal{E}_{t-s}^{\lambda^\top} ds \end{aligned} \quad (14)$$

and

$$\begin{aligned} \phi_t(Q_1) - \phi_t(Q_2) &\leq \mathcal{E}_t^\lambda(Q_1 - Q_2)\mathcal{E}_t^{\lambda^\top} \\ &+ \lambda \int_0^t \mathcal{E}_{t-s}^\lambda \left[(I - \mathcal{K}_2 C)(\phi_s(Q_1) - \phi_s(Q_2))(I - \mathcal{K}_2 C)^\top \right] \mathcal{E}_{t-s}^{\lambda^\top} ds. \end{aligned} \quad (15)$$

Proof. For the ease of notation, for a given $s \geq 0$, let us denote $\phi_s(Q_1)$ by P_1 and $\phi_s(Q_2)$ by P_2 . Let the associated optimal gains be \mathcal{K}_1 and \mathcal{K}_2 respectively. Let us first prove the following which will be useful in the rest of the proof:

$$(I - \mathcal{K}_1 C)P_1 C^\top (\mathcal{K}_1 - \mathcal{K}_2)^\top \geq (I - \mathcal{K}_2 C)P_2 C^\top (\mathcal{K}_1 - \mathcal{K}_2)^\top \quad (16)$$

Indeed, from Lemma 3 we have

$$\begin{aligned} (I - \mathcal{K}_1 C)P_1 C^\top (\mathcal{K}_1 - \mathcal{K}_2)^\top &- (I - \mathcal{K}_2 C)P_2 C^\top (\mathcal{K}_1 - \mathcal{K}_2)^\top \\ &= \mathcal{K}_1 V (\mathcal{K}_1 - \mathcal{K}_2)^\top - \mathcal{K}_2 V (\mathcal{K}_1 - \mathcal{K}_2)^\top \\ &= (\mathcal{K}_1 - \mathcal{K}_2) V (\mathcal{K}_1 - \mathcal{K}_2)^\top \\ &\geq 0 \end{aligned}$$

Now, notice,

$$\begin{aligned} (I - \mathcal{K}_1 C)(P_1 - P_2)(I - \mathcal{K}_2 C)^\top &- (I - \mathcal{K}_1 C)(P_1 - P_2)(I - \mathcal{K}_1 C)^\top \\ &= (I - \mathcal{K}_1 C)(P_1 - P_2)C^\top (\mathcal{K}_1 - \mathcal{K}_2)^\top \\ &= ((I - \mathcal{K}_1 C)P_1 - (I - \mathcal{K}_1 C)P_2)C^\top (\mathcal{K}_1 - \mathcal{K}_2)^\top \\ &= (I - \mathcal{K}_1 C)P_1 C^\top (\mathcal{K}_1 - \mathcal{K}_2)^\top - (I - \mathcal{K}_1 C)P_2 C^\top (\mathcal{K}_1 - \mathcal{K}_2)^\top \\ &\stackrel{(a)}{\geq} (I - \mathcal{K}_2 C)P_2 C^\top (\mathcal{K}_1 - \mathcal{K}_2)^\top - (I - \mathcal{K}_1 C)P_2 C^\top (\mathcal{K}_1 - \mathcal{K}_2)^\top \\ &= ((I - \mathcal{K}_2 C)P_2 - (I - \mathcal{K}_1 C)P_2)C^\top (\mathcal{K}_1 - \mathcal{K}_2)^\top \\ &= (\mathcal{K}_1 - \mathcal{K}_2)C P_2 C^\top (\mathcal{K}_1 - \mathcal{K}_2)^\top \\ &\geq 0, \end{aligned}$$

where inequality (a) is obtained using (16). Thus, we have $(I - \mathcal{K}_1 C)(P_1 - P_2)(I - \mathcal{K}_2 C)^\top \geq (I - \mathcal{K}_1 C)(P_1 - P_2)(I - \mathcal{K}_1 C)^\top$. Putting back $\phi_s(Q_1)$ and $\phi_s(Q_2)$ in place of P_1 and P_2 respectively we have the first statement of our Proposition 2. That is:

$$\begin{aligned} \phi_t(Q_1) - \phi_t(Q_2) &= \mathcal{E}_t^\lambda(Q_1 - Q_2)\mathcal{E}_t^{\lambda^\top} \\ &+ \lambda \int_0^t \mathcal{E}_{t-s}^\lambda \left[(I - \mathcal{K}_1 C)(\phi_s(Q_1) - \phi_s(Q_2))(I - \mathcal{K}_2 C)^\top \right] \mathcal{E}_{t-s}^{\lambda^\top} ds \\ &\geq \mathcal{E}_t^\lambda(Q_1 - Q_2)\mathcal{E}_t^{\lambda^\top} \\ &+ \lambda \int_0^t \mathcal{E}_{t-s}^\lambda \left[(I - \mathcal{K}_1 C)(\phi_s(Q_1) - \phi_s(Q_2))(I - \mathcal{K}_1 C)^\top \right] \mathcal{E}_{t-s}^{\lambda^\top} ds \end{aligned}$$

The proof of the second statement of the Proposition 2 follows from a similar argument. \square

3.2. Existence of fixed point

We can relate the solutions of (6) with the mapping $\mathcal{F}_\lambda : \mathbb{R}^{n \times n} \rightarrow \mathbb{R}^{n \times n}$ defined as

$$\mathcal{F}_\lambda(P) := \lambda \int_0^\infty \mathcal{E}_t^\lambda (I - K_P C) P \mathcal{E}_t^{\lambda \top} dt + \int_0^\infty \mathcal{E}_t^\lambda G G^\top \mathcal{E}_t^{\lambda \top} dt \quad (17)$$

where $K_P = PC^\top(CPC^\top + V)^{-1}$. In this subsection, we derive conditions for the fixed point of \mathcal{F}_λ . To do so, we introduce the mapping $\mathcal{T} : \mathbb{R} \rightarrow \mathbb{R}$, defined as

$$\mathcal{T}(\alpha) := f(\alpha) + \bar{\theta}_\lambda^G, \quad (18)$$

where the constant $\bar{\theta}_\lambda^G$ satisfies

$$\int_0^\infty \mathcal{E}_t^\lambda G G^\top \mathcal{E}_t^{\lambda \top} dt \leq \bar{\theta}_\lambda^G I$$

and the function $f : \mathbb{R} \rightarrow \mathbb{R}$ is defined as

$$f(\alpha) := \frac{\lambda \alpha}{\lambda - 2\mu(A)} - \frac{\lambda \bar{\theta}_\lambda^G \alpha^2}{\beta \alpha + 1}.$$

The existence of a finite positive $\bar{\theta}_\lambda^G$ is guaranteed from the condition (11) in Theorem 1 which implies that $\lambda > 2\mu(A)$. To see that, we can write

$$\int_0^\infty \mathcal{E}_t^\lambda G G^\top \mathcal{E}_t^{\lambda \top} dt \leq \lambda_{\max}(G G^\top) \int_0^\infty \mathcal{E}_t^\lambda \mathcal{E}_t^{\lambda \top} dt.$$

Now using [15, Fact 11.15.6] we have

$$\mathcal{E}_t^\lambda \mathcal{E}_t^{\lambda \top} \leq e^{(2\mu - A)t} I.$$

Thus,

$$\begin{aligned} \lambda_{\max}(G G^\top) \int_0^\infty \mathcal{E}_t^\lambda \mathcal{E}_t^{\lambda \top} dt &\leq \lambda_{\max}(G G^\top) \frac{1}{\lambda - 2\mu(A)} I \\ &=: \bar{\theta}_\lambda^G I. \end{aligned}$$

Lemma 1. *If the sequence $\{\mathcal{T}^{(k)}(\alpha)\}_{k \in \mathbb{N}}$ converges to a unique fixed point for each $\alpha \in \mathbb{R}_+$, then there exists a fixed point to \mathcal{F}_λ , which is unique and the sequence $\{\mathcal{F}_\lambda^{(k)}(P_0)\}_{k \in \mathbb{N}}$ converges to the fixed point for each $P_0 \in \mathbb{S}_+^{n \times n}$.*

Proof of Lemma 1. We provide the proof in three parts. First, we will show that the convergence of $\{\mathcal{T}^{(k)}(\alpha)\}_{k \in \mathbb{N}}$ implies that a fixed point \mathcal{Q} exists for the operator \mathcal{F}_λ ; Second, we show that the fixed point of \mathcal{F}_λ is unique; and third, we show that the successive iterations of the mapping \mathcal{F}_λ on any positive definite matrix converge to the fixed point \mathcal{Q} , thus guaranteeing that $\lim_{k \rightarrow \infty} \mathcal{F}_\lambda^{(k)}(P_0) = \mathcal{Q}$ for any $P_0 \in \mathbb{S}_+^{n \times n}$.

Existence: We know from the monotonicity property of \mathcal{F}_λ , that if $\mathcal{F}_\lambda^{(n_1)}(\alpha I) \geq \mathcal{F}_\lambda^{(n_2)}(\alpha I)$, then $\mathcal{F}_\lambda^{(n_1+k)}(\alpha I) \geq$

$\mathcal{F}_\lambda^{(n_2+k)}(\alpha I)$ for all $k \geq 0$. To show the existence of a fixed point of \mathcal{F}_λ , we look at the iterations $\mathcal{F}_\lambda^{(k)}(\alpha I)$ for $\alpha = 0$. Clearly, $\mathcal{F}_\lambda(0) > 0$ and thus $\mathcal{F}_\lambda^{(k+1)}(0) \geq \mathcal{F}_\lambda^{(k)}(0)$ for all $k \geq 0$. Also, $\mathcal{F}_\lambda^{(k)}(0) \leq \mathcal{T}^{(k)}(0)I$ and thus if \mathcal{T} is convergent, $\lim_{k \rightarrow \infty} \mathcal{F}_\lambda^{(k)}(0) = \mathcal{Q}$ exists, since it is both monotone increasing and upper bounded by a convergent operator. Thus there exists a fixed point \mathcal{Q} of \mathcal{F}_λ .

Uniqueness: Next we show that if there exists a fixed point of \mathcal{F}_λ , it is unique. We do so by using contradiction. Let us assume that there exists two distinct fixed points $\mathcal{Q}_1, \mathcal{Q}_2 \in \mathbb{S}_+^{n \times n}$ of \mathcal{F}_λ and define $K_{\mathcal{Q}_2} = \mathcal{Q}_2 C^\top (C \mathcal{Q}_2 C^\top + V)^{-1}$, $K_{\mathcal{Q}_1} = \mathcal{Q}_1 C^\top (C \mathcal{Q}_1 C^\top + V)^{-1}$. Then we can write

$$\begin{aligned} \mathcal{Q}_2 - \mathcal{Q}_1 &= \mathcal{F}_\lambda(\mathcal{Q}_2) - \mathcal{F}_\lambda(\mathcal{Q}_1) \\ &= \lambda \int_0^\infty \mathcal{E}_t^\lambda (I - K_{\mathcal{Q}_2} C) (\mathcal{Q}_2 - \mathcal{Q}_1) (I - K_{\mathcal{Q}_1} C)^\top \mathcal{E}_t^{\lambda \top} dt \\ &\stackrel{(a)}{\leq} \lambda \int_0^\infty \mathcal{E}_t^\lambda (I - K_{\mathcal{Q}_1} C) (\mathcal{Q}_2 - \mathcal{Q}_1) (I - K_{\mathcal{Q}_1} C)^\top \mathcal{E}_t^{\lambda \top} dt, \end{aligned}$$

where inequality (a) results from Proposition 2. Now rearranging the above and using (10) we have

$$\begin{aligned} \mathcal{Q}_2 &\leq \mathcal{Q}_1 + \lambda \int_0^\infty \mathcal{E}_t^\lambda (I - K_{\mathcal{Q}_1} C) (\mathcal{Q}_2 - \mathcal{Q}_1) (I - K_{\mathcal{Q}_1} C)^\top \mathcal{E}_t^{\lambda \top} dt \\ &= \int_0^\infty \mathcal{E}_t^\lambda \left[\lambda (I - K_{\mathcal{Q}_1} C) (\mathcal{Q}_1) (I - K_{\mathcal{Q}_1} C)^\top + \lambda K_{\mathcal{Q}_1} V K_{\mathcal{Q}_1}^\top \right. \\ &\quad \left. + G G^\top \right] \mathcal{E}_t^{\lambda \top} dt \\ &\quad + \lambda \int_0^\infty \mathcal{E}_t^\lambda (I - K_{\mathcal{Q}_1} C) (\mathcal{Q}_2 - \mathcal{Q}_1) (I - K_{\mathcal{Q}_1} C)^\top \mathcal{E}_t^{\lambda \top} dt \\ &= \lambda \int_0^\infty \mathcal{E}_t^\lambda (I - K_{\mathcal{Q}_1} C) (\mathcal{Q}_2) (I - K_{\mathcal{Q}_1} C)^\top \mathcal{E}_t^{\lambda \top} dt \\ &\quad + \lambda \int_0^\infty \mathcal{E}_t^\lambda K_{\mathcal{Q}_1} V K_{\mathcal{Q}_1}^\top \mathcal{E}_t^{\lambda \top} dt + \int_0^\infty \mathcal{E}_t^\lambda G G^\top \mathcal{E}_t^{\lambda \top} dt \\ &=: \mathcal{G}_{\mathcal{Q}_1}(\mathcal{Q}_2). \end{aligned}$$

See that applying the operator \mathcal{F}_λ once on both sides of $\mathcal{Q}_2 \leq \mathcal{G}_{\mathcal{Q}_1}(\mathcal{Q}_2)$ results in $\mathcal{Q}_2 = \mathcal{F}_\lambda(\mathcal{Q}_2) \leq \mathcal{F}_\lambda(\mathcal{G}_{\mathcal{Q}_1}(\mathcal{Q}_2))$. Using the comparison argument from [10, Lemma 5.1], it is easy to see that $\mathcal{F}_\lambda(\mathcal{G}_{\mathcal{Q}_1}(\mathcal{Q}_2)) \leq \mathcal{G}_{\mathcal{Q}_1}^{(2)}(\mathcal{Q}_2)$. In general for $k > 0$ we can write $\mathcal{Q}_2 \leq \mathcal{G}_{\mathcal{Q}_1}^{(k)}(\mathcal{Q}_2)$. Notice from [16, Theorem 4.1, Lemma 5.1] for the same class of operators as $\mathcal{G}_{\mathcal{Q}}$, controllability of (A, G) in addition to the existence of a fixed point for the operator $\mathcal{G}_{\mathcal{Q}}$ guarantees the uniqueness and convergence to the same. We can clearly see that \mathcal{Q}_1 is indeed the fixed point of the operator $\mathcal{G}_{\mathcal{Q}_1}$ and is unique, from the previous argument. As a result, $\mathcal{Q}_2 \leq \lim_{k \rightarrow \infty} \mathcal{G}_{\mathcal{Q}_1}^{(k)}(\mathcal{Q}_2) = \mathcal{Q}_1$. Now, since in the entire argument \mathcal{Q}_1 and \mathcal{Q}_2 did not have any pre-defined ordering, we can repeat the arguments the other way around to get $\mathcal{Q}_1 \leq \mathcal{Q}_2$, which can simultaneously hold only when $\mathcal{Q}_1 = \mathcal{Q}_2$. This shows that if there exists a fixed point of \mathcal{F}_λ , it is indeed unique.

Convergence: Let us denote the fixed point of \mathcal{T} as the scalar α^* . To show that \mathcal{F}_λ is convergent for every initial positive definite matrix, notice that for any initial

positive definite matrix P_0 , we can always find a scalar $z > 1$ such that $P_0 \leq z\alpha^*I$. In that case, $\mathcal{F}_\lambda^{(k)}(0) \leq \mathcal{F}_\lambda^{(k)}(P_0) \leq \mathcal{F}_\lambda^{(k)}(z\alpha^*I)$, for all $k \geq 0$. To show the convergence of $\mathcal{F}_\lambda^{(k)}(P_0)$, it is enough to show that $\mathcal{F}_\lambda^{(k)}(z\alpha^*I)$ is convergent for any $z > 1$ since due to the uniqueness of the fixed point of \mathcal{F}_λ , if $\mathcal{F}_\lambda^{(k)}(z\alpha^*I)$ converges, it converges to the same point as $\mathcal{F}_\lambda^{(k)}(0)$. We know that $\mathcal{F}_\lambda^{(k)}(z\alpha^*I) \leq \mathcal{T}^{(k)}(z\alpha^*)I$. If \mathcal{T} is convergent, then, for any $e > 0$, there exists a k_0 such that $\mathcal{T}^{(k)}(z\alpha^*) \leq \alpha^* + e$ for all $k > k_0$. Let e be chosen such that $e < (z-1)\alpha^*$. Then, we will have $\mathcal{F}_\lambda^{(k)}(z\alpha^*I) \leq \mathcal{T}^{(k)}(z\alpha^*)I < z\alpha^*I$ for all $k > k_0$. Now, invoking the monotone property of \mathcal{F}_λ , we see that $\mathcal{F}_\lambda^{(k+k_0)}(z\alpha^*I) < \mathcal{F}_\lambda^{(k)}(z\alpha^*I)$ for every $k \geq 0$. Thus, $\mathcal{F}_\lambda^{(k)}(z\alpha^*I)$ is monotone decreasing after every k_0 iterations and since these iterations are lower bounded by the convergent $\mathcal{F}_\lambda^{(k)}(0)$, it can be seen that $\lim_{k \rightarrow \infty} \mathcal{F}_\lambda^{(k)}(z\alpha^*I) = \mathcal{Q}$. Since this was done for a general $z > 1$, it is valid for all $z > 1$. Recalling, $\mathcal{F}_\lambda^{(k)}(0) \leq \mathcal{F}_\lambda^{(k)}(P_0) \leq \mathcal{F}_\lambda^{(k)}(z\alpha^*I)$, we thus have $\lim_{k \rightarrow \infty} \mathcal{F}_\lambda^{(k)}(P_0) = \mathcal{Q}$ for any positive definite matrix P_0 . \square

We next study the convergence of the mapping \mathcal{T} towards a fixed point. To do so, we let $\bar{\alpha}$, and α^* , be the fixed points of the mappings f and \mathcal{T} respectively, that is,

$$f(\bar{\alpha}) - \bar{\alpha} = 0 \quad \text{and} \quad f(\alpha^*) - \alpha^* + \bar{\theta}_\lambda^G = 0.$$

By definition,

$$\begin{aligned} f(\alpha) - \alpha &= \frac{\lambda\alpha}{\lambda - 2\mu(A)} - \frac{\lambda\theta_\lambda^C \alpha^2}{\beta\alpha + 1} - \alpha \\ &= \alpha \left(\frac{(2\mu(A)\beta - \lambda(\lambda - 2\mu(A))\theta_\lambda^C)\alpha + 2\mu(A)}{(\lambda - 2\mu(A))(\beta\alpha + 1)} \right). \end{aligned}$$

It is easy to see that when the condition in (11) holds, there exists a unique fixed point $\bar{\alpha} = \frac{2\mu(A)}{\lambda(\lambda - 2\mu(A))\theta_\lambda^C - 2\mu(A)\beta}$ to f (and thus to \mathcal{T}) in the interval $(0, \infty)$. We next observe that

- For $\alpha > \bar{\alpha}$, we have $f(\alpha) < \alpha$,

- For $0 < \alpha \leq \bar{\alpha}$, we have $f(\alpha) \geq \alpha$.

With these properties of the mapping f , we can now study the convergence of the sequence obtained by successive iterations of the mapping \mathcal{T} .

Lemma 2. For a given $\alpha_0 \geq 0$, consider the sequence $\alpha_k = \mathcal{T}^{(k)}(\alpha_0)$, for $k \in \mathbb{N}$. If the inequality (11) holds, then

$$\lim_{k \rightarrow \infty} \mathcal{T}^{(k)}(\alpha_0) = \alpha^*.$$

Proof of Lemma 2. The proof is carried out in three parts. We examine the evolution of $\mathcal{T}^{(k)}(\alpha_0)$, $k \geq 0$ and show the following:

- for every $\alpha_0 \in (0, \bar{\alpha}]$, there exists \bar{k} such that $\mathcal{T}^{\bar{k}}(\alpha_0) > \bar{\alpha}$;
- the set $(\bar{\alpha}, \infty)$ is forward invariant under the mapping \mathcal{T} , that is, if $\alpha_0 \in (\bar{\alpha}, \infty)$ then $\mathcal{T}^{(k)}(\alpha_0) \in (\bar{\alpha}, \infty)$ for all $k \geq 0$;
- the mapping \mathcal{T} restricted to $(\bar{\alpha}, \infty)$ is contractive.

The convergence of $\mathcal{T}^{(k)}$ clearly follows from the above three properties. To show these properties, we first note the monotonicity of the operator \mathcal{T} in its arguments.

$$\begin{aligned} \mathcal{T}(a) - \mathcal{T}(b) &= \left(\frac{\lambda a}{\lambda - 2\mu(A)} - \frac{\lambda\theta_\lambda^C a^2}{\beta a + 1} \right) - \left(\frac{\lambda b}{\lambda - 2\mu(A)} - \frac{\lambda\theta_\lambda^C b^2}{\beta b + 1} \right) \\ &= \frac{\lambda(a-b)}{\lambda - 2\mu(A)} - \lambda\theta_\lambda^C \left(\frac{a^2}{\beta a + 1} - \frac{b^2}{\beta b + 1} \right) \\ &= \left(\frac{\lambda}{\lambda - 2\mu(A)} - \frac{\lambda\theta_\lambda^C(\beta ab + a + b)}{(\beta a + 1)(\beta b + 1)} \right) (a - b) \end{aligned} \quad (19)$$

Let $\rho = \left(\frac{\lambda}{\lambda - 2\mu(A)} - \frac{\lambda\theta_\lambda^C(\beta ab + a + b)}{(\beta a + 1)(\beta b + 1)} \right)$. Then we have

$$\mathcal{T}(a) - \mathcal{T}(b) = \rho(a - b).$$

To prove the monotonicity of the operator \mathcal{T} in its arguments what remains to show is that $\rho > 0$ for all $a, b \geq 0$. For this, we first show that $\theta_\lambda^C \leq \frac{\beta}{\lambda - 2\mu(A)}$. Notice that

$$\begin{aligned} \int_0^\infty \mathcal{E}_t^\lambda C^\top V^{-1} C \mathcal{E}_t^{\lambda^\top} dt &\leq \lambda_{\max}(V^{-1}) \int_0^\infty \mathcal{E}_t^\lambda C^\top C \mathcal{E}_t^{\lambda^\top} dt \\ &= \frac{1}{\lambda_{\min}(V)} \int_0^\infty \mathcal{E}_t^\lambda C^\top C \mathcal{E}_t^{\lambda^\top} dt \\ &\stackrel{(a)}{\leq} \frac{\beta}{\lambda_{\max}(CC^\top)} \int_0^\infty \mathcal{E}_t^\lambda C^\top C \mathcal{E}_t^{\lambda^\top} dt \\ &\leq \frac{\beta \lambda_{\max}(C^\top C)}{\lambda_{\max}(CC^\top)} \int_0^\infty \mathcal{E}_t^\lambda \mathcal{E}_t^{\lambda^\top} dt \\ &\leq \frac{\beta}{\lambda - 2\mu(A)} I, \end{aligned}$$

where inequality (a) holds since $CC^\top \leq \beta V$. Now since $\theta_\lambda^C \leq \int_0^\infty \mathcal{E}_t^\lambda C^\top V^{-1} C \mathcal{E}_t^{\lambda^\top} dt$, we have that the inequality $\theta_\lambda^C \leq \frac{\beta}{\lambda - 2\mu(A)}$ holds. Now using the same we have,

$$\begin{aligned} \rho &= \left(\frac{\lambda}{\lambda - 2\mu(A)} - \frac{\lambda\theta_\lambda^C(\beta ab + a + b)}{(\beta a + 1)(\beta b + 1)} \right) \\ &\geq \left(\frac{\lambda}{\lambda - 2\mu(A)} - \frac{\lambda\beta(\beta ab + a + b)}{(\lambda - 2\mu(A))(\beta a + 1)(\beta b + 1)} \right) \\ &= \frac{\lambda}{\lambda - 2\mu(A)} \left(\frac{1}{(\beta a + 1)(\beta b + 1)} \right) \\ &> 0. \end{aligned}$$

This shows that for all $a \geq b$, $\mathcal{T}(a) \geq \mathcal{T}(b)$. Putting $b = \mathcal{T}(a)$, we can easily arrive at the fact that if $\mathcal{T}(a) \geq a$,

then $\mathcal{T}^{(k+1)}(a) \geq \mathcal{T}^{(k)}(a)$ for all $k \in \mathbb{N}$ and vice versa when the inequalities are reversed.

First part: We now look at the first part of the proof to show that for every $\alpha_0 \in [0, \bar{\alpha}]$, there exists \bar{k} such that $\mathcal{T}^{\bar{k}}(\alpha_0) > \bar{\alpha}$. Using the properties of f above, we have that for any $\alpha_0 \in (0, \bar{\alpha}]$, we have $f(\alpha_0) \geq \alpha_0$ and consequently, $\mathcal{T}(\alpha_0) > \alpha_0$. Moreover, note that for this choice of α_0 ,

$$\begin{aligned} \mathcal{T}(\alpha_0) - \alpha_0 &= f(\alpha_0) - \alpha_0 + \bar{\theta}_\lambda^G \\ &\geq \bar{\theta}_\lambda^G, \end{aligned}$$

which shows that $\mathcal{T}^{(k+1)}(\alpha_0) > \mathcal{T}^{(k)}(\alpha_0)$ for all $k \in \mathbb{N}$ and also that whenever $\mathcal{T}^{(k)}(\alpha_0) \leq \bar{\alpha}$, $\mathcal{T}^{(k+1)}(\alpha_0) - \mathcal{T}^{(k)}(\alpha_0) \geq \bar{\theta}_\lambda^G$. Now using contradiction we can show that there exists \bar{k} such that $\mathcal{T}^{(\bar{k})}(\alpha_0) > \bar{\alpha}$. Let us assume that for all $k > 0$, $\mathcal{T}^{(k)}(\alpha_0) \leq \bar{\alpha}$, then using a telescopic sum on the above inequality implies that $\mathcal{T}^{(k+1)}(\alpha_0) - \alpha_0 \geq (k+1)\bar{\theta}_\lambda^G$, which is an increasing function in k . $\bar{\alpha}$ being finite, the above implies that there has to exist a \bar{k} for which $\mathcal{T}^{(\bar{k})}(\alpha_0) > \bar{\alpha}$ which is a contradiction and thus, the sequence $\mathcal{T}^{(k)}(\alpha_0)$ cannot be in the region $(0, \bar{\alpha}]$ for all k and has to escape into $(\bar{\alpha}, \infty)$ at some iteration.

Second part: We move to the next part of the proof where we show that there does not exist an iteration for any $k \in \mathbb{N}$ and some $\alpha_0 \in (0, \infty)$, such that $\mathcal{T}^{(k)}(\alpha_0) > \bar{\alpha}$ and $\mathcal{T}^{(k+1)}(\alpha_0) \leq \bar{\alpha}$ hold. The proof is simple thanks to the monotonicity property of \mathcal{T} . $\mathcal{T}^{(k_0)}(\alpha_0) > \bar{\alpha}$ and $\mathcal{T}^{(k_0+1)}(\alpha_0) \leq \bar{\alpha}$ for some $k_0 \in \mathbb{N}$ implies that $\mathcal{T}^{(k+1)}(\alpha_0) < \mathcal{T}^{(k)}(\alpha_0)$ for all $k \in \mathbb{N}$ using the monotonicity of \mathcal{T} . However, we know from the proof of the previous part that whenever $\mathcal{T}^{(k)}(\alpha_0) \leq \bar{\alpha}$, $\mathcal{T}^{(k+1)}(\alpha_0) > \mathcal{T}^{(k)}(\alpha_0)$. This would mean that $\mathcal{T}^{(k_0)}(\alpha_0) > \mathcal{T}^{(k_0+1)}(\alpha_0) < \mathcal{T}^{(k_0+2)}(\alpha_0)$, which is not possible since \mathcal{T} is monotonous and this completes the proof to show that once the sequence $\mathcal{T}^{(k)}(\alpha_0)$, $k \in \mathbb{N}$, $\alpha_0 \in (0, \infty)$ is in the region $(\bar{\alpha}, \infty)$, all future iterations are restricted to the same region.

Third part: Now we proceed to the final part of the proof to show that the mapping \mathcal{T} restricted to $(\bar{\alpha}, \infty)$, is contractive. Notice that

$$\begin{aligned} 1 - \rho &= \frac{-2\mu(A)}{\lambda - 2\mu(A)} + \frac{\lambda \underline{\theta}_\lambda^C (\beta ab + a + b)}{(\beta a + 1)(\beta b + 1)} \\ &= \frac{\left(\lambda(\lambda - 2\mu(A)) \underline{\theta}_\lambda^C - 2\mu(A)\beta \right) (\beta ab + a + b) - 2\mu(A)}{(\lambda - 2\mu(A))(\beta a + 1)(\beta b + 1)}, \end{aligned} \quad (20)$$

where, clearly the expression (20) takes positive values if and only if condition (11) holds, which also implies that $\lambda > 2\mu(A)$, thus making the denominator positive for all positive pairs a, b . Moreover, notice that for any $a, b \geq 0$, $\beta ab + a + b$ is an increasing function in both a and b and for some fixed a , the expression $\beta ab + a + b$ takes on the minimum value at $b = 0$. It is easy to show that for any $a > \bar{\alpha}$ and $b = 0$, the expression in the numerator is

positive and it remains positive for $a > \bar{\alpha}$ and $b > \bar{\alpha}$ due to the increasing nature of $\beta ab + a + b$. This implies that for any $a > \bar{\alpha}$ and $b > \bar{\alpha}$, $1 - \rho > 0$, or in other words $\rho < 1$. Thus, \mathcal{T} is a contraction mapping for $\alpha_0 > \bar{\alpha}$, using (19). \square

3.3. Proof of Theorem 1

Having the results on the existence, uniqueness and convergence to a fixed point of operator \mathcal{F}_λ , we prove Theorem 1 by relating the solution (6) to the evolution of the operator \mathcal{F}_λ . From Lemma 3 it is possible to rewrite (10) as

$$\begin{aligned} \phi_t(Q) &= \mathcal{E}_t^\lambda Q \mathcal{E}_t^{\lambda^\top} + \int_0^t \mathcal{E}_{t-s}^\lambda \left[\lambda(I - \mathcal{K}_s C) \phi_s(Q) \right. \\ &\quad \left. + GG^\top \right] \mathcal{E}_{t-s}^{\lambda^\top} ds. \end{aligned} \quad (21)$$

Using the above, it is easy to see that a solution \bar{P} of (12) can be equivalently written as the solution of

$$P = \int_0^\infty \mathcal{E}_{t-s}^\lambda \left[\lambda(I - K_P C) P + GG^\top \right] \mathcal{E}_{t-s}^{\lambda^\top} ds. \quad (22)$$

Now, a fixed point Q of \mathcal{F}_λ , satisfies $\mathcal{F}_\lambda(Q) = Q$ which implies the existence of a solution to (22). We define the sequence $P_t^{(k)}$, $k = 0, 1, \dots$, $t \geq 0$ with $P_t^{(0)} = 0$

$$\begin{aligned} P_t^{(k+1)} &= \mathcal{E}_t^\lambda Q \mathcal{E}_t^{\lambda^\top} + \int_0^t \mathcal{E}_{t-s}^\lambda \left[\lambda(I - \mathcal{K}_s C) P_s^{(k)} \right. \\ &\quad \left. + GG^\top \right] \mathcal{E}_{t-s}^{\lambda^\top} ds. \end{aligned} \quad (23)$$

Now following the exact methodology as in [17, Proof of Lemma 3.1 (ii)] we can relate $\phi_t(Q)$ to $P_t^{(k)}$ as $\lim_{k \rightarrow \infty} P_t^{(k)} = \phi_t(Q)$. Now in the limit $t \rightarrow \infty$, notice that the first term on the right hand side of (23) goes to 0 (owing to $\lambda > 2\mu(A)$) and the second term takes the exact form as \mathcal{F}_λ :

$$\begin{aligned} P^{(k+1)} &:= \lim_{t \rightarrow \infty} P_t^{(k+1)} \\ &= \int_0^\infty \mathcal{E}_{t-s}^\lambda \left[\lambda(I - K_{P^{(k)}} C) P^{(k)} + GG^\top \right] \mathcal{E}_{t-s}^{\lambda^\top} ds \\ &= \mathcal{F}_\lambda(P^{(k)}). \end{aligned}$$

Thus we have $\lim_{t \rightarrow \infty} \phi_t(Q) = \lim_{k \rightarrow \infty} \mathcal{F}_\lambda^{(k)}(Q)$. This ends the proof of the theorem.

3.4. Approximating the lower bound in (A1)

We provide an algorithm to compute the range of λ satisfying the condition (11) by numerically constructing a lower bound on $\underline{\theta}_\lambda^C$. In general, we do not know how $\underline{\theta}_\lambda^C$ depends on the sampling rater λ which makes it difficult to describe the range of λ for which (11) holds. The procedure outlined in this section will provide a lower bound on $\underline{\theta}_\lambda^C$ as an explicit function of λ . We will refer to such an approximation as $\underline{\theta}_\lambda^{C^*}$ and describe the procedure to compute it in Algorithm 1.

Algorithm 1 Computing $\underline{\theta}_\lambda^{C^*}$

Input: $\delta \in \mathbb{R}_{>0}$, matrices A, C, G, V .

Output: $\underline{\theta}_\lambda^{C^*}$, where $\underline{\theta}_\lambda^{C^*} I \leq \int_0^\infty \mathcal{E}_t^\lambda C^\top V^{-1} C \mathcal{E}_t^\lambda dt$ for all $\lambda \geq 2\mu(A) + \delta$

- 1: Compute $M_{\lambda^*} = \int_0^\infty \mathcal{E}_t^\lambda C^\top V^{-1} C \mathcal{E}_t^\lambda dt$ with $\lambda^* = 2\mu(A) + \delta$ and calculate its minimum eigenvalue $m_{\lambda^*} := \lambda_{\min}(M_{\lambda^*})$.
 - 2: Compute L^* , where L^* is the smallest positive integer such that $\int_0^{L^*} e^{(\frac{\lambda^*}{2}I-A)t} C^\top V^{-1} C e^{(\frac{\lambda^*}{2}I-A^\top)t} dt \geq m_{\lambda^*} I$.
 - 3: Choose the pair K, L such that $K \leq m_{\lambda^*}$ and $L \geq L^*$.
 - 4: Compute the scalar c such that $c = \frac{1}{2} \log \left(\lambda_{\min}(e^{AL} e^{A^\top L}) \right)$
 - 5: Set $\underline{\theta}_\lambda^{C^*} = K e^{-L(\lambda-2c)}$ for all $\lambda \geq 2\mu(A) + \delta$.
-

Proposition 3. For some $\delta > 0$ and $\lambda \geq 2\mu(A) + \delta$, the constant $\underline{\theta}_\lambda^{C^*}$ obtained from Algorithm 1 satisfies

$$\underline{\theta}_\lambda^{C^*} I \leq \int_0^\infty \mathcal{E}_t^\lambda C^\top V^{-1} C \mathcal{E}_t^\lambda dt, \quad \forall \lambda \geq 2\mu(A) + \delta. \quad (24)$$

Proof. Algorithm 1 yields $\underline{\theta}_\lambda^{C^*} = K e^{-L(\lambda-2c)}$, for some choice of K, L , and c . For $\lambda^* = 2\mu(A) + \delta$, $\underline{\theta}_{\lambda^*}^{C^*}$ takes the value $K e^{-2L(\mu(A)-c+\delta)}$. Notice that due to the choice $K < \lambda_{\min}(M_{\lambda^*})$, we have that $\underline{\theta}_{\lambda^*}^{C^*} < \underline{\theta}_{\lambda^*}^C$ for $\lambda = \lambda^*$. In what follows, we will show that by choosing L and c as in Algorithm 1, we have $\underline{\theta}_\lambda^{C^*} I \leq \int_0^\infty \mathcal{E}_t^\lambda C^\top V^{-1} C \mathcal{E}_t^\lambda dt$ for all $\lambda \geq 2\mu(A) + \delta$. Towards this end, we note that, for every $\lambda \geq 2\mu(A) + \delta$, we have

$$\begin{aligned} \frac{\underline{\theta}_\lambda^C}{\underline{\theta}_\lambda^{C^*}} I &\geq \frac{e^{L(\lambda-2\lambda_{\min}(A))}}{m_{\lambda^*}} \int_0^\infty \mathcal{E}_t^\lambda C^\top V^{-1} C \mathcal{E}_t^\lambda dt \\ &= \frac{1}{m_{\lambda^*}} \left(\int_0^\infty e^{L(\frac{\lambda}{2}-c)} \mathcal{E}_t^\lambda C^\top V^{-1} C \mathcal{E}_t^\lambda e^{L(\frac{\lambda}{2}-c)} dt \right) \\ &\stackrel{(a)}{\geq} \frac{1}{m_{\lambda^*}} \left(\lambda_{\min} \left(\int_0^L e^{(\frac{\lambda}{2}I-A)t} C^\top V^{-1} C e^{(\frac{\lambda}{2}I-A^\top)t} dt \right) \right. \\ &\quad \left. + \int_L^\infty e^{L(\frac{\lambda}{2}-c)} \mathcal{E}_t^\lambda C^\top V^{-1} C \mathcal{E}_t^\lambda e^{L(\frac{\lambda}{2}-c)} dt \right) \end{aligned}$$

To show how inequality (a) holds, note that

$$\begin{aligned} &\int_0^L e^{L(\frac{\lambda}{2}-c)} \mathcal{E}_t^\lambda C^\top V^{-1} C \mathcal{E}_t^\lambda e^{L(\frac{\lambda}{2}-c)} dt \\ &= \int_0^L e^{L(\frac{\lambda}{2}-c)} e^{(A-\frac{\lambda}{2}I)(L-t)} C^\top V^{-1} C e^{(A^\top-\frac{\lambda}{2}I)(L-t)} e^{L(\frac{\lambda}{2}-c)} dt \\ &= e^{(A-cI)L} \left(\int_0^L e^{(\frac{\lambda}{2}I-A)t} C^\top V^{-1} C e^{(\frac{\lambda}{2}I-A^\top)t} dt \right) e^{(A^\top-cI)L} \\ &\geq \left(e^{(A-cI)L} e^{(A^\top-cI)L} \right) \lambda_{\min} \left(\int_0^L e^{(\frac{\lambda}{2}I-A)t} C^\top V^{-1} C e^{(\frac{\lambda}{2}I-A^\top)t} dt \right) \end{aligned}$$

Now given the choice of c in Algorithm 1, it readily follows that $\left(e^{(A-cI)L} e^{(A^\top-cI)L} \right) \geq e^{-2c} \left(e^{AL} e^{A^\top L} \right) \geq e^{-2c} \lambda_{\min} \left(e^{AL} e^{A^\top L} \right) \geq I$. As a result,

$$\begin{aligned} &\int_0^L e^{L(\frac{\lambda}{2}-c)} \mathcal{E}_t^\lambda C^\top V^{-1} C \mathcal{E}_t^\lambda e^{L(\frac{\lambda}{2}-c)} dt \\ &\geq \lambda_{\min} \left(\int_0^L e^{(\frac{\lambda}{2}I-A)t} C^\top V^{-1} C e^{(\frac{\lambda}{2}I-A^\top)t} dt \right), \end{aligned}$$

and proceeding from inequality (a), we have

$$\begin{aligned} \frac{\underline{\theta}_\lambda^C}{\underline{\theta}_\lambda^{C^*}} &\geq \frac{1}{m_{\lambda^*}} \lambda_{\min} \left(\int_0^L e^{(\frac{\lambda}{2}I-A)t} C^\top V^{-1} C e^{(\frac{\lambda}{2}I-A^\top)t} dt \right) \\ &\stackrel{(b)}{\geq} \frac{1}{m_{\lambda^*}} \lambda_{\min} \left(\int_0^L e^{(\frac{\lambda}{2}I-A)t} C^\top V^{-1} C e^{(\frac{\lambda}{2}I-A^\top)t} dt \right) \\ &\geq I, \end{aligned}$$

where inequality (b) holds since

$$\int_0^L e^{(\frac{\lambda}{2}I-A)t} C^\top V^{-1} C e^{(\frac{\lambda}{2}I-A^\top)t} dt$$

is clearly an increasing function in λ . The foregoing inequalities, hence, show that $\underline{\theta}_\lambda^C \geq \underline{\theta}_\lambda^{C^*}$. \square

Remark 2. In step 2 of Algorithm 1 existence of an L^* is guaranteed from the fact that for any $\lambda > 2\mu(A)$, $(\frac{\lambda}{2}I-A)$ is not Hurwitz and as a result as L^* becomes larger, the integral starts growing and eventually explodes as $L^* \rightarrow \infty$. Thus, there always exists a particular value of L^* at which the integral goes beyond the finite valued m_{λ^*} .

Algorithm 1 thus provides us an approximate by performing computations just with one specific value of λ . Choosing a λ^* arbitrarily close to $2\mu(A)$ will lead to $m_{\lambda^*} \geq \underline{\theta}_{\lambda^*}^C$ for all $\lambda \geq 2\mu(A)$. Using this $\underline{\theta}_{\lambda^*}^{C^*}$, condition (11) can be used to compute an exact range of λ which guarantees the existence of a fixed point of \mathcal{F}_λ . This can be simply done by solving for values of λ which satisfy

$$\lambda(\lambda - 2\mu(A)) K e^{-(\lambda-2c)L} > 2\mu(A)\beta, \quad (25)$$

where K, L, C are computed using Algorithm 1. The values of the mean sampling rate which satisfy the above inequality above lie within a range

$$\underline{\lambda} < \lambda < \bar{\lambda},$$

where $\underline{\lambda}$ and $\bar{\lambda}$ are the roots of

$$\lambda(\lambda - 2\mu(A)) K e^{-(\lambda-2c)L} = 2\mu(A)\beta.$$

In the scalar case, ($n = 1$), when all the matrices A, C, G, V are scalar, $\underline{\theta}_\lambda^C$ can be directly computed as

$\frac{\theta_\lambda^C}{\lambda} = \frac{\beta}{\lambda - 2A}$. Thus, the sufficient condition for convergence can be expressed as

$$\lambda(\lambda - 2A) \frac{\beta}{\lambda - 2A} > 2A\beta$$

or, $\lambda > 2A$ (in the scalar case),

which is also a necessary condition for the convergence as shown earlier. Thus in the scalar case, $\lambda > 2A$ is a necessary and sufficient condition for the existence of a fixed point for the expected error covariance equation.

4. Bounds using continuous-time Riccati flows

We perform a comparison analysis of the expected error covariance $\phi_t(Q)$ with that of standard Kalman-Bucy filters with the same (A, C, G) but with the covariance scaled by a constant τ so that the system is represented by $(A, C, G, \tau V)$. We denote the error covariance flow associated with the standard Kalman Bucy error covariance $(\psi_t^\tau(Q))$ for a continuous time system described by $(A, C, G, \tau V)$ and with initial error covariance Q as:

$$\begin{aligned} \partial_t \psi_t^\tau(Q) &= A\psi_t^\tau(Q) + \psi_t^\tau(Q)A^\top + GG^\top \\ &\quad - \frac{\psi_t^\tau(Q)C^\top V^{-1}C\psi_t^\tau(Q)}{\tau} \end{aligned} \quad (26)$$

4.1. Lower bound

We can compute a lower bound on $\phi_t(Q)$ in terms of $\psi_t^\tau(Q)$ as stated in the following result:

Proposition 4. *For any $Q \in \mathbb{S}_+^{n \times n}$ and $\tau \leq \frac{1}{\lambda}$, it holds that $\phi_t(Q) \geq \psi_t^\tau(Q)$ for all $t \geq 0$.*

Proof. Notice that,

$$\begin{aligned} &\partial_t(\phi_t(Q) - \psi_t^\tau(Q)) \\ &= A\phi_t(Q) + \phi_t(Q)A^\top - \lambda\phi_t(Q)C^\top(C\phi_t(Q)C^\top + V)^{-1}C\phi_t(Q) \\ &\quad - A\psi_t^\tau(Q) - \psi_t^\tau(Q)A^\top + \frac{\psi_t^\tau(Q)C^\top V^{-1}C\psi_t^\tau(Q)}{\tau} \\ &\geq A\phi_t(Q) + \phi_t(Q)A^\top - \lambda\phi_t(Q)C^\top(V)^{-1}C\phi_t(Q) \\ &\quad - A\psi_t^\tau(Q) - \psi_t^\tau(Q)A^\top + \frac{\psi_t^\tau(Q)C^\top V^{-1}C\psi_t^\tau(Q)}{\tau} \\ &\stackrel{(a)}{\geq} A\phi_t(Q) + \phi_t(Q)A^\top - \lambda\phi_t(Q)C^\top(V)^{-1}C\phi_t(Q) \\ &\quad - A\psi_t^\tau(Q) - \psi_t^\tau(Q)A^\top + \lambda\psi_t^\tau(Q)C^\top V^{-1}C\psi_t^\tau(Q) \\ &\stackrel{(b)}{=} \left(A - \frac{\phi_t(Q) + \psi_t^\tau(Q)}{2} \lambda C^\top V^{-1} C \right) (\phi_t(Q) - \psi_t^\tau(Q)) \\ &\quad + (\phi_t(Q) - \psi_t^\tau(Q)) \left(A - \frac{\phi_t(Q) + \psi_t^\tau(Q)}{2} \lambda C^\top V^{-1} C \right)^\top, \end{aligned}$$

where inequality (a) holds by using $\tau \leq \frac{1}{\lambda}$ and equality (b) holds by using the following identity:

$$\begin{aligned} &MSM - NSN \\ &= \left(\frac{M+N}{2} S \right) (M-N) + (M-N) \left(S \frac{M+N}{2} \right), \end{aligned} \quad (27)$$

where M, N and S are all symmetric and positive semidefinite matrices of appropriate dimension. Thus,

$$\begin{aligned} \phi_t(Q) - \psi_t^\tau(Q) &\geq e^{\int_0^t M_s ds} (\phi_0(Q) - \psi_0^\tau(Q)) e^{\int_0^t M_s^\top ds} \\ &= 0, \end{aligned} \quad (28)$$

where $M_t = \left(A - \frac{\phi_t(Q) + \psi_t^\tau(Q)}{2} \lambda C^\top V^{-1} C \right)$ for $Q \in \mathbb{S}_+^{n \times n}$. \square

Let us define the controllability and observability Gramians associated with ψ_t^τ as

$$\begin{aligned} \mathcal{C}(t, t-t_0) &:= \int_{t-t_0}^t e^{As} (GG^\top) e^{A^\top s} ds, \\ \mathcal{O}^\tau(t, t-t_0) &:= \int_{t-t_0}^t e^{As} (C^\top (\tau V)^{-1} C) e^{A^\top s} ds \end{aligned}$$

respectively, where the superscript τ is to remind the reader that the Gramians depend on the particular choice of τ . If the continuous-time linear system defined by $(A, C, G, \tau V)$, $\tau \leq \frac{1}{\lambda}$ is uniformly completely controllable and uniformly completely observable, it means that there exists a $t_0 \geq 0$ such that for all $t \geq t_0$, we have

$$0 < \underline{\gamma}_C I \leq \mathcal{C}(t, t-t_0) \leq \bar{\gamma}_C I, \quad (29a)$$

$$0 < \underline{\gamma}_O^\tau I \leq \mathcal{O}^\tau(t, t-t_0) \leq \bar{\gamma}_O^\tau I, \quad (29b)$$

respectively. Note that the t_0 need not be the same for both the controllability and observability Gramians. However, here for the ease of notation we consider t_0 as the maximum among the two for \mathcal{C} and \mathcal{O}^τ respectively. Then, we have the following result.

Proposition 5. *Suppose that there exist positive scalars $\underline{\gamma}_C, \bar{\gamma}_C, \underline{\gamma}_O^\tau, \bar{\gamma}_O^\tau$, and t_0 such that (29) is satisfied. Then, for every $Q \in \mathbb{S}_+^{n \times n}$ and any $\tau \leq \frac{1}{\lambda}$, the following inequality holds for $t \geq 0$:*

$$\begin{aligned} \phi_t(Q) &\geq (C^{-1}(t, t-t_0) + \mathcal{O}^\tau(t, t-t_0))^{-1} \\ &\geq \left(\frac{\underline{\gamma}_C}{1 + \underline{\gamma}_C \bar{\gamma}_O^\tau} \right) I. \end{aligned} \quad (30)$$

Proof. We know from [9, Lemma 7.2] that given uniform complete controllability and observability that the error covariance flow ψ_t^τ for a continuous-time linear system described by system parameters $(A, C, G, \tau V)$ satisfies

$$\psi_t^\tau(Q) \geq (C^{-1}(t, t-t_0) + \mathcal{O}^\tau(t, t-t_0))^{-1} \geq \left(\frac{\underline{\gamma}_C}{1 + \underline{\gamma}_C \bar{\gamma}_O^\tau} \right) I,$$

for all initial $Q \in \mathbb{S}_+^{n \times n}$ (see [9, Chapter 7] for the proof). From Proposition 4 we know that for all $\tau \leq \frac{1}{\lambda}$, we have $\phi_t(Q) \geq \psi_t^\tau(Q)$ for any initial $Q \in \mathbb{S}_+^{n \times n}$ and $t \geq 0$. \square

4.2. Upper bound

To get the upper bound on the flow of expected error covariance, we consider $\alpha = \alpha^* + \delta$ for some $\delta > 0$ and α^* being the fixed point of the mapping \mathcal{T} introduced in (18). We let $\Lambda_\alpha := \max_{t \geq 0} \lambda_{\max}(\phi_t(\alpha I))$ and let K_α be such that $C\Lambda_\alpha C^\top \leq K_\alpha V$ and $L_\alpha := \frac{\lambda}{1+K_\alpha}$.

Proposition 6. For every $Q \in \mathbb{S}_+^{n \times n}$ and $\tau > \frac{1}{L_\alpha}$, there exists $t_Q > 0$ and $Q_1 \in \mathbb{S}_+^{n \times n}$ such that

$$\phi_t(Q) \leq \psi_t^\tau(Q_1), \quad \text{for } t \geq t_Q.$$

In particular, Q_1 is chosen such that $\psi_{t_Q}^\tau(Q_1) \geq \Lambda_\alpha I$.

Proof. We will proceed with the proof as follows: first, we define a set of matrices for which we show that irrespective of the initial expected error covariance Q , there always exists a $t = t_Q > 0$ at which $\phi_t(Q) \in \mathbb{Q}$ where \mathbb{Q} is a set that will be defined soon. Second, we show that all flows starting from \mathbb{Q} can be upper bounded by a continuous-time Riccati flow for all times.

We start with the first part of the proof here. Recall the operator \mathcal{T} from Section 3.2. Let us define the set $\mathbb{Q} := \{P \in \mathbb{S}_+^{n \times n} : P \leq \alpha^* I\}$ and the fixed point of ϕ_t as \mathcal{P}_∞ . We know that $\lim_{t \rightarrow \infty} \phi_t(Q) = \mathcal{P}_\infty \leq \alpha^* I < \alpha I$ for any initial $Q \in \mathbb{S}_+^{n \times n}$. Considering the evolution of the real part of the maximum eigenvalue of $\phi_t(Q)$ we see that $\lim_{t \rightarrow \infty} \lambda_{\max}(\phi_t(Q)) < \alpha$. As a result, there exists a time $t = t_Q$ at which $\lambda_{\max}(\phi_t(Q)) < \alpha$ occurs for the first time. As a result at $t = t_Q$ we have $\phi_t(Q) < \alpha I$ occurring for the first time too. It is easy to see that $\phi_{t_Q}(Q) \in \mathbb{Q}$. Thus, starting from any $Q \in \mathbb{S}_+^{n \times n}$ there exists a time at which the flow enters the set \mathbb{Q} .

For the second part, we start with the known fact from the monotonicity property (Proposition 1) that $Q_1 \leq Q_2$ implies that $\phi_t(Q_1) \leq \phi_t(Q_2)$ for all times $t \geq 0$. Thus for all matrices $P \in \mathbb{Q}$ it holds that $\phi_t(P) \leq \phi_t(\alpha I)$ for all $t \geq 0$. The flow $\phi_t(\alpha I)$ is continuous and converges to \mathcal{P}_∞ . Thus, $\Lambda_\alpha = \max_{t \geq 0} \lambda_{\max}(\phi_t(\alpha I))$ exists and is finite.

We then have that $\phi_t(P) \leq \phi_t(\alpha I) \leq \Lambda_\alpha I$ for all $t \geq 0$ and for a general $Q \in \mathbb{S}_+^{n \times n}$, using the first part of the proof, the same can be re-written as $\phi_t(Q) \leq \Lambda_\alpha I$ holds for all $t \geq t_Q$. Now, notice that for all $t \geq t_Q$ and some initial $Q_1 \in \mathbb{S}_+^{n \times n}$ for the continuous-time Riccati flow, which we will talk more about later, we have

$$\begin{aligned} & \partial_t(\psi_t^\tau(Q_1) - \phi_t(Q)) \\ &= A\psi_t^\tau(Q_1) + \psi_t^\tau(Q_1)A^\top - \frac{\psi_t^\tau(Q_1)C^\top V^{-1}C\psi_t^\tau(Q_1)}{\tau} \\ & \quad - A\phi_t(Q) - \phi_t(Q)A^\top + \lambda\phi_t(Q)C^\top(C\phi_t(Q)C^\top + V)^{-1}C\phi_t(Q) \\ & \stackrel{(a)}{\geq} A\psi_t^\tau(Q_1) + \psi_t^\tau(Q_1)A^\top - \frac{\psi_t^\tau(Q_1)C^\top V^{-1}C\psi_t^\tau(Q_1)}{\tau} \\ & \quad - A\phi_t(Q) - \phi_t(Q)A^\top + \lambda\phi_t(Q)C^\top(C\Lambda_\alpha C^\top + V)^{-1}C\phi_t(Q) \\ & \stackrel{(b)}{\geq} A\psi_t^\tau(Q_1) + \psi_t^\tau(Q_1)A^\top - L_\alpha\psi_t^\tau(Q_1)C^\top V^{-1}C\psi_t^\tau(Q_1) \\ & \quad - A\phi_t(Q) - \phi_t(Q)A^\top + L_\alpha\phi_t(Q)C^\top V^{-1}C\phi_t(Q) \\ & \stackrel{(c)}{=} \left(A - \frac{\phi_t(Q) + \psi_t^\tau(Q_1)}{2} L_\alpha C^\top V^{-1} C \right) (\psi_t^\tau(Q_1) - \phi_t(Q)) \\ & \quad + (\psi_t^\tau(Q_1) - \phi_t(Q)) \left(A - \frac{\phi_t(Q) + \psi_t^\tau(Q_1)}{2} L_\alpha C^\top V^{-1} C \right)^\top, \end{aligned}$$

where inequality (a) is an application of $\phi_t(Q) \leq \Lambda_\alpha I$ for all $t > t_Q$. Inequality (b) can be seen as follows: we

find the smallest scalar K_α such that $C\Lambda_\alpha C^\top \leq K_\alpha V$ and as a result we have $(C\Lambda_\alpha C^\top + V)^{-1} \geq \frac{V^{-1}}{K_\alpha + 1}$ and finally put $L_\alpha = \frac{\lambda}{K_\alpha + 1}$. Together with this we use $\tau > \frac{1}{L_\alpha}$ to get the desired expression. Equality (c) is simply the application of the identity (27). Next, integrating both sides, we can write for all $t \geq t_Q$

$$\begin{aligned} \psi_t^\tau(Q_1) - \phi_t(Q) & \geq e^{\int_{t_Q}^t N_s \, ds} (\psi_{t_Q}^\tau(Q_1) - \phi_{t_Q}(Q)) e^{\int_{t_Q}^t N_s^\top \, ds} \\ & \geq e^{\int_{t_Q}^t N_s \, ds} (\psi_{t_Q}^\tau(Q_1) - \alpha I) e^{\int_{t_Q}^t N_s^\top \, ds}, \end{aligned} \quad (31)$$

where $N_t = \left(A - \frac{\phi_t(Q) + \psi_t^\tau(Q_1)}{2} L_\alpha C^\top V^{-1} C \right)$ for all $t \geq t_Q$. Notice that for $\psi_t^\tau(Q_1) \geq \phi_t(Q)$ to hold for all times bigger than t_Q , all we need to ensure is $\psi_{t_Q}^\tau(Q_1) \geq \alpha I$ since that automatically implies $\psi_t^\tau(Q_1) \geq \phi_{t_Q}(Q)$. Using the backward Riccati flow as in [18, Section 5.7], we know that we can always find a \mathcal{Q} such that $\psi_{t_Q}^\tau(\mathcal{Q}) = \alpha I$ and choosing $Q_1 \geq \mathcal{Q}$ gives us $\psi_t^\tau(Q_1) - \phi_t(Q) \geq 0$ for all $t \geq t_Q$, which completes the proof. \square

Remark 3. From the proof of the Proposition 6, it is clear we could in practice define the set \mathbb{Q} in a similar manner with any matrix that upper bounds the fixed point of the flow, that is, using any $R > \mathcal{P}_\infty$. However, we use αI in relation to the fixed point of the operator \mathcal{T} since its computation is easy as seen in Section 3.2 and moreover because it allows computation of an upper bound of \mathcal{P}_∞ without actually computing \mathcal{P}_∞ which might not be so easy.

Thus, to derive the upper bounds, we work with $\tau \geq \frac{1}{L_\alpha}$ and to avoid any confusion, we consider the following bounds on observability Gramian \mathcal{O}^τ , for $\tau \leq \frac{1}{L_\alpha}$:

$$0 < \underline{\gamma}_O^\tau I \leq \mathcal{O}^\tau(t, t - t_0) \leq \bar{\gamma}_O^\tau I. \quad (32)$$

Using t_0 in (29a) and (32), and t_Q as in Proposition 6, we consider

$$\bar{t} := \max\{t_0, t_Q\}. \quad (33)$$

Proposition 7. Let $\tau \geq \frac{1}{L_\alpha}$ and suppose that there exist positive scalars $\underline{\gamma}_C, \bar{\gamma}_C, \underline{\gamma}_O^\tau, \bar{\gamma}_O^\tau$, and t_0 such that (29a) and (32) are satisfied. Then, for every $Q \in \mathbb{S}_+^{n \times n}$ and \bar{t} satisfying (33), the following inequality holds for $t \geq 0$:

$$\phi_t(Q) \leq [\mathcal{O}^\tau(t, t - t_0)]^{-1} + \mathcal{C}(t, t - t_0) \leq \left(\frac{1 + \underline{\gamma}_O^\tau \bar{\gamma}_C}{\underline{\gamma}_O^\tau} \right) I. \quad (34)$$

Proof. The proof is similar to the proof of Proposition 4. We can write

$$\psi_t^\tau(Q) \leq [\mathcal{O}^\tau(t, t - t_0)]^{-1} + \mathcal{C}(t, t - t_0) \leq \left(\frac{1 + \underline{\gamma}_O^\tau \bar{\gamma}_C}{\underline{\gamma}_O^\tau} \right) I,$$

for all initial $Q \in \mathbb{S}_+^{n \times n}$ (see [9, Chapter 7] for the proof). From Proposition 6 we know that for all $\tau \geq \frac{1}{L_\alpha}$, we have

$\phi_t(Q) \leq \psi_t^\tau(Q_1)$ for any initial $Q \in \mathbb{S}_+^{n \times n}$, for $Q_1 \geq Q$ and for all times $t \geq t_Q$. Now choose $\bar{t} = \max\{t_0, t_Q\}$ and we get the bound in (34) for $t \geq \bar{t}$. \square

Example 1. To demonstrate the upper and lower bounds derived in Proposition 4 and Proposition 6, we present a numerical example here. The system parameters are $A = \begin{pmatrix} -3 & 2 \\ 1 & 1 \end{pmatrix}$, $C = \begin{pmatrix} 0 & 1 \end{pmatrix}$, $G = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and $V = 0.1$. The mean sampling rate of the Poisson process is $\lambda = 4$. We simulate the expected error covariance (6) with the initial expected error covariance $Q = \begin{pmatrix} 1.58 & 0.7 \\ 0.7 & 6 \end{pmatrix}$. The trace of the expected error covariance is shown by the blue curve in Figure 1. We simulate the flow of the trace of the solution of continuous-time Riccati equation (26) with $\tau = 0.2 < \frac{1}{\lambda}$ for the lower bound; and $\tau = 5 > 1/L_\alpha > 4$ for the upper bound. The trace of the solution of Riccati equation associated with the lower and upper bounds are shown by red curves in the figure below.

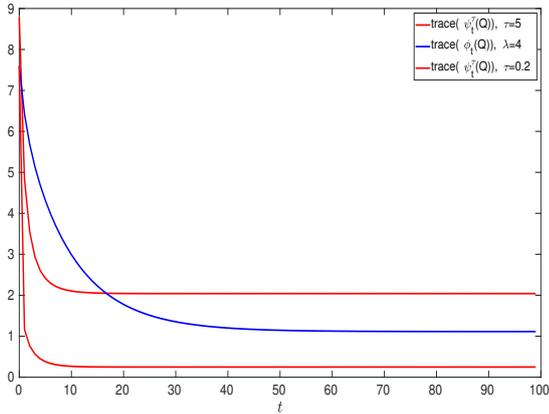


Figure 1: Time evolution of trace of the expected error covariance (blue) in comparison with the trace of the solution of Riccati equation (26), which provide the upper and lower bounds (red).

5. Fluctuation Analysis

The results from the preceding sections can now be used for analyzing the difference in the flow of error covariance matrices resulting from different initial conditions. Our main result in this direction is stated next:

Theorem 2. *Suppose that assumptions **(A2)**, **(A1)**, and condition (11) are satisfied. Let $\phi_t(Q_1), \phi_t(Q_2)$ denote the respective expected error covariance flows for $t \geq 0$, and $Q_1, Q_2 \in \mathbb{S}_+^{n \times n}$. Then there exists $m_{Q_2} > 0$ (depending on Q_2) such that the following holds*

$$\|\phi_t(Q_1) - \phi_t(Q_2)\| \leq \exp(-at) (1 + J_{Q_2} t) \|Q_1 - Q_2\|, \quad (35)$$

where $\|\cdot\|$ denotes the induced Euclidean norm, $a := 2\left(\frac{\lambda}{2} - \mu(A)\right)$ and $J_{Q_2} := \lambda m_{Q_2}^2 \exp\left(\frac{\lambda m_{Q_2}^2}{a}\right)$.

Proof. Recalling (15) from Proposition 2, we can write

$$\begin{aligned} \phi_t(Q_1) - \phi_t(Q_2) &\leq \mathcal{E}_t^\lambda(Q_1 - Q_2) \mathcal{E}_t^{\lambda \top} \\ &+ \lambda \int_0^t \mathcal{E}_{t-s}^\lambda M_{\phi_s(Q_2)} (\phi_s(Q_1) - \phi_s(Q_2)) M_{\phi_s(Q_2)}^\top \mathcal{E}_{t-s}^{\lambda \top} ds, \end{aligned}$$

where $M_{\phi_t(Q_2)} := (I - \phi_t(Q_2)C^\top(C\phi_t(Q_2)C^\top + V)^{-1}C)$. Introducing norm on both sides, the above inequality can be re-written as

$$\begin{aligned} \|\phi_t(Q_1) - \phi_t(Q_2)\| &\leq \|\mathcal{E}_t^\lambda\|^2 \|Q_1 - Q_2\| \\ &+ \lambda \int_0^t \|\mathcal{E}_{t-s}^\lambda\|^2 \|M_{\phi_s(Q_2)}\|^2 \|\phi_s(Q_1) - \phi_s(Q_2)\| ds. \end{aligned} \quad (36)$$

We would like to show that for any $Q_2 \in \mathbb{S}_+^{n \times n}$, it is possible to find a constant scalar m_{Q_2} such that $\|M_{\phi_t(Q_2)}\| \leq m_{Q_2}$, for $t \in [0, \infty)$. To do so, we use the fact that for all $t \in [\bar{t}, \infty)$ (with \bar{t} as in (33)), the flow $\phi_t(Q_2)$ is bounded uniformly by constants as in Propositions 5 and 7. As a result, there exists a certain matrix $R > 0$ (lying between the uniform upper and lower bounds established in Section 4) which maximizes the norm $\|M_{\phi_t(Q_2)}\| = \|(I - RC^\top(CRC^\top + V)^{-1}C)\|$. In other words, there exists a maximum for $\|M_{\phi_t(Q_2)}\|$ over $[\bar{t}, \infty)$. This maximum is indeed independent of the initial Q_2 for the upper and lower bounds defining the search space is uniform. It only remains to show that $\|M_{\phi_t(Q_2)}\|$ achieves a finite maximum in the interval $t \in [0, \bar{t})$, for then, the maxima over the entire range of $t \in [0, \infty)$ will simply be the maximum among the two finite maximum values obtained. The existence of the maximum over $[0, \bar{t}]$ can be proved from the facts that \bar{t} is finite and that the flow $\phi_t(Q_2)$ is continuous over a finite interval starting from Q_2 (see [10, Proposition 5.2] for the proof of the continuity of $\phi_t(Q)$ over finite intervals). Thus, $\phi_t(Q_2)$ must achieve a maximum over the finite interval $t \in [0, \bar{t}]$. Note that this maximum, in general, depends on the initial condition Q_2 . Thus, there exists a maximum m_{Q_2} such that $\|M_{\phi_t(Q_2)}\| \leq m_{Q_2}$ for all $t \geq 0$.

We know that $\|\mathcal{E}_t^\lambda\| \leq \exp\left(\left(\mu(A) - \frac{\lambda}{2}\right)t\right)$. Let us take $a := 2\left(\frac{\lambda}{2} - \mu(A)\right) > 0$. Then (36) can be written as

$$\begin{aligned} \|\phi_t(Q_1) - \phi_t(Q_2)\| &\leq \exp(-at) \|Q_1 - Q_2\| \\ &+ \lambda m_{Q_2}^2 \int_0^t \exp(-a(t-s)) \|\phi_s(Q_1) - \phi_s(Q_2)\| ds. \end{aligned}$$

Applying Grönwall's inequality to the above we have

$$\begin{aligned} \|\phi_t(Q) - \phi_t(Q_2)\| &\leq \exp(-at) \|Q_1 - Q_2\| \\ &+ \lambda m_{Q_2}^2 \exp(-at) \exp\left(\frac{\lambda m_{Q_2}^2}{a}\right) \|Q_1 - Q_2\| E_t(\lambda, m_{Q_2}, a), \end{aligned}$$

where¹ $E_t(\lambda, m_{Q_2}, a) = \int_0^t \exp\left(\frac{-\lambda m_{Q_2}^2 \exp(-a(t-s))}{a}\right) ds$.

¹Notice that it is possible to exactly compute the definite integral in the definition of $E_t(\lambda, m_{Q_2}, a)$. However, we do not write the exact expression here for brevity.

We note that $\exp\left(\frac{-\lambda m_{Q_2}^2 \exp(-a(t-s))}{a}\right) \leq 1$ (which holds since the exponent is always negative) and replace it by the same. On doing so, we obtain

$$\begin{aligned} \|\phi_t(Q) - \phi_t(Q_2)\| &\leq \exp(-at) \|Q_1 - Q_2\| \\ &+ \lambda m_{Q_2}^2 \exp(-at) \exp\left(\frac{\lambda m_{Q_2}^2}{a}\right) \|Q_1 - Q_2\| \int_0^t ds \\ &\leq \exp(-at) \|Q_1 - Q_2\| \left(1 + \lambda m_{Q_2}^2 \exp\left(\frac{\lambda m_{Q_2}^2}{a}\right) t\right). \end{aligned}$$

It is easy to see that in the limit as $t \rightarrow \infty$, we have the expression $\exp(-at) \left(1 + \left(\lambda m_{Q_2}^2 \exp\left(\frac{\lambda m_{Q_2}^2}{a}\right)\right) t\right) \rightarrow 0$, the quantity within the parantheses being linear in t , which is indeed the expected result. Setting $J_{Q_2} := \lambda m_{Q_2}^2 \exp\left(\frac{\lambda m_{Q_2}^2}{a}\right)$ completes the proof. \square

Remark 4. If we choose $Q_2 = \mathcal{P}_\infty$, $M_{\phi_t(Q_2)}$ is a constant matrix for all $t \geq 0$ and as a result, we can simply write $m_{Q_2} = \|(I - \mathcal{P}_\infty C^\top (C\mathcal{P}_\infty C^\top + V)^{-1}C)\|$.

6. Expected Estimation Error

Now, we turn our attention towards analysing the estimation error. The main result can be studied as follows:

Theorem 3. *Suppose that assumptions (A2) (A1) and the condition (11) are satisfied. Let $\underline{\gamma}_C, \bar{\gamma}_C$ be defined as in (29a), $\bar{\gamma}_O^\zeta$ as in (29b), and $\underline{\gamma}_O^\zeta$ be as in (32). Then, for every initial expected covariance $Q \in \mathbb{S}_+^{n \times n}$, there exists $\zeta > 0$ such that the time-varying matrix $(A - \lambda \mathcal{K}_t C)$ satisfies*

$$\begin{aligned} &\exp\left(\int_{t'}^t (A - \lambda \mathcal{K}_s C) ds\right) \\ &\leq \frac{\underline{\gamma}_C \bar{\gamma}_O^\zeta}{\left(1 + \underline{\gamma}_C \bar{\gamma}_O^\zeta\right) \left(1 + \underline{\gamma}_O^\zeta \bar{\gamma}_C\right)} \exp\left(-\frac{\zeta \underline{\gamma}_C}{1 + \underline{\gamma}_C \bar{\gamma}_O^\zeta} (t - t')\right) \end{aligned} \quad (37)$$

for each $t \geq t' \geq \bar{t}$.

Proof. We start by taking the derivative of the inverse of the flow matrix and observe that,

$$\partial_t \phi_t^{-1} = -\phi_t^{-1} (\partial_t \phi_t) \phi_t^{-1}. \quad (38)$$

Using the above, we have for any initial $Q \in \mathbb{S}_+^{n \times n}$,

$$\begin{aligned} \partial_t \phi_t^{-1}(Q) &= -\phi_t^{-1}(Q)A - A^\top \phi_t^{-1}(Q) - \phi_t^{-1}(Q)GG^\top \phi_t^{-1}(Q) \\ &+ \lambda C^\top (C\phi_t(Q)C^\top + V)^{-1}C. \end{aligned}$$

Notice that

$$\begin{aligned} \partial_t \phi_t(Q) &= (A - \lambda \mathcal{K}_t C)\phi_t(Q) + \phi_t(Q)(A - \lambda \mathcal{K}_t C)^\top + GG^\top \\ &+ \lambda \phi_t(Q)C^\top (C\phi_t(Q)C^\top + V)^{-1}C\phi_t(Q) \\ &\stackrel{(a)}{\geq} (A - \lambda \mathcal{K}_t C)\phi_t(Q) + \phi_t(Q)(A - \lambda \mathcal{K}_t C)^\top + GG^\top \\ &+ L_\alpha \phi_t(Q)C^\top V^{-1}C\phi_t(Q) \end{aligned} \quad (39)$$

where inequality (a) holds for all $t \geq t_Q$ (recalling Proposition 6) and the scalar L_α is introduced in Proposition 6. We let $S := L_\alpha C^\top V^{-1}C$ in the calculation that follows. Using (39) in (38), we have

$$\begin{aligned} \partial_t \phi_t^{-1}(Q) &\leq -\phi_t^{-1}(Q)(A - \lambda \mathcal{K}_t C) - (A - \lambda \mathcal{K}_t C)^\top \phi_t^{-1}(Q) \\ &- \phi_t^{-1}(Q)GG^\top \phi_t^{-1}(Q) - S. \end{aligned}$$

Rearranging the above, we have

$$\begin{aligned} \partial_t \phi_t^{-1}(Q) + \phi_t^{-1}(Q)(A - \lambda \mathcal{K}_t C) + (A - \lambda \mathcal{K}_t C)^\top \phi_t^{-1}(Q) \\ \leq -\phi_t^{-1}(Q)GG^\top \phi_t^{-1}(Q) - S. \end{aligned} \quad (40)$$

Now, using an argument similar as in the proof of [19, Theorem 4.8], we can write, for $t \geq t_Q$,

$$\begin{aligned} \partial_t (E_{s,t}^\top(Q) \phi_t^{-1}(Q) E_{s,t}(Q)) &= E_{s,t}^\top(Q) [\partial_t \phi_t^{-1}(Q) + \\ &\phi_t^{-1}(Q)(A - \lambda \mathcal{K}_t C) + (A - \lambda \mathcal{K}_t C)^\top \phi_t^{-1}(Q)] E_{s,t}(Q), \end{aligned} \quad (41)$$

where $E_{s,t}(Q) = \exp\left(\int_s^t (A - \lambda \mathcal{K}_r C) dr\right)$. Substituting (40) into (41) yields,

$$\begin{aligned} \partial_t (E_{s,t}^\top(Q) \phi_t^{-1}(Q) E_{s,t}(Q)) &\leq -E_{s,t}^\top(Q) [S + \\ &\phi_t^{-1}(Q)GG^\top \phi_t^{-1}(Q)] E_{s,t}(Q). \end{aligned}$$

Using the last inequality, we can write for all $t \geq t' \geq \bar{t}$,

$$\begin{aligned} E_{t',t}^\top(Q) \phi_t^{-1}(Q) E_{t',t}(Q) &\leq \phi_{t'}^{-1}(Q) - \\ &\int_{t'}^t E_{s,t}^\top(Q) (\phi_s^{-1}(Q)GG^\top \phi_s^{-1}(Q) + S) E_{s,t}(Q) ds. \end{aligned} \quad (42)$$

Now, the integral above is positive definite owing to the observability of (A, C) and the controllability of (A, G) (see [16, Lemma 4.1]). Notice that the equality in (39) can be also written as

$$\begin{aligned} \phi_t(Q) &= E_{t',t}(Q) \phi_{t'}(Q) E_{t',t}^\top(Q) + \int_{t'}^t E_{s,t}(Q)GG^\top E_{s,t}^\top(Q) ds \\ &+ \lambda \int_{t'}^t E_{s,t}(Q) \phi_s(Q) C^\top (C\phi_s(Q)C^\top + V)^{-1}C\phi_s(Q) E_{s,t}^\top(Q) ds. \end{aligned}$$

When assumptions (A1), (A2) and condition (11) hold, we already know that for $t \geq t' \geq \bar{t}$, $\phi_t(Q)$ will be uniformly bounded below and above by the expressions as in Prop. 5 and Prop. 7, respectively. Thus $\int_{t'}^t E_{s,t}(Q)GG^\top E_{s,t}^\top(Q) ds$ and for that matter $\int_{t'}^t E_{s,t}(Q)E_{s,t}^\top(Q) ds$ exists and is bounded for all $t \geq \bar{t}$. Thus, we can always compute $\mathbb{R} \ni \zeta_{t',t} > 0$ such that

$$\begin{aligned} &\int_{t'}^t E_{s,t}^\top(Q) (\phi_s^{-1}(Q)GG^\top \phi_s^{-1}(Q) + S) E_{s,t}(Q) ds \\ &\geq \zeta_{t',t} \int_{t'}^t E_{s,t}(Q) E_{s,t}^\top(Q) ds, \end{aligned}$$

holds for all $t \geq t' \geq \bar{t}$. Again, using the fact that $\phi_t(Q)$ remains uniformly bounded for all times greater than a

certain value, and that the integrals exist and is finite for all such times, it is evident that there exists a global constant $\zeta = \min_{t,t',Q} \zeta_{t',t} > 0$. Then, we can write

$$\begin{aligned} & \int_{t'}^t E_{s,t}^\top(Q) (\phi_s^{-1}(Q) G G^\top \phi_s^{-1}(Q) + S) E_{s,t}(Q) ds \\ & \geq \zeta \int_{t'}^t E_{s,t}(Q) E_{s,t}^\top(Q) ds \\ & \stackrel{(a)}{\geq} \frac{\zeta \underline{\gamma}_C}{1 + \underline{\gamma}_C \bar{\gamma}_O^\#} \int_{t'}^t E_{s,t}(Q) \phi_s^{-1}(Q) E_{s,t}^\top(Q) ds, \end{aligned}$$

where inequality (a) is a simple application of the lower bound in Proposition 5. Using this, we can rewrite (42) as

$$\begin{aligned} & E_{t',t}^\top(Q) \phi_t^{-1}(Q) E_{t',t}(Q) \\ & \leq \phi_{t'}^{-1}(Q) - \frac{\zeta \underline{\gamma}_C}{1 + \underline{\gamma}_C \bar{\gamma}_O^\#} \int_{t'}^t E_{s,t}(Q) \phi_s^{-1}(Q) E_{s,t}^\top(Q) ds \\ & \leq \frac{\underline{\gamma}_C}{1 + \underline{\gamma}_C \bar{\gamma}_O^\#} I - \frac{\zeta \underline{\gamma}_C}{1 + \underline{\gamma}_C \bar{\gamma}_O^\#} \int_{t'}^t E_{s,t}(Q) \phi_s^{-1}(Q) E_{s,t}^\top(Q) ds. \end{aligned} \quad (43)$$

Applying Grönwall's lemma to the above we have

$$\begin{aligned} & \|E_{t',t}^\top(Q) \phi_t^{-1}(Q) E_{t',t}(Q)\| \\ & \leq \frac{\underline{\gamma}_C}{1 + \underline{\gamma}_C \bar{\gamma}_O^\#} \exp\left(-\frac{\zeta \underline{\gamma}_C}{1 + \underline{\gamma}_C \bar{\gamma}_O^\#} (t - t')\right). \end{aligned} \quad (44)$$

Using Proposition 6, we have

$$\|E_{t',t}^\top(Q) \phi_t^{-1}(Q) E_{t',t}(Q)\| \geq \frac{1 + \underline{\gamma}_O^\# \bar{\gamma}_C}{\underline{\gamma}_O^\#} \|E_{t',t}(Q)\|^2.$$

Combining this with (44) we have

$$\begin{aligned} & \|E_{t',t}(Q)\|^2 \\ & \leq \frac{\underline{\gamma}_C \underline{\gamma}_O^\#}{(1 + \underline{\gamma}_C \bar{\gamma}_O^\#)(1 + \underline{\gamma}_O^\# \bar{\gamma}_C)} \exp\left(-\frac{\zeta \underline{\gamma}_C}{1 + \underline{\gamma}_C \bar{\gamma}_O^\#} (t - t')\right), \end{aligned}$$

and this completes the proof. \square

Example 2. Let us revisit Example 1 using the same system parameters and mean sampling rate. This time, we simulate the Euclidean norm of the expected error (8) in figure 2. The initial value of the expected error is chosen as $\bar{x}_0 = \begin{pmatrix} 3 \\ -3 \end{pmatrix}$.

7. Conclusions

We studied the filtering problem for continuous-time linear stochastic systems with Poisson-sampled discrete observations. The focus was in particular on deriving conditions for existence of fixed point and studying analytical bounds on the expected error covariance matrix. The

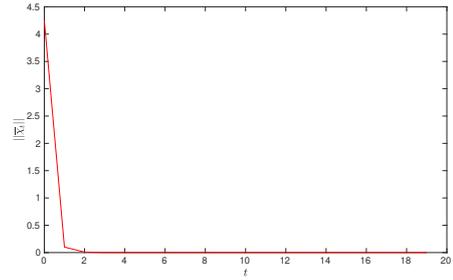


Figure 2: Time evolution of the expected error norm $\|\bar{x}_t\|$.

analytical bounds on the error covariance matrix and the first moment of estimation error provided in Theorem 2 and Theorem 3 allow us to analyze the performance of the proposed filter.

Based on the analysis presented in this paper, several interesting directions could be investigated. Analyzing filters with Poisson-sampled observations has also been investigated in the distributed setting where multiple filters interact with each other based on the information being exchanged between them [20], and it would be interesting to see whether bounds on sampling rate allow us to study the optimal solution for such cases. Similarly, when looking at ensemble filters, some recent work has also investigated convergence of such filters using the system class and observation model studied here [21], and one could again draw connections with the sampling bounds. In the long run, we can study the filtering while taking other models of the communication channel into consideration, such as, quantization and decoding errors.

Appendix

In this Appendix, we collect two lemmas which describe the matrix identities that have been used in the results developed in this paper.

Lemma 3. Let $P \in \mathbb{R}^{n \times n}$, $V \in \mathbb{R}^{p \times p}$, and $C \in \mathbb{R}^{p \times n}$ be constant matrices. If $P \geq 0$ and $V > 0$, then the following matrix equalities hold:

$$\begin{aligned} (I + PC^\top V^{-1}C)^{-1} &= I - PC^\top (CPC^\top + V)^{-1}C \quad (.1a) \\ (I + PC^\top V^{-1}C)^{-1} PC^\top V^{-1} &= PC^\top (CPC^\top + V)^{-1} \quad (.1b) \end{aligned}$$

Proof. Proof of the identities in Lemma 3 follows from [9, Appendix 7B]. For (.1a) it can be easily proved by multiplying the inverse of the expression of the left-hand side on both sides of the identity, thus giving

$$\begin{aligned} & (I + PC^\top V^{-1}C)(I - PC^\top (CPC^\top + V)^{-1}C) \\ &= I + PC^\top V^{-1}C - PC^\top V^{-1}V(CPC^\top + V)^{-1}C \\ & \quad - PC^\top V^{-1}CPC^\top (CPC^\top + V)^{-1}C \\ &= I. \end{aligned}$$

On the other hand, (.1b) can be proved by multiplying $PC^\top V^{-1}$ on the right-hand side of (.1a). From there, it follows that

$$\begin{aligned} & (I + PC^\top V^{-1}C)^{-1}PC^\top V^{-1} \\ &= PC^\top V^{-1} - PC^\top (CPC^\top + V)^{-1}CPC^\top V^{-1} \\ &+ [PC^\top (CPC^\top + V)^{-1} - PC^\top (CPC^\top + V)^{-1}VV^{-1}] \\ &= PC^\top (CPC^\top + V)^{-1}, \end{aligned}$$

which is the desired equality. \square

Lemma 4. Let $K_1 = P_1C^\top (CP_1C^\top + V)^{-1}$ and $K_2 = P_2C^\top (CP_2C^\top + V)^{-1}$, then it holds that

$$(I - K_1C)P_1 - (I - K_2C)P_2 = (I - K_1C)(P_1 - P_2)(I - K_2C)^\top.$$

Proof. Let $K = PC^\top (CPC^\top + V)^{-1}$. It then follows from (.1a) and (.1b) that

$$\begin{aligned} 0 &= (I - KC)PC^\top V^{-1} - K = (I - KC)PC^\top - KV \\ &= (I - KC)PC^\top K^\top - KVK^\top \\ &= (I - KC)P(I - KC)^\top + KVK^\top - (I - KC)P. \end{aligned} \quad (.2)$$

Using (.2), and the expressions for K_1 and K_2 given in Lemma 4, it follows that

$$\begin{aligned} & (I - K_1C)P_1 - (I - K_2C)P_2 \\ &= (I - K_1C)P_1 - K_1VK_2^\top + K_1VK_2^\top - (I - K_2C)P_2 \\ &= (I - K_1C)P_1 - (I - K_1C)P_1C^\top K_2^\top + K_1CP_2(I - K_2C)^\top \\ &\quad - P_2(I - K_2C)^\top \\ &= (I - K_1C)P_1(I - K_2C)^\top - (I - K_1C)P_2(I - K_2C)^\top \\ &= (I - K_1C)(P_1 - P_2)(I - K_2C)^\top \end{aligned}$$

which completes the proof. \square

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