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► To cite this version:

Arthur Scolari Fagundes, João Manoel Gomes da Silva, Sophie Tarbouriech. A hybrid approach to the stability analysis of sampled-data Lur'e systems. *Systems and Control Letters*, 2024, 184, pp.105710. <10.1016/j.sysconle.2023.105710>. <hal-04866566>

HAL Id: hal-04866566

<https://laas.hal.science/hal-04866566v1>

Submitted on 6 Jan 2025

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A Hybrid Approach to the Stability Analysis of Sampled-Data Lur'e Systems.*

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Abstract

This work deals with the stability analysis of Lur'e systems under sampled-data control, where the Lur'e nonlinearity is assumed to be both sector and slope restricted. The stability conditions are derived by using a hybrid system representation and a generalized Lur'e type timer-dependent Lyapunov function. Considering a polynomial timer-dependence, the stability conditions are cast in sum-of-squares optimization problems aiming at computing the largest range of sampling intervals or the largest sector bounds on the nonlinearity for which the origin of the closed-loop system is globally asymptotically stable.

Keywords: Lur'e systems; Sampled-data systems; Hybrid systems;

1. Introduction

Lur'e systems and the problem of absolute stability have been extensively studied since 1944, when A. I. Lur'e and V. N. Postnikov published their seminal work [1]. In the early stages of research, the advances in the field were summarized by attempts of A.I. Lur'e and other authors to find necessary and sufficient conditions on the parameters of the system for the existence of a Lyapunov function to certify the global asymptotic stability of the Lur'e system for any nonlinearity that lies in a given sector [2], [3]. More recently, works focusing on stability analysis and stabilization, considering linear matrix inequalities (LMI) formulations have been proposed (see, for instance, [4], [5], [6], [7], [8] and references therein). On the other hand, the advent of networked control systems has caused a renewal in the interest in sampled-data control, mainly regarding nonlinear systems and the aperiodic sampling [9].

The first results to provide stability conditions for Lur'e systems under sampled-data control were obtained in the last decade, based on Lyapunov-Krasovskii Functionals. In [10] and [11], the stability of a sampled-data Lur'e system with aperiodic sampling was studied and certified with the proposal of LMI conditions. To obtain these conditions, the authors have considered a

*This work was supported by Coordenação de Aperfeiçoamento de Pessoal de Nível Superior (CAPES) - Finance Code 001 and Conselho Nacional de Desenvolvimento Científico e Tecnológico (CNPq) (grant PQ 307449/2019-0) from Brazil. It was also supported by Project Stic-AmSud NetConHybSDP, code 22-STIC-09, funded by CAPES (Brazil) and CNRS (France).

Lyapunov-Krasovskii functional proposed in [12]. The results were extended for robust control in [13]. Furthermore, in [14] the conditions were derived based on passivity arguments. It should be noticed that all of the aforementioned references follow the Lyapunov-Krasovskii approach for the time-delay representation of sampled-data systems [15]. In fact, this approach was also adopted to deal with the synchronization of Lur'e systems with sampled-data control in [16], [17], [18] and [19]. Less conservative conditions, by using a more generic looped-functional approach inspired in the work from [20], have been recently proposed in [21]. Regarding other approaches, conditions to assess the stability when a periodic sampled-data control law is designed from the Euler approximated discrete-time Lur'e system were proposed in [22]. We can also cite [23], which proposed the design of a stabilizing sampled-data control law aiming at guaranteeing cost certification using the impulsive formulation of [24] and a classical Lur'e-Postnikov type Lyapunov function, where only the quadratic term is time-dependent. Considering only the periodic sampling case, the conditions are cast as differential LMIs (i.e. DLMI - see for instance [25] and [26]) and their solution is carried out by appropriately gridding the intersampling interval and considering a piecewise affine time dependence for the quadratic term of the Lyapunov function .

Regarding the choice of Lyapunov function candidates to assess stability of Lur'e type systems (not necessarily with sampled-data control), several structures have been considered in the literature. The simpler one is a quadratic function depending on the plant state ([27], [5]), which are also associated to the Circle criterion [28]. The Lur'e-Postnikov function, a candidate associated to the Popov criterion [28], is used for instance in [29], [30], [31], [32], [4]. Other studies consider quadratic forms that depend not only on the state but also on the nonlinearity ϕ . For instance, [7] proposes a candidate with crossed terms between the state and the nonlinearity for the local stability analysis of discrete-time Lur'e systems, which is strategically structured to avoid nonconvex stability conditions. For continuous-time Lur'e system, assuming bounds on the slope of the nonlinearity, a structure composed by a full quadratic form involving the state and the non-linearity and also some extended integral terms is considered in [33]. Thereafter, considering also slope bounds, [34] and [8] propose a generalized Lur'e-Postnikov type function, in the sense that it relaxes the necessity of both the positivity of the quadratic term and the nonnegativity of the multipliers of the integral terms, which are present in the classical version. Clearly, this function encompasses all the aforementioned structures. More recently, [35] presents a generalization of this structure for discrete-time systems.

This paper proposes a method to assess the global asymptotic stability of the origin of a Lur'e system with aperiodic sampled-data control. The closed-loop system is represented as an impulsive model, based on the hybrid systems framework of [36]. In this framework the system state is augmented with the control signal and a timer that counts the time elapsed since the last sampling instant. Thus, the aperiodic sampling is represented by a jump (i.e. an impulsive update) in the state that occurs in intervals of time ranging from a lower to an upper bound. We consider that the nonlinearity is sector bounded with slope restrictions. From this setup we propose conditions to assess the global asymptotic stability of the origin of the closed-loop system. For this, we consider as Lyapunov function candidate a timer-dependent version of the generalized Lur'e-Postnikov function proposed in [8]. In this case, the quadratic term depends also on the nonlinearity and is timer-dependent. Thus, assuming a polynomial timer dependence, the conditions are cast as sum-of-squares (SOS) constraints and optimization problems are proposed to compute the largest range of sampling intervals or the largest sector bounds on the nonlinearity, for which the global asymptotic stability of the origin can be ensured.

The main contributions of the paper can therefore be summarized as follows:

- The use of a hybrid system framework to address the stability analysis of sampled-data Lur'e systems;
- The generalization of the Lur'e-Postnikov function proposed in [8] considering it dependent on a timer variable;
- The development of timer-dependent matrix inequalities that guarantee the global asymptotic stability of the origin of the closed-loop system, and further formulation as sum-of-squares expressions;
- The formulation of two sum-of-squares optimization problems: one related to the maximization of the upper bound of the inter-sampling interval and another related to the maximization of the sector and/or slope bounds of the nonlinearity, for which it is possible to ensure the asymptotic stability of the origin under aperiodic sampled-data control;
- The demonstration, by means of numerical examples, of the conservatism reduction by considering the use of the generalized Lur'e-Postnikov Lyapunov function with polynomial timer-dependent terms of appropriate degrees.

The paper is organized as follows. In Section 2, we present the Lur'e system with sampled-data control, and the problems of interest. In Section 3, the hybrid system framework is introduced as a representation of the sampled-data system, along with a timer-dependent generalized Lur'e-type Lyapunov function. In Section 4, based on the proposed Lyapunov function, stability conditions are developed in the form of matrix inequalities. In Section 5, the timer dependence of the Lyapunov function candidate is assumed to be polynomial and the stability conditions are cast as sum-of-squares (SOS) constraints. In Section 6, optimization problems are proposed to estimate the maximum upper bound on the intersampling interval, or the sector of the Lur'e system nonlinearity, while ensuring that the sampled-data system is globally asymptotically stable. Numerical examples follow in Section 7. Finally, concluding remarks are presented in Section 8.

Notation

\mathbb{N} is the set of natural numbers, \mathbb{R} is the set of real numbers, and $\mathbb{R}_{\geq 0}$ is the set of nonnegative real numbers. The i th element of a vector $v \in \mathbb{R}^n$ is denoted by $v_{(i)}$. The vector v has Euclidean norm given by $|v| = \sqrt{v_{(1)}^2 + \dots + v_{(n)}^2}$. The distance of a vector v to a closed set \mathcal{A} is denoted $|v|_{\mathcal{A}}$ and it is defined as $|v|_{\mathcal{A}} = \inf_{y \in \mathcal{A}} |v - y|$. The induced 2-norm of a matrix M is represented by $|M|$. \mathbb{S}^n is the set of symmetric matrices of order n , and for a symmetric matrix $S \in \mathbb{S}^n$, $S \succ 0$ means that S is positive definite. \mathbb{D}^n and $\mathbb{D}_{\geq 0}^n$ are the sets of the diagonal and positive semidefinite diagonal matrices of order n , respectively. M^\top denotes the transpose of M , and $\text{He}\{M\} = M^\top + M$. The matrices I_n and $0_{n \times m}$ denote an identity matrix of order n and a $n \times m$ matrix of zeros, respectively.

2. Problem statement

Consider a Lur'e system, given by the following set of equations:

$$\begin{cases} \dot{x}(t) = Ax(t) + B_u u(t) + B_\phi \phi(y(t)) \\ y(t) = Cx(t) \end{cases} \quad (1)$$

where $A \in \mathbb{R}^{n \times n}$, $B_u \in \mathbb{R}^{n \times p}$, $B_\phi \in \mathbb{R}^{n \times m}$, and $C \in \mathbb{R}^{m \times n}$. The vector $x \in \mathbb{R}^n$ represents the plant state, $u \in \mathbb{R}^p$ is the input and $y \in \mathbb{R}^m$ is the argument of the function ϕ , which depends linearly on the state.

The nonlinear function $\phi : \mathbb{R}^m \rightarrow \mathbb{R}^m$ characterizes the Lur'e system, being a decentralized vector valued function, that is:

$$\phi(y(t)) \triangleq [\phi_{(1)}(y_{(1)}(t)), \dots, \phi_{(m)}(y_{(m)}(t))]^\top. \quad (2)$$

We assume that the nonlinear function ϕ is sector bounded with slope restrictions, that is, for all $y \in \mathbb{R}^m$ it satisfies

$$\phi_{(i)}(0) = 0, \quad (3)$$

$$\frac{\phi_{(i)}(y)}{y_{(i)}} \in [\underline{\delta}_{(i)}, \bar{\delta}_{(i)}], \quad (4)$$

$$\frac{d\phi_{(i)}(y)}{dy_{(i)}} \in [\underline{\gamma}_{(i)}, \bar{\gamma}_{(i)}], \quad (5)$$

for all $i = 1, 2, \dots, m$, where $\underline{\delta} \in \mathbb{R}^m$ and $\bar{\delta} \in \mathbb{R}^m$ are, respectively, the vectors of lower and upper sector bounds, and $\underline{\gamma} \in \mathbb{R}^m$, $\bar{\gamma} \in \mathbb{R}^m$ are, respectively, the vectors of lower and upper slope bounds. Therefore, $\underline{\delta}_{(i)} \leq \bar{\delta}_{(i)}$ and $\underline{\gamma}_{(i)} \leq \bar{\gamma}_{(i)} \forall i \in \{1, \dots, m\}$. Along the text, we consider the following diagonal matrices defined from these bounds:

$$\underline{\Delta} \triangleq \text{diag}\{\underline{\delta}_{(1)}, \dots, \underline{\delta}_{(m)}\},$$

$$\underline{\Gamma} \triangleq \text{diag}\{\underline{\gamma}_{(1)}, \dots, \underline{\gamma}_{(m)}\},$$

$$\bar{\Delta} \triangleq \text{diag}\{\bar{\delta}_{(1)}, \dots, \bar{\delta}_{(m)}\},$$

$$\bar{\Gamma} \triangleq \text{diag}\{\bar{\gamma}_{(1)}, \dots, \bar{\gamma}_{(m)}\}.$$

We consider that the control signal $u(t)$ is given as a sampled-data feedback control law composed by terms depending on the state, on the previous value of the control signal, and on the nonlinearity ϕ , as follows:

$$u(t) = K_x x(t_k) + K_u u(t_{k-1}) + K_\phi \phi(y(t_k)) \quad (6)$$

$$= K_x x(t_k) + K_u u(t_{k-1}) + K_\phi \phi(Cx(t_k)), \quad \forall t \in [t_k, t_{k+1}) \quad (7)$$

where $K_x \in \mathbb{R}^{p \times n}$, $K_u \in \mathbb{R}^{p \times p}$ and $K_\phi \in \mathbb{R}^{p \times m}$ are the controller gains, and t_k , $k \in \mathbb{N}$ denote the sampling instants.

Remark 1. *The gains K_u and K_ϕ can be seen as additional degrees of freedom for the controller. Beyond the fact of being more generic, the use of the nonlinear term $K_\phi \phi$ can be useful to convexify synthesis conditions in a continuous-time or discrete-time framework (see for instance [37], [5] and [27]). On the other hand, when a hybrid system framework is considered to model a sampled-data system, the state is augmented with the control signal and then the use of information about the control applied at the last sampling instant (i.e. $u(t_{k-1})$) can also be helpful to convexify synthesis conditions (see for instance [38]). Note that the control law (6) can be reduced to a linear state feedback if $K_u = 0_{p \times p}$ and $K_\phi = 0_{p \times m}$, or a nonlinear state feedback if only $K_u = 0_{p \times p}$.*

It is assumed that the sampling intervals are possibly aperiodic, ranging from a lower value T_1 to an upper value T_2 , that is

$$0 < T_1 \leq t_{k+1} - t_k \leq T_2. \quad (8)$$

From the setup above, considering a given closed-loop system formed by the connection of the plant (1) with the controller given by (6), we are interested in the following problems:

P1. Given the system (1) and controller (6) parameters, the sector and slope bounds $\underline{\delta}_{(i)}$, $\underline{\gamma}_{(i)}$, $\bar{\delta}_{(i)}$, and $\bar{\gamma}_{(i)}$ for $i = 1, \dots, m$, and the lower bound T_1 on the sampling interval, find an estimate for the maximum T_2 such that the origin of the closed-loop system given by (1) and (6) is globally asymptotically stable.

P2. Given the system (1) and controller (6) parameters, the sampling interval bounds T_1 and T_2 , and the lower sector and slope bounds $\underline{\delta}_{(i)}$, $\underline{\gamma}_{(i)}$, find estimates for the maximum $\bar{\delta}_{(i)}$, and $\bar{\gamma}_{(i)}$ for $i = 1, \dots, m$ such that the origin of the closed-loop system given by (1) and (6) is globally asymptotically stable.

To tackle these two problems we consider a hybrid system framework as described next.

3. Hybrid System Representation

Under the considered sampled-data policy, the control signal $u(t)$ experiences a jump (i.e. its value is instantaneously updated) at each sampling time and it is held constant between two successive sampling times. The input then follows a jump dynamics, whereas the state x follows a continuous-time evolution. The resulting closed-loop system can therefore be elegantly represented by the hybrid dynamical systems framework described in [36]. In this context, considering $z = [x^\top \ u^\top]^\top$, we define the hybrid system state $\eta = [z^\top \ \tau]^\top$, where the state τ is a timer that counts the time elapsed since the last sampling instant. In this case, a jump can be triggered (i.e. the update of the control signal with the sampled value of the state) when $\tau \in [T_1, T_2]$. After the jump, the value of τ is reset to 0.

Hence, the closed-loop system described by (1) and (6) is cast as an hybrid system \mathcal{H} , described generically as follows:

$$\mathcal{H} \begin{cases} \dot{\eta} = \begin{bmatrix} \dot{z} \\ \dot{\tau} \end{bmatrix} = f(\eta), & \forall \eta \in \mathcal{C} \\ \eta^+ = \begin{bmatrix} z^+ \\ \tau^+ \end{bmatrix} = g(\eta), & \forall \eta \in \mathcal{D} \end{cases} \quad (9)$$

The sets $\mathcal{C} = \mathbb{R}^q \times [0, T_2]$ and $\mathcal{D} = \mathbb{R}^q \times [T_1, T_2]$, with $q = n + p$, are the *flow* and the *jump sets*, respectively, while the function $f : \mathbb{R}^h \rightarrow \mathbb{R}^h$, with $h = q + 1$, is the *flow map* and the function $g : \mathbb{R}^h \rightarrow \mathbb{R}^h$ is the *jump map*. In our case, functions f and g are defined as follows:

$$f(\eta) = \begin{bmatrix} A_F z + B_F \phi(C_z z) \\ 1 \end{bmatrix} \quad (10)$$

$$g(\eta) = \begin{bmatrix} A_J z + K_J \phi(C_z z) \\ 0 \end{bmatrix} \quad (11)$$

with:

$$A_F = \begin{bmatrix} A & B_u \\ 0_{p \times n} & 0_{p \times p} \end{bmatrix}, \quad B_F = \begin{bmatrix} B_\phi \\ 0_{p \times m} \end{bmatrix}, \quad C_z = [C \quad 0_{m \times p}],$$

$$A_J = \begin{bmatrix} I_n & 0_{n \times p} \\ K_x & K_u \end{bmatrix}, \quad K_J = \begin{bmatrix} 0_{n \times m} \\ K_\phi \end{bmatrix}.$$

The system \mathcal{H} has solutions given in terms of a hybrid arc $\eta(t, k)$ defined in the domain $\text{dom } \eta = \cup_{k=0}^{\infty} ([t_k, t_{k+1}], k)$, which is complete and, because $T_1 > 0$, without Zeno behavior. In this case, system (9) satisfies the basic conditions for well-posedness (i.e closedness of \mathcal{C} and \mathcal{D} , and continuity of f and g) [36]. Note that, with this formalism, η is in fact a function of the continuous-time t and the discrete-time k .

The stability of system \mathcal{H} is associated to a compact set \mathcal{A} , containing the origin and the domain of the timer. The set is defined as follows:

$$\mathcal{A} = \{0\} \times [0, T_2]. \quad (12)$$

As introduced above, τ is in fact an auxiliary variable to model the sampling phenomenon: during the flow, $\tau(t, k) = t - t_k$, and, at jumps, $\tau(t_k, k) = 0$. Since we are assuming that $t_0 = 0$, it follows that $\tau(0, 0) = 0$. Thus, we consider that the initial conditions are expressed as $\eta(0, 0) = \begin{bmatrix} z(0, 0) \\ 0 \end{bmatrix}$.

The next theorem provides three conditions to be satisfied by a Lyapunov function V in order to ensure the uniform global asymptotic stability of the attractor \mathcal{A} . This theorem corresponds to relaxed conditions of Theorem 3.18 in [36], since system (9) does not involve set-valued maps and the attractor \mathcal{A} is compact.

Theorem 1. *If there exist a function $V : \mathbb{R}^h \rightarrow \mathbb{R}_{\geq 0}$, class \mathcal{K}_∞ functions α_1 and α_2 , and a continuous positive definite function ω (namely a function such that $\omega(0) = 0$ and $\omega(s) > 0$ for all $s > 0$) such that*

$$\alpha_1(|\eta|_{\mathcal{A}}) \leq V(\eta) \leq \alpha_2(|\eta|_{\mathcal{A}}) \quad \forall \eta \in (C \cup D) \quad (13)$$

$$\langle \nabla V(\eta), f(\eta) \rangle \leq -\omega(|\eta|_{\mathcal{A}}) \quad \forall \eta \in C \setminus \mathcal{A} \quad (14)$$

$$V(g(\eta)) - V(\eta) \leq -\omega(|\eta|_{\mathcal{A}}) \quad \forall \eta \in D \quad (15)$$

then the compact set \mathcal{A} is uniformly globally asymptotically stable.

Note that if \mathcal{A} is uniformly globally asymptotically stable for the hybrid system (9) with flow and jump maps given by (10) and (11), it follows that 0 (the origin) is a globally asymptotically stable equilibrium point for the closed-loop system given by (1) and (6). Thus, the conditions of Theorem 1 allow us to address the problems P1 and P2. With that in mind, based on a generalized timer-dependent Lur'e-Postnikov function, we cast the conditions of Theorem 1 as timer-dependent LMIs, in order to develop a method to numerically evaluate these stability conditions.

4. Preliminary lemmas

From properties (3), (4) and (5), the following lemmas regarding the nonlinearity ϕ can be stated [8]:

Lemma 1. *If ϕ satisfies (3)-(5), then the relation*

$$(\phi(y) - \underline{\Delta}y)^\top U(\overline{\Delta}y - \phi(y)) \geq 0 \quad (16)$$

is satisfied for all $y \in \mathbb{R}^m$, with any matrix $U \in \mathbb{D}_{\geq 0}^m$.

Lemma 2. *If ϕ satisfies (3)-(5), then the relation*

$$(\dot{\phi}(y) - \underline{\Gamma}\dot{y})^\top U(\overline{\Gamma}\dot{y} - \dot{\phi}(y)) \geq 0 \quad (17)$$

is satisfied for all $y \in \mathbb{R}^m$, with any matrix $U \in \mathbb{D}_{\geq 0}^m$.

The Lyapunov function candidate is defined here as a timer-dependent generalized Lur'e-Postnikov function, given by:

$$V(\eta) \triangleq V(z, \tau) = V_0(z, \tau) + \sum_{i=1}^m \lambda_{(i)} \int_0^{C_{z(i)}z} (\phi_{(i)}(s) - \underline{\delta}_{(i)}s) ds, \quad (18)$$

where the term $V_0 : \mathbb{R}^h \rightarrow \mathbb{R}$ is a timer-dependent quadratic function on z and $\phi(C_z z)$ as follows

$$V_0(z, \tau) \triangleq \begin{bmatrix} z \\ \phi(C_z z) \end{bmatrix}^\top P(\tau) \begin{bmatrix} z \\ \phi(C_z z) \end{bmatrix}, \quad (19)$$

P being a matrix function $P : [0, T_2] \rightarrow \mathbb{S}^{q+m}$ with each element being a continuous Lipschitz function of τ . Compared to a classical Lur'e-Postnikov function [28], this generalized version presents a quadratic term that depends also on the nonlinearity ϕ . Furthermore, differently from the classical one, the matrix P is not necessarily positive definite and the multipliers $\lambda_{(i)}$ are not necessarily nonnegative. In this case, the positive definiteness of the Lyapunov function should be ensured by other means. Of course, this would be straightforwardly achieved by forcing $P(\tau) \succ 0$ and $\lambda_{(i)} \geq 0$, $i = 1, \dots, m$. However, proceeding this way can be conservative. A less conservative condition to ensure the positivity of $V(\eta)$ and to satisfy condition (13) can be obtained by considering a timer-dependent version of the Lemma 4 proposed in [8], given as follows:

Lemma 3. *Consider $V(\eta)$ as in (18), where ϕ satisfies (3)-(5), and define $\Lambda \triangleq \text{diag}\{\lambda_{(1)}, \dots, \lambda_{(m)}\}$. If there exist matrix functions $U_0 : [0, T_2] \rightarrow \mathbb{D}_{\geq 0}^m$ and $\tilde{\Lambda} : [0, T_2] \rightarrow \mathbb{D}_{\geq 0}^m$ such that*

$$\Psi_\Lambda(\tau) \succeq 0, \quad \forall \tau \in [0, T_2] \quad (20)$$

$$\Psi_V(\tau) \succ 0, \quad \forall \tau \in [0, T_2] \quad (21)$$

where

$$\begin{aligned} \Psi_\Lambda(\tau) &= \Lambda + \tilde{\Lambda}(\tau), \\ \Psi_V(\tau) &= P(\tau) - \frac{1}{2} \begin{bmatrix} C_z^\top \\ 0_{m \times m} \end{bmatrix} (\overline{\Delta} - \underline{\Delta}) \tilde{\Lambda}(\tau) [C_z \ 0_{m \times m}] + \text{He} \left\{ \frac{1}{2} \begin{bmatrix} (\underline{\Delta} C_z)^\top \\ -I_m \end{bmatrix} U_0(\tau) [\overline{\Delta} C_z \ -I_m] \right\}, \end{aligned}$$

then there exist class \mathcal{K}_∞ functions α_1 and α_2 for which condition (13) holds.

Proof. To prove that α_2 exists, consider $P(\tau)$, without loss of generality, expressed as:

$$P(\tau) = \begin{bmatrix} P_1(\tau) & P_2(\tau) \\ \star & P_3(\tau) \end{bmatrix}. \quad (22)$$

From(22), the function candidate (18)-(19) is rewritten as:

$$V(\eta) = z^\top P_1(\tau)z + \text{He}\{z^\top P_2(\tau)\phi(C_z z)\} + \phi(C_z z)^\top P_3(\tau)\phi(C_z z) + \sum_{i=1}^m \lambda_{(i)} \int_0^{C_{z(i)}z} (\phi_{(i)}(s) - \underline{\delta}_{(i)}s)ds.$$

From (4), note that

$$0 \leq \phi_{(i)}(s) - \underline{\delta}_{(i)}s \leq \bar{\delta}_{(i)}s - \underline{\delta}_{(i)}s = (\bar{\delta}_{(i)} - \underline{\delta}_{(i)})s.$$

Hence, it follows that

$$|\phi(s) - \underline{\Delta}s| \leq |(\bar{\Delta} - \underline{\Delta})s|, \quad (23)$$

and, taking into account the norm relation $|a| - |b| \leq |a - b|$, we have that

$$|\phi(s)| - |\underline{\Delta}s| \leq |\phi(s) - \underline{\Delta}s|.$$

Combining the last expression with (23), it follows that

$$|\phi(s)| \leq |(\bar{\Delta} - \underline{\Delta})s| + |\underline{\Delta}s|.$$

Considering now the norm relation $|ab| \leq |a||b|$, an upper bound for $V(\eta)$ is thus given by

$$\begin{aligned} V(\eta) &\leq |P_1(\tau)||z|^2 + 2|P_2(\tau)|(|(\bar{\Delta} - \underline{\Delta})C_z| + |\underline{\Delta}C_z|)|z|^2 \\ &\quad + |P_3(\tau)|(|(\bar{\Delta} - \underline{\Delta})C_z| + |\underline{\Delta}C_z|)^2|z|^2 + \frac{1}{2}|C_z^\top \Lambda(\bar{\Delta} - \underline{\Delta})C_z||z|^2 \\ &= \bar{\Psi}(\tau)|z|^2 \leq \max_{\tau \in [0, T_2]} \{\bar{\Psi}(\tau)\}|z|^2. \end{aligned} \quad (24)$$

Note that from the definition of \mathcal{A} in (12) it follows that $|z| = |\eta|_{\mathcal{A}}$ for all $\tau \in [0, T_2]$. Therefore, one can conclude that

$$V(\eta) \leq \alpha_2(|\eta|_{\mathcal{A}}) = \hat{\alpha}_2|\eta|_{\mathcal{A}}^2 = \hat{\alpha}_2|z|^2,$$

with

$$\hat{\alpha}_2 = \max_{\tau \in [0, T_2]} \{\bar{\Psi}(\tau)\},$$

which proves the existence of a class \mathcal{K}_∞ function α_2 .

Now we prove the existence of a class \mathcal{K}_∞ function α_1 verifying (13). With condition (20) verified, a lower bound for the $V(\eta)$ given by (18) is provided as follows [8]:

$$V(\eta) = V_0(\eta) + \sum_{i=1}^m \lambda_{(i)} \int_0^{C_{z(i)}z} (\phi_{(i)}(s) - \underline{\delta}_{(i)}s)ds \geq V_0(\eta) - \sum_{i=1}^m \tilde{\lambda}_{(i)}(\tau) \int_0^{C_{z(i)}z} (\phi_{(i)}(s) - \underline{\delta}_{(i)}s)ds. \quad (25)$$

Then, taking into account the function candidate (19), and that $\int_0^{C_{z(i)}z} (\phi_{(i)}(s) - \underline{\delta}_{(i)}s)ds = \frac{1}{2}z^\top C_{z(i)}^\top (\bar{\delta}_{(i)} - \underline{\delta}_{(i)})C_{z(i)}z - \int_0^{C_{z(i)}z} (\bar{\delta}_{(i)}s - \phi_{(i)}(s))ds$, the inequality (25) is equivalently expressed as

$$V(\eta) \geq \begin{bmatrix} z \\ \phi \end{bmatrix}^\top \underline{\Psi}(\tau) \begin{bmatrix} z \\ \phi \end{bmatrix} + \sum_{i=1}^m \tilde{\lambda}_{(i)}(\tau) \int_0^{C_{z(i)}z} (\bar{\delta}_{(i)}s - \phi_{(i)}(s))ds, \quad (26)$$

where

$$\underline{\Psi}(\tau) = \Psi_V(\tau) - \text{He} \left\{ \frac{1}{2} \begin{bmatrix} (\underline{\Delta}C_z)^\top \\ -I_m \end{bmatrix} U_0(\tau) [\overline{\Delta}C_z \quad -I_m] \right\}.$$

From the positivity of $\Psi_V(\tau)$ given by (21), and the positivity of the last term given by relation (16) in Lemma 1, it follows that $\underline{\Psi}(\tau) \succ 0$, $\forall \tau \in [0, T_2]$. Furthermore, as $\tilde{\lambda}_{(i)}(\tau) \geq 0$ and $\int_0^{C_{z(i)}z} (\bar{\delta}_{(i)}s - \phi_{(i)}(s))ds \geq 0$, $i = 1, \dots, m$, one can conclude from (26) that

$$V(\eta) \geq \begin{bmatrix} z \\ \phi \end{bmatrix}^\top \underline{\Psi}(\tau) \begin{bmatrix} z \\ \phi \end{bmatrix} + \sum_{i=1}^m \tilde{\lambda}_{(i)}(\tau) \int_0^{C_{z(i)}z} (\bar{\delta}_{(i)}s - \phi_{(i)}(s))ds \geq \begin{bmatrix} z \\ \phi \end{bmatrix}^\top \underline{\Psi}(\tau) \begin{bmatrix} z \\ \phi \end{bmatrix}.$$

Considering $\hat{\alpha}_1$ the minimum eigenvalue of $\underline{\Psi}(\tau)$ for $\tau \in [0, T_2]$, we have that

$$V(\eta) \geq \hat{\alpha}_1 \left\| \begin{bmatrix} z \\ \phi \end{bmatrix} \right\|^2 \geq \hat{\alpha}_1 |z|^2 = \alpha_1(|\eta|_{\mathcal{A}}),$$

which proves the existence of a class \mathcal{K}_∞ function α_1 , finishing the proof. \square

Remark 2. Lemma 3 proposes a condition of positivity for the generalized Lur'e-Postnikov function defined in (18). Compared to a classical Lur'e type Lyapunov function, Lemma 3 provides some relaxation about the positivity necessity of the generalized quadratic term and the nonnegativity necessity of the integral multipliers $\lambda_{(i)}$. Indeed, the positivity of the Lyapunov function is satisfied only for nonlinearities ϕ satisfying (16) in Lemma 1. Furthermore, for other nonlinearities, such as deadzone and saturation functions, the integral part of function (18) assumes a generalized quadratic form on z and $\phi(z)$, leading to a sign-indefinite quadratic form $V(\eta)$ which could be described as in (19). In this particular case, more suitable relaxations can be employed (see for instance [39]).

5. Stability conditions

The next theorem states sufficient conditions for the stability of the closed-loop system, considering the Lyapunov function candidate (18)-(19). Before presenting the theorem, we define the following matrices:

$$\begin{aligned} M_1 &\triangleq [I_q \quad 0_{q \times m} \quad 0_{q \times m}], \quad M_2 \triangleq [0_{m \times q} \quad I_m \quad 0_{m \times m}] \\ \mathbb{M}_{F1} &\triangleq A_F M_1 + B_F M_2, \quad M_3 \triangleq [0_{m \times q} \quad 0_{m \times m} \quad I_m] \\ M_{12} &\triangleq \begin{bmatrix} M_1 \\ M_2 \end{bmatrix}, \quad \mathbb{M}_J \triangleq \begin{bmatrix} A_J & K_J \\ 0_{m \times q} & I_m \end{bmatrix}, \quad \mathbb{M}_{F2} \triangleq \begin{bmatrix} A_F & B_F & 0_{q \times m} \\ 0_{m \times q} & 0_{m \times m} & I_m \end{bmatrix} \end{aligned}$$

Theorem 2. If there exist matrix functions $P : [0, T_2] \rightarrow \mathbb{S}^{q+m}$, $\tilde{\Lambda} : [0, T_2] \rightarrow \mathbb{D}_{\geq 0}^m$ and $U_j : [0, T_2] \rightarrow \mathbb{D}_{\geq 0}^m$, $j = 0, 1, 2$, and a matrix $\Lambda \in \mathbb{D}^m$ such that

$$\Psi_\Lambda(\tau) \succeq 0 \quad \forall \tau \in [0, T_2] \quad (27)$$

$$\Psi_V(\tau) \succ 0 \quad \forall \tau \in [0, T_2] \quad (28)$$

$$\Psi_F(\tau) \prec 0 \quad \forall \tau \in [0, T_2] \quad (29)$$

$$\Psi_J(\tau) \prec 0 \quad \forall \tau \in [T_1, T_2] \quad (30)$$

where

$$\begin{aligned}
\Psi_\Lambda(\tau) &= \Lambda + \tilde{\Lambda}(\tau), \\
\Psi_V(\tau) &= P(\tau) - \frac{1}{2} \begin{bmatrix} C_z^\top \\ 0_{m \times m} \end{bmatrix} (\bar{\Delta} - \underline{\Delta}) \tilde{\Lambda}(\tau) [C_z \ 0_{m \times m}] + \text{He} \left\{ \begin{bmatrix} (\underline{\Delta} C_z)^\top \\ -I_m \end{bmatrix} U_0(\tau) [\bar{\Delta} C_z \ -I_m] \right\}, \\
\Psi_F(\tau) &= M_{12}^\top \dot{P}(\tau) M_{12} + \text{He} \{ M_{12}^\top P(\tau) \mathbb{M}_{F2} \} - 0.5 \text{He} \{ (M_1^\top (\underline{\Delta} C_z)^\top - M_2^\top) \Lambda C_z \mathbb{M}_{F1} \} \\
&\quad + \text{He} \{ (M_2^\top - M_1^\top (\underline{\Delta} C_z)^\top) U_1(\tau) (\bar{\Delta} C_z M_1 - M_2) \} \\
&\quad + \text{He} \{ (M_3^\top - \mathbb{M}_{F1}^\top (\underline{\Gamma} C_z)^\top) U_2(\tau) (\bar{\Gamma} C_z \mathbb{M}_{F1} - M_3) \}, \\
\Psi_J(\tau) &= \mathbb{M}_J^\top P(0) \mathbb{M}_J - P(\tau),
\end{aligned}$$

then the origin of the closed-loop system given by (1) and (6) is globally asymptotically stable.

Proof. From Lemma 3, it follows that (27) and (28) ensure that (13) is verified.

Taking into account that

$$\frac{d}{dt} \left[\sum_{i=1}^m \lambda_{(i)} \int_0^{C_z(i)z} (\phi_{(i)}(s) - \delta_{(i)} s) ds \right] = -(z^\top (\underline{\Delta} C_z)^\top - \phi^\top(C_z z)) \Lambda C_z \dot{z}, \quad (31)$$

and denoting $\langle \nabla V(\eta), f(\eta) \rangle \triangleq \nabla V f$, one obtains

$$\begin{aligned}
\nabla V f &= \text{He} \left\{ \begin{bmatrix} z \\ \phi(C_z z) \end{bmatrix}^\top P(\tau) \begin{bmatrix} \dot{z} \\ \dot{\phi}(C_z z) \end{bmatrix} \right\} \\
&\quad + \begin{bmatrix} z \\ \phi(C_z z) \end{bmatrix}^\top \frac{\partial P(\tau)}{\partial \tau} \begin{bmatrix} z \\ \phi(C_z z) \end{bmatrix} \dot{\tau} - (z^\top (\underline{\Delta} C_z)^\top - \phi^\top(C_z z)) \Lambda C_z \dot{z},
\end{aligned} \quad (32)$$

Consider now $U_1, U_2 : [0, T_2] \rightarrow \mathbb{D}_{\geq 0}^m$, and recall that the nonlinearity ϕ satisfies the sector bounds (4) and slope bounds (5). If

$$\begin{aligned}
\nabla V f + 2(\phi(C_z z) - \underline{\Delta} C_z z)^\top U_1(\tau) (\bar{\Delta} C_z z - \phi(C_z z)) \\
+ 2(\dot{\phi}(C_z z) - \underline{\Gamma} C_z \dot{z})^\top U_2(\tau) (\bar{\Gamma} C_z \dot{z} - \dot{\phi}(C_z z)) < 0
\end{aligned} \quad (33)$$

for all $\tau \in [0, T_2]$, we can conclude from Lemmas 1 and 2 that $\nabla V f < 0$ for all $\tau \in [0, T_2]$, and therefore the condition (14) of Theorem 1 is satisfied. Recalling from (10) that $\dot{\tau} = 1$ and $\dot{z} = A_F z + B_F \phi(C_z z)$, and defining $\zeta = [z^\top \ \phi^\top(C_z z) \ \dot{\phi}^\top(C_z z)]^\top$, an equivalent expression for (33) is given as follows:

$$\begin{aligned}
\zeta^\top \left(M_{12}^\top \dot{P}(\tau) M_{12} + \text{He} \{ M_{12}^\top P(\tau) \mathbb{M}_{F2} \} - 0.5 \text{He} \{ (M_1^\top (\underline{\Delta} C_z)^\top - M_2^\top) \Lambda C_z \mathbb{M}_{F1} \} \right. \\
+ \text{He} \{ (M_2^\top - M_1^\top (\underline{\Delta} C_z)^\top) U_1(\tau) (\bar{\Delta} C_z M_1 - M_2) \} \\
\left. + \text{He} \{ (M_3^\top - \mathbb{M}_{F1}^\top (\underline{\Gamma} C_z)^\top) U_2(\tau) (\bar{\Gamma} C_z \mathbb{M}_{F1} - M_3) \} \right) \zeta < 0.
\end{aligned} \quad (34)$$

Thus, from (34), we conclude that (14) holds provided that condition (29) is satisfied.

Note now, from the definition of $g(\eta) \triangleq g(z, \tau)$ in (11), that (15) can be rewritten as follows:

$$\Delta V \triangleq V(g(z, \tau)) - V(z, \tau) = \begin{bmatrix} z^+ \\ \phi(C_z z^+) \end{bmatrix}^\top P(0) \begin{bmatrix} z^+ \\ \phi(C_z z^+) \end{bmatrix} - \begin{bmatrix} z \\ \phi(C_z z) \end{bmatrix}^\top P(\tau) \begin{bmatrix} z \\ \phi(C_z z) \end{bmatrix}.$$

Recalling that $C_z = [C \ 0_{m \times p}]$, from (11) we have that $C_z z = C_z z^+$. Thus, it follows that

$$\begin{bmatrix} z^+ \\ \phi(C_z z^+) \end{bmatrix} = \mathbb{M}_J \begin{bmatrix} z \\ \phi(C_z z) \end{bmatrix}. \quad (35)$$

Then we can conclude that $\Delta V < 0 \forall \tau \in [T_1, T_2]$, provided that condition (30) holds, which ensures that (15) is verified at the jumps. Note that since LMIs (29) and (30) are strict, it is always possible to determine a quadratic positive definite function ω formally satisfying (14) and (15).

Hence, the conditions of Theorem 2 guarantee the verification of conditions of Theorem 1 with V as defined in (18)-(19), which concludes the proof. \square

Remark 3. *Since the LMIs in Theorem 2 depend on $P(\tau)$ and $\dot{P}(\tau) = \frac{\partial P(\tau)}{\partial \tau} \dot{\tau}$, with τ being a timer that counts the time between the last sampling instant, they can be seen as DLMI in the sense presented in references [25] and [26]. Note, however, that in the hybrid system framework, τ is considered as a state with $\dot{\tau} = 1$. In this case, they can also be seen as “timer” parameter dependent LMIs, similar to the ones that appear in the literature of linear parameter varying systems (LPV) (see for instance [40] and [41]) and also referred as “clock” dependent conditions in [42].*

5.1. Polynomial dependence on τ

Theorem 2 is quite generic in the sense that the stability can be certified if we are able to find some matrix functions depending on τ . However, nothing is specified about the suitable structure of these functions. Furthermore, once we fix a certain structure (or dependence on τ), the conditions must be verified for all τ in the considered intervals, which leads to an infinite dimensional problem. To overcome this problem, we could consider, for instance, gridding techniques similar to the ones performed in the context of linear parameter varying (LPV) systems [40], but loosing in this case the formal certification of stability. Another possibility is to consider an affine [43], [44], [45] or piecewise affine [23] dependence leading to a finite test of LMIs. On the other hand, as any continuous nonlinear function can be approximated by a polynomial function of appropriate degree, we propose here to consider matrix functions with polynomial dependence on τ , which can be generically described as follows:

$$M(\tau) = \sum_{i=0}^d M_i \tau^i = M_0 + \tau M_1 + \dots + \tau^d M_d, \quad (36)$$

where d is the degree of the matrix polynomial function and M_i , $i = 1, \dots, d$ are constant matrices. In this case, conditions in Theorem 2 can be expressed as sum-of-squares (SOS) conditions [46], [47], [42] and efficiently solved by available software packages such as SOSTOOLS [48]. We recall that a matrix $M(\tau)$ is said to be a SOS matrix, or simply “SOS”, if it can be written as $M(\tau) = H(\tau)^\top H(\tau)$. Hence, if $M(\tau)$ is SOS it follows that $M(\tau) \succeq 0$, $\forall \tau$.

The matrix polynomial structure of finite degree on τ , given in equation (36) can therefore approximate a generic nonlinear function, leading to more degrees of freedom in the search for an appropriate Lyapunov function. Note that in the SOS framework, for each matrix variable $M(\tau)$, one has M_0, \dots, M_d as free matrices to be determined, which can be used to “fit” a suitable nonlinear function leading to the stability certification. Of course, there is a trade-off between the degree of the polynomial, that should be chosen a priori, and the numerical complexity.

The next theorem exploits this structure and presents the stability conditions cast in the sum-of-squares programming framework.

Theorem 3. *If there exist matrix polynomials $P : [0, T_2] \rightarrow \mathbb{S}^{q+m}$, $\Lambda \in \mathbb{D}^m$, $\tilde{\Lambda} : [0, T_2] \rightarrow \mathbb{D}^m$, $Q_j : [0, T_2] \rightarrow \mathbb{S}^m$, $j \in \{0, 1, 2, 3, 4\}$, $Q_5 : [0, T_2] \rightarrow \mathbb{S}^{q+m}$, $Q_6 : [0, T_2] \rightarrow \mathbb{S}^{q+2m}$, $Q_7 : [0, T_2] \rightarrow \mathbb{S}^{q+m}$, $U_j : [0, T_2] \rightarrow \mathbb{D}^m$, $j \in \{0, 1, 2\}$, and a scalar $\xi > 0$, such that*

$$Q_j(\tau), j \in \{0, 1, 2, 3, 4\}, \quad \text{are SOS} \quad (37)$$

$$U_0(\tau) - Q_0(\tau)\tau(T_2 - \tau) \quad \text{is SOS} \quad (38)$$

$$U_1(\tau) - Q_1(\tau)\tau(T_2 - \tau) \quad \text{is SOS} \quad (39)$$

$$U_2(\tau) - Q_2(\tau)\tau(T_2 - \tau) \quad \text{is SOS} \quad (40)$$

$$\tilde{\Lambda}(\tau) - Q_3(\tau)\tau(T_2 - \tau) \quad \text{is SOS} \quad (41)$$

$$\Psi_\Lambda - Q_4(\tau)\tau(T_2 - \tau) \quad \text{is SOS} \quad (42)$$

$$\Psi_V(\tau) - Q_5(\tau)\tau(T_2 - \tau) - \xi I_{q+m} \quad \text{is SOS} \quad (43)$$

$$-\Psi_F(\tau) - Q_6(\tau)\tau(T_2 - \tau) - \xi I_{2(q+m)} \quad \text{is SOS} \quad (44)$$

$$-\Psi_J(\tau) - Q_7(\tau)(\tau - T_1)(T_2 - \tau) - \xi I_{q+m} \quad \text{is SOS} \quad (45)$$

then, the origin of the closed-loop system (1) and (6) is globally asymptotically stable.

Proof. If the conditions in (37) are verified, then

$$Q_j(\tau) \succeq 0, j \in \{0, 1, 2, 3, 4\}, \quad \forall \tau.$$

Noting that $\tau(T_2 - \tau) \succeq 0$ for $\tau \in [0, T_2]$, provided that $Q_0(\tau) \succeq 0 \forall \tau$, it follows that (38) ensures that

$$U_0(\tau) \succeq 0 \quad \tau \in [0, T_2]. \quad (46)$$

Similarly, as $Q_1(\tau), Q_2(\tau), Q_3(\tau), Q_4(\tau) \succeq 0 \forall \tau$, the conditions (39), (40) and (41) imply that $U_1(\tau), U_2(\tau), \tilde{\Lambda}(\tau)$ and $(\Lambda - \tilde{\Lambda}(\tau)) \succeq 0 \forall \tau \in [0, T_2]$. Moreover, since $Q_5(\tau) \succeq 0 \forall \tau$ and $\xi > 0$, the condition (43) implies that

$$\Psi_V(\tau) \succeq \xi I_{q+m} \succ 0 \quad \forall \tau \in [0, T_2], \quad (47)$$

and therefore that (28) is satisfied. Similarly, since $Q_6(\tau) \succeq 0 \forall \tau$, one has that (44) implies (29).

Finally, noting that $(\tau - T_1)(T_2 - \tau) \succeq 0$ for $\tau \in [T_1, T_2]$, provided that $Q_7(\tau) \succeq 0 \forall \tau$, it follows that (45) ensures (30). \square

6. Optimization problems

Here we formulate optimization problems to solve problems P1 and P2 defined in Section 2, based on the constraints of Theorem 3. In fact, P1 and P2 can be cast as the following optimization problems

$$\text{P1} \sim \begin{cases} \max T_2 \\ \text{subject to (37) - (45)} \end{cases} \quad (48)$$

$$\text{P2} \sim \begin{cases} \max f(\bar{\Delta}, \bar{\Gamma}) \\ \text{subject to (37) - (45)} \end{cases} \quad (49)$$

Observe that in optimization problem (49), a generic objective function $f(\bar{\Delta}, \bar{\Gamma})$ depending on $\bar{\Delta}$ and $\bar{\Gamma}$, that is, on the upper bounds of the sector and/or the slope of the nonlinearity, is considered. The basic idea is to enlarge the sector for which the stability can be certified. For instance, we can consider $f(\bar{\Delta}, \bar{\Gamma}) = \sum_{i=1}^m (\bar{\delta}_{(i)} + \bar{\gamma}_{(i)})$ and $\bar{\delta}_{(i)} = \bar{\gamma}_{(i)} = \beta, \forall i$ and maximize β .

On the other hand, it should be noticed that the objective variable T_2 of problem (48) and $\bar{\Delta}$ and $\bar{\Gamma}$ of (49) cannot be directly considered as decision variables, because they are multiplied by other decision variables in (45) and (44). Thus, to solve (48) or (49), an SOS feasibility problem is solved repeatedly, each time increasing the value of T_2 or β , respectively, until the constraints become unfeasible. Hence, solutions to both (48) and (49) can be obtained with the SOSstools toolbox [48].

7. Numerical examples

Example 1

Consider system (1) with the following parameters

$$A = \begin{bmatrix} -0.5 & -6.2 & -0.105 & -1.2 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad B_u = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad B_\phi = \begin{bmatrix} 0.5 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad C = [0 \quad 0.2 \quad 0 \quad 0],$$

and with nonlinearity sector and slope bounds given by $\bar{\Delta} = \beta, \underline{\Delta} = 0, \bar{\Gamma} = \beta, \underline{\Gamma} = -\beta$, where $\beta = \sqrt{2}/2$, and a control law (6) with:

$$K_x = [0.1 \quad 0.2 \quad 0.005 \quad 0.2], \quad K_u = 0 \quad \text{and} \quad K_\phi = 0.5.$$

Defining d_X the degree of a polynomial decision variable $X(\tau)$, we consider $d_{\bar{\Delta}} = d_{U_0} = d_{U_1} = d_{U_2} = d_{Q_0} = d_{Q_1} = d_{Q_2} = d_{Q_3} = d_{Q_4} = 0$, and $d_{Q_5} = d_{Q_6} = d_{Q_7} = 2$, and $d_P = 4$. In this case, the values of T_2 resulting from the solution of (48) considering different values of T_1 are shown in Table 1. As a comparison, the results obtained with the recent method of [21] based on a looped-functional approach are also shown in the same table. The table also displays the results considering only the quadratic part $V_0(\eta)$ of the Lyapunov function (that is, for $V(\eta) = V_0(\eta)$), that results in conditions with reduced complexity. It can be noted that the present method provides less conservative estimates of T_2 , surpassing the limit of $T_1 = T_2 = 5$ reached by the looped-functional based method, and that the more general Lur'e-type candidate significantly improves the estimates.

Table 1: Example 1 - Obtained values of T_2 for different values of T_1

T_1 (Given)	0.5	1	2	3	4	5	17.9
T_2 from (48)	6.8	7.5	12.4	15.2	16.6	17.9	17.9
T_2 with $V_0(\eta)$	5.7	6.0	6.4	6.6	6.7	6.8	unfeas.
T_2 [21]	2.9	3.3	3.9	4.4	4.8	5	unfeas.

Now, consider the control law (6) with following gains

$$K_x = [-2.8322 \quad -2.5547 \quad -6.6458 \quad -1.3226], \quad K_u = 0 \quad \text{and} \quad K_\phi = 0.$$

In this case, considering $T_1 = T_2 = T$, i.e., a periodic sampling case, the maximum estimate obtained for the sampling period ensuring the global stability of the origin, obtained with the conditions in Theorem 2, is given by $T = 0.607$. To illustrate how near this estimate is to a value that results in divergent trajectories, the system is simulated with a nonlinearity belonging to the considered sector with $\beta = \sqrt{2}/2$, defined as $\phi(y(t)) = 0.1\sin(5y(t)) + 0.15y(t)$. Figure 1 shows the simulation of the system for the initial condition $x(0) = [1 \ 1 \ 1 \ 1]^T$ considering $T = 0.607$ and $T = 0.61$. Note that with $T = 0.61$, the origin is not globally asymptotically stable, which means that a small increase of 0.5% in the estimated value of T is enough to result in divergent trajectories for the plant states $x(1)$, $x(2)$, $x(3)$ and $x(4)$.

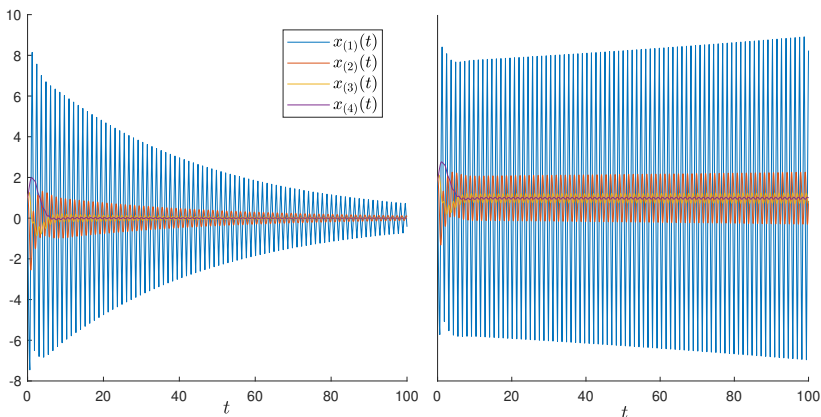


Figure 1: Example 1 - Simulations of the closed-loop system for $x(0) = [1 \ 1 \ 1 \ 1]^T$ considering $T = 0.607$ (left), and $T = 0.61$ (right).

Consider now $T_1 = 0.5$. Solving the problem (49) to maximize the sector and slope bounds, the objective function defined as f parameterized by β , i.e. considering the maximization of β . For different values of T_2 , Figure 2 shows the maximum obtained β considering different polynomial degrees for the variables.

The results can be seen as a short blanket dilemma: larger values of T_2 lead to smaller admissible sectors, (i.e. smaller is the optimal β), and vice-versa. Furthermore, it can be observed that less conservative results are obtained with some polynomial variables of larger degree. When the degrees are 2 or less, for example, no estimate greater than $T_2 = 8$ could be obtained. Another conclusion is that the degree of P significantly reduces the conservatism, compared to the polynomial degrees of other variables.

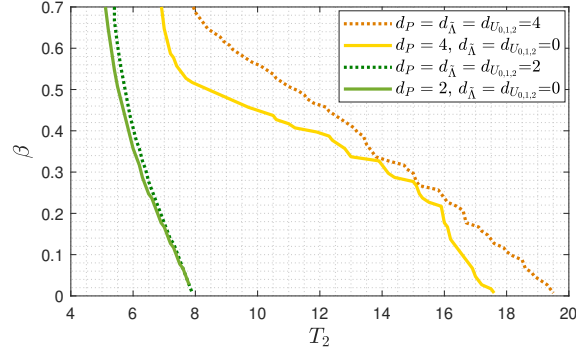


Figure 2: Example 1 - Trade-off between β and T_2 . The lines in this plot correspond to different combinations for the polynomial degrees of d_P , $d_{\bar{\lambda}}$, d_{U_0} , d_{U_1} and d_{U_2} .

Example 2

Consider now the system (1) tackled in [14], where

$$A = \begin{bmatrix} -2 & 1 \\ 1 & 1 \end{bmatrix}, \quad B_u = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad B_\phi = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad \text{and} \quad C = [0 \quad 1].$$

The nonlinearity is defined as $\phi(y(t)) = \sin(y(t)) - y(t)$, which satisfies (3), (4), and (5) with $\bar{\Delta} = 1.2173$, $\underline{\Delta} = 0$, $\bar{\Gamma} = 0$ and $\underline{\Gamma} = -2$. In [14], the control law (6) is parameterized by a single value κ , as follows:

$$K_x = [0 \quad \kappa], \quad K_u = 0 \quad \text{and} \quad K_\phi = 0. \quad (50)$$

In Figure 3, we compare now the results obtained with the proposed approach with $T_1 = 0.1$ and the ones of [14], which is based on Lyapunov-Krasovskii functionals, regarding P1. The graph shows the estimates of T_2 for varying values of κ , ranging from -1.9 to -16. We can observe that larger bounds for T_2 are obtained with the proposed method. The conservatism reduction is in particular greater for smaller values of κ .

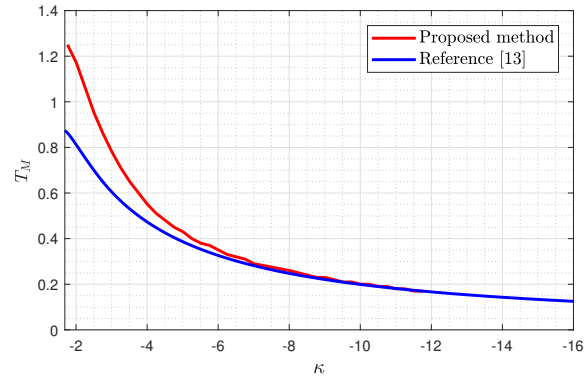


Figure 3: Example 2 - Estimate of the maximal admissible T_2 for different values of parameter κ .

8. Conclusion

This paper proposed theoretical conditions for the stability analysis of Lur'e systems with sampled-data control by using the hybrid dynamical system framework. A generalized Lur'e-Postnikov function depending on a timer variable was considered. Then, timer-dependent matrix inequalities were proposed to guarantee the global asymptotic stability of the origin of the closed-loop system. Considering a polynomial timer dependence, these conditions were tackled through sum-of-squares expressions, useful to deal with some optimization problems, such as the maximization of the upper bound of the inter-sampling interval and the maximization of the sector and/or slope bounds of the nonlinearity, for which it is possible to ensure the asymptotic stability of the origin. Finally, the conservatism reduction was illustrated by considering the use of the generalized Lur'e-Postnikov Lyapunov function with polynomial timer-dependent terms of appropriate degrees. The results pave the way for future works, such as the case where particular nonlinearities are considered (such as saturation or deadzone, in which case it would be possible to switch Lemma 3 for Lemma 2 in [39]) or the case where the plant inputs are also affected by nonlinearities issued from the actuators modeling.

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