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RANK CONDITIONS FOR EXACTNESS OF SEMIDEFINITE RELAXATIONS IN POLYNOMIAL OPTIMIZATION

JEAN B. LASSERRE

ABSTRACT. We consider the Moment-SOS hierarchy in polynomial optimization. We first provide a sufficient condition to solve the truncated \mathbf{K} -moment problem associated with a given degree- $2n$ pseudo-moment sequence ϕ^n and a semi-algebraic set $\mathbf{K} \subset \mathbb{R}^d$. Namely, let $2v$ be the maximum degree of the polynomials that describe \mathbf{K} . If the rank r of its associated moment matrix is less than $n - v + 1$, then ϕ^n has an atomic representing measure supported on at most r points of \mathbf{K} . When used at step- n of the Moment-SOS hierarchy, it provides a sufficient condition to guarantee its finite convergence (i.e., the optimal value of the corresponding degree- n semidefinite relaxation of the hierarchy is the global minimum). For Quadratic Constrained Quadratic Problems (QCQPs) one may also recover global minimizers from the optimal pseudo-moment sequence. Our condition is in the spirit of Blekherman's rank condition and while on the one-hand it is more restrictive, on the other hand it applies to constrained POPs as it provides a *localization* on \mathbf{K} for the representing measure.

1. INTRODUCTION

Consider the *polynomial* optimization problem (POP):

$$(1.1) \quad \mathbf{P} : \quad f^* = \min_{\mathbf{x}} \{ f(\mathbf{x}) : \mathbf{x} \in \mathbf{K} \}$$

$$(1.2) \quad \text{with } \mathbf{K} := \{ \mathbf{x} \in \mathbb{R}^n : g_j(\mathbf{x}) \geq 0, \quad j = 1, \dots, m \},$$

where $f, g_j \in \mathbb{R}[\mathbf{x}]$ are polynomials. In particular, if $\deg(f) \leq 2$ and $\deg(g_j) \leq 2$ for all $j = 1, \dots, m$, then \mathbf{P} is called a quadratic constrained quadratic problem (QCQP). This latter class contains many important problems for which computing (or even approximating) f^* “efficiently” is a scientific challenge; indeed \mathbf{P} is NP-hard in general.

A popular strategy to compute (or approximate) f^* is to provide a monotone non decreasing sequence of lower bounds that converges to f^* from below. Some LP and semidefinite (SDP) relaxations introduced in the nineties [17] and the 2000's [10, 11] provide an example of such a strategy and the interested reader is referred to e.g. [12, 13] for an analysis of their respective advantages and drawbacks.

This paper is concerned with the *Moment-SOS hierarchy* [5, 8] which applies to solve not only POPs but also many important problems in Science & Engineering, provided that they are modeled as instances of the Generalized Moment Problem

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(GMP) with algebraic data; the interested reader is also referred to e.g. [9] for a recent exposition of such applications.

The Moment-SOS hierarchy for solving \mathbf{P} consists of a nested sequence of semidefinite relaxations $(\mathbf{Q}_n)_{n \in \mathbb{N}}$ of \mathbf{P} , whose size increases with n and whose associated sequence of optimal values $(\rho_n)_{n \in \mathbb{N}}$ is monotone non decreasing and converges to f^* as n increases. Moreover it has been shown [15, 14] that generically (i) its convergence is finite, i.e., $\rho_n = f^*$ at some step- n of the hierarchy, and (ii) extraction of global minimizers can be done by exploiting a flatness condition due to Curto & Fialkow [2] (and related to a certain rank condition on moment matrices). When $\rho_n = f^*$ the semidefinite relaxation \mathbf{Q}_n is said to be *exact*.

On rank conditions. Let $\phi = (\phi_\alpha)_{\alpha \in \mathbb{N}_{2n}^d}$, be a real sequence (up to degree $2n$) with positive semidefinite (psd) moment matrix $\mathbf{M}_n(\phi) \succeq 0$ (see definition in §2). Then ϕ has a representing measure if there exists a measure ϕ on \mathbb{R}^d such that $\phi_\alpha = \int \mathbf{x}^\alpha d\phi$ for all $\alpha \in \mathbb{N}_{2n}^d$. To identify whether a sequence ϕ has a representing measure, an important result is the (unconstrained) flatness condition of Curto and Fialkow [2, 3] which states that if

$$(1.3) \quad \text{rank}(\mathbf{M}_n(\phi)) = \text{rank}(\mathbf{M}_{n-1}(\phi)),$$

then ϕ has an atomic representing measure on \mathbb{R}^d supported on $\text{rank}(\mathbf{M}_n(\phi))$ atoms. Similarly with $d_j := \lceil \deg(g_j)/2 \rceil$, and $g_j(\mathbf{x}) = \sum_{\beta} g_{j,\beta} \mathbf{x}^\beta$, $j = 1, \dots, m$, let $g_j \phi = (g_j \phi)_\alpha$, $\alpha \in \mathbb{N}_{2n}^d$, be the sequence where $(g_j \phi)_\alpha = \sum_{\beta} g_{j,\beta} \phi_{\alpha+\beta}$ for all $\alpha \in \mathbb{N}_{2n}^d$. Then suppose that ϕ also satisfies $\mathbf{M}_{n-d_j}(g_j \phi) \succeq 0$, $j = 1, \dots, m$, and let $v := \max_j d_j$. The constrained flatness condition of Curto and Fialkow [2, 3] states that if

$$(1.4) \quad \text{rank}(\mathbf{M}_n(\phi)) = \text{rank}(\mathbf{M}_{n-v}(\phi)),$$

then ϕ has an atomic representing measure supported on $\text{rank}(\mathbf{M}_n(\phi))$ atoms in \mathbf{K} . Finally, by a result of Blekherman [1], it turns out that if

$$(1.5) \quad \text{rank}(\mathbf{M}_n(\phi)) \leq \begin{cases} 3n-3 & \text{if } n \geq 3 \\ 6 & \text{if } n = 2, \end{cases}$$

then the subsequence $\phi^{\hat{n}} (:= (\phi_\alpha)_{\alpha \in \mathbb{N}_{2n-1}^d})$ of ϕ has a representing measure on \mathbb{R}^d ; see [3]¹.

Remark 1.1. *Importantly, notice that in contrast to (1.3), on the one hand the condition (1.5) is only concerned with the single moment matrix $\mathbf{M}_n(\phi)$, but on the other hand there is no localization of the support of its measure. Moreover only the subsequence $\phi^{\hat{n}}$ of ϕ (and not ϕ) has a representing measure.*

Contribution. We are concerned with practical sufficient rank-conditions for finite convergence of the Moment-SOS hierarchy. With $g_0(\mathbf{x}) = 1$ for all \mathbf{x} , the degree- n semidefinite relaxation \mathbf{Q}_n of the Moment-SOS hierarchy associated with \mathbf{P} , reads:

$$(1.6) \quad \mathbf{Q}_n : \quad \rho_n = \min_{\phi} \{ \phi(f) : \phi(1) = 1 ; \mathbf{M}_{n-d_j}(g_j \phi) \succeq 0, j = 0, \dots, m \},$$

where $\phi = (\phi_\alpha)_{\alpha \in \mathbb{N}_{2n}^d}$, and $\mathbf{M}_{n-d_j}(g_j \phi)$ is the localizing matrix associated with ϕ and the polynomial $g_j \in \mathbb{R}[\mathbf{x}]$. $(\mathbf{M}_{n-d_0}(g_0 \phi) = \mathbf{M}_n(\phi)$ is the moment matrix

¹In [7, Theorem 2.36] Blekherman's result (1.5) is incorrectly stated. Indeed only the subsequence $\phi^{n, 2n-1}$ of moments up to degree $2n-1$ has a representing measure and not the whole sequence ϕ of moments up to degree $2n$ in general.

associated with ϕ .) With ϕ (in bold) is associated the Riesz linear functional $\phi \in \mathbb{R}[\mathbf{x}]_{2n}^*$ (not in bold face) defined by:

$$p = \sum_{\alpha \in \mathbb{N}_{2n}^d} p_{\alpha} \mathbf{x}^{\alpha} \mapsto \phi(p) = \sum_{\alpha \in \mathbb{N}_{2n}^d} p_{\alpha} \phi_{\alpha}, \quad \forall p \in \mathbb{R}[\mathbf{x}]_{2n}.$$

The truncated \mathbf{K} -moment problem is concerned with conditions on ϕ to guarantee that ϕ has a representing measure on \mathbf{K} , i.e., $\phi(p) = \int_{\mathbf{K}} p d\phi$ for all $p \in \mathbb{R}[\mathbf{x}]_{2n}$, for some measure ϕ on \mathbf{K} ; see e.g. [16, 7].

Let $d_f := \lceil \deg(f)/2 \rceil$ and $d_j := \lceil \deg(g_j)/2 \rceil$, for all $j = 1, \dots, m$, and $v := \max\{d_1, \dots, d_m\}$. Then our first result provides a sufficient condition to solve the truncated \mathbf{K} -moment problem.

Theorem 1.2. *With $n \geq v$, let $\phi^n = (\phi_{\alpha})_{\alpha \in \mathbb{N}_{2n}^d}$ be such that*

$$(1.7) \quad \phi(1) = 1; \quad \mathbf{M}_n(\phi^n) \succeq 0; \quad \mathbf{M}_{n-d_j}(g_j \phi^n) \succeq 0, \quad j = 1, \dots, m.$$

If $s := \text{rank}(\mathbf{M}_n(\phi^n)) \leq n - v + 1$ then $\phi^{\hat{n}} := (\phi_{\alpha}^n)_{\alpha \in \mathbb{N}_{2n-1}^d}$ has a representing measure supported on at most $r (\leq s)$ points of \mathbf{K} . In addition, if $r = s$ then ϕ^n has a representing measure supported on s points of \mathbf{K} .

Our second result investigates the impact of Theorem 1.2 on the Moment-SOS hierarchy.

Theorem 1.3. *Let \mathbf{P} be as in (1.1). For every n with $n \geq v$ and $2n - 1 \geq \deg(f)$, let $\phi^n = (\phi_{\alpha}^n)_{\alpha \in \mathbb{N}_{2n}^d}$ be an optimal solution of the semidefinite relaxation \mathbf{Q}_n :*

(i) If $\text{rank}(\mathbf{M}_n(\phi^n)) \leq n - v + 1$ then $\phi^n(f) = f^$, that is, $\rho_n = f^*$ and the relaxation \mathbf{Q}_n is exact.*

(ii) Next assume that $\deg(f) \leq 2$ and $\deg(g_j) \leq 2$ for all j , so that $v = 1$ (and \mathbf{P} is a QCQP). Then one may recover a probability measure supported on global minimizers of \mathbf{P} (e.g., via the extraction procedure of [6]).

Remark 1.4. *It is well-known that if $\text{rank}(\mathbf{M}_n(\phi^n)) = 1$ then ϕ^n is the moment vector of the Dirac measure $\delta_{\mathbf{x}^*}$ for some global minimizer $\mathbf{x}^* \in \mathbf{K}$. So it is fair to say that the rank-condition in Theorem 1.2 and Theorem 1.3 provides an extension of this result.*

This rank condition is in the spirit of Blekherman's condition (1.5) (i.e., with no flatness condition as in (1.3)). While on the one hand it is more restrictive than (1.5), on the other hand it provides an additional localization on \mathbf{K} of the support of representing measure. This localization feature is crucial for polynomial optimization as described in Theorem 1.3.

In fact, in [1, p. 72] it is wrongly stated that (1.5) provides a stopping criterion for exactness of the hierarchy of SOS relaxations associated with POPs. The reason why it is incorrect is because again there is no localization of the support of the representing measure. On the other hand, in Section 4 we prove that indeed (1.5) provides a sufficient condition to detect whether the *single* semidefinite relaxation associated with the *unconstrained* POP: $\inf\{f(\mathbf{x}) : \mathbf{x} \in \mathbb{R}^d\}$ (where f is an even degree polynomial) is exact. The proof is not trivial because if $\text{rank}(\mathbf{M}_n(\phi)) \leq 3n - 3$ then only the subsequence of moments up to degree $2n - 1$ has a representing measure, say μ , on \mathbb{R}^d , and f is of degree $2n$ and not of degree $2n - 1$; hence $\phi(f)$ is not necessarily equal to $\int f d\mu$.

2. NOTATION, DEFINITIONS AND PRELIMINARY RESULTS

Let $\mathbb{R}[\mathbf{x}]$ denote the ring of polynomials in the variables $\mathbf{x} = (x_1, \dots, x_d)$ and let $\mathbb{R}[\mathbf{x}]_n$ be the vector space of polynomials of degree at most n (whose dimension is $s(n) := \binom{n+d}{n}$). For every $n \in \mathbb{N}$, let $\mathbb{N}_n^d := \{\alpha \in \mathbb{N}^d : |\alpha| (= \sum_i \alpha_i) \leq n\}$, and let $\mathbf{v}_n(\mathbf{x}) = (\mathbf{x}^\alpha)_{\alpha \in \mathbb{N}_n^d}$, $\alpha \in \mathbb{N}^d$, be the vector of monomials of the canonical basis (\mathbf{x}^α) of $\mathbb{R}[\mathbf{x}]_n$. A polynomial $f \in \mathbb{R}[\mathbf{x}]_n$ is written

$$\mathbf{x} \mapsto f(\mathbf{x}) = \sum_{\alpha \in \mathbb{N}_n^d} f_\alpha \mathbf{x}^\alpha = \langle \mathbf{f}, \mathbf{v}_n(\mathbf{x}) \rangle,$$

where $\mathbf{f} = (f_\alpha)_{\alpha \in \mathbb{N}_n^d} \in \mathbb{R}^{s(n)}$ is its vector of coefficients in the canonical basis of monomials $(\mathbf{x}^\alpha)_{\alpha \in \mathbb{N}_n^d}$. For real symmetric matrices, let $\langle \mathbf{B}, \mathbf{C} \rangle := \text{trace}(\mathbf{B}\mathbf{C})$ while the notation $\mathbf{B} \succeq 0$ stands for \mathbf{B} is positive semidefinite (psd) whereas $\mathbf{B} \succ 0$ stands for \mathbf{B} is positive definite (pd).

The Riesz linear functional. Given a sequence $\phi = (\phi_\alpha)_{\alpha \in \mathbb{N}^d}$ (with ϕ in bold), the Riesz functional is the linear mapping $\phi : \mathbb{R}[\mathbf{x}] \rightarrow \mathbb{R}$ (with ϕ not in bold) defined by:

$$(2.1) \quad f = \sum_{\alpha} f_\alpha \mathbf{x}^\alpha \quad \mapsto \phi(f) = \sum_{\alpha \in \mathbb{N}^d} f_\alpha \phi_\alpha, \quad \forall f \in \mathbb{R}[\mathbf{x}].$$

Moment matrix. The degree- n *moment matrix* associated with a sequence $\phi = (\phi_\alpha)_{\alpha \in \mathbb{N}^d}$, $\alpha \in \mathbb{N}^d$, is the real symmetric matrix $\mathbf{M}_n(\phi)$ with rows and columns indexed by \mathbb{N}_n^d , and whose entry (α, β) is just $\phi_{\alpha+\beta}$, for every $\alpha, \beta \in \mathbb{N}_n^d$. Alternatively, let $\mathbf{v}_n(\mathbf{x}) \in \mathbb{R}^{s(n)}$ be the vector $(\mathbf{x}^\alpha)_{\alpha \in \mathbb{N}_n^d}$, and define the real symmetric matrices (\mathbf{B}_α^1) by

$$(2.2) \quad \mathbf{v}_n(\mathbf{x}) \mathbf{v}_n(\mathbf{x})^T = \sum_{\alpha \in \mathbb{N}_{2n}^d} \mathbf{B}_\alpha^1 \mathbf{x}^\alpha, \quad \forall \mathbf{x} \in \mathbb{R}^d.$$

Then $\mathbf{M}_n(\phi) = \sum_{\alpha \in \mathbb{N}_{2n}^d} \phi_\alpha \mathbf{B}_\alpha^1$. If ϕ has a representing measure ϕ then $\mathbf{M}_n(\phi) \succeq 0$ because $\langle \mathbf{f}, \mathbf{M}_n(\phi) \mathbf{f} \rangle = \int f^2 d\phi \geq 0$, for all $f \in \mathbb{R}[\mathbf{x}]_n$; in this case ϕ_α is the α -moment of ϕ .

A measure whose all moments are finite, is *moment determinate* if there is no other measure with same moments. The support of a Borel measure ϕ on \mathbb{R}^d (denoted $\text{supp}(\phi)$) is the smallest closed set Ω such that $\phi(\mathbb{R}^d \setminus \Omega) = 0$.

In the TCS community, a vector $\phi = (\phi_\alpha)$ whose moment matrix is psd, is called a vector of *pseudo-moments* (and *moments* if ϕ has a representing measure).

Localizing matrix. With ϕ as above and $g \in \mathbb{R}[\mathbf{x}]$ (with $g(\mathbf{x}) = \sum_{\gamma} g_\gamma \mathbf{x}^\gamma$), the degree- n *localizing matrix* associated with ϕ and g is the real symmetric matrix $\mathbf{M}_n(g\phi)$ with rows and columns indexed by \mathbb{N}_n^d , and whose entry (α, β) is just $\sum_{\gamma} g_\gamma \phi_{\alpha+\beta+\gamma}$, for every $\alpha, \beta \in \mathbb{N}_n^d$. Alternatively, let \mathbf{B}_α^g be the real symmetric matrices defined by:

$$(2.3) \quad g(\mathbf{x}) \mathbf{v}_n(\mathbf{x}) \mathbf{v}_n(\mathbf{x})^T = \sum_{\alpha \in \mathbb{N}_{2n+\deg g}^d} \mathbf{B}_\alpha^g \mathbf{x}^\alpha, \quad \forall \mathbf{x} \in \mathbb{R}^d.$$

Then $\mathbf{M}_n(g\phi) = \sum_{\alpha \in \mathbb{N}_{2n+\deg g}^d} \phi_\alpha \mathbf{B}_\alpha^g$. Importantly,

$$(2.4) \quad \mathbf{M}_n(g_j \phi) \succeq 0 \quad \Leftrightarrow \quad \phi(f^2 g_j) \geq 0, \quad \forall f \in \mathbb{R}[\mathbf{x}]_n.$$

Next, if ϕ has a representing measure ϕ whose support is contained in the set $\{\mathbf{x} : g(\mathbf{x}) \geq 0\}$ then $\mathbf{M}_n(g\phi) \succeq 0$ for all n because $\langle \mathbf{f}, \mathbf{M}_n(g\phi)\mathbf{f} \rangle = \int f^2 g d\phi \geq 0$, for all $f \in \mathbb{R}[\mathbf{x}]_n$.

Homogenization. Let $\phi^n = (\phi_\alpha)_n$. Its homogenization is the vector $\tilde{\phi}^n = (\tilde{\phi}_{i,\alpha}^n)_{i+|\alpha|=2n}$ defined by:

$$\tilde{\phi}_{2n-|\alpha|,\alpha}^n = \phi_\alpha^n, \quad \alpha \in \mathbb{N}_{2n}^d,$$

and the homogenization $\widetilde{\mathbf{M}}_n(\tilde{\phi}^n)$ of the moment matrix $\mathbf{M}_n(\phi^n)$ is defined by:

$$\widetilde{\mathbf{M}}_n(\tilde{\phi}^n)((i, \alpha), (j, \beta)) = \mathbf{M}_n(\phi^n)(\alpha, \beta) = \tilde{\phi}_{i+j, \alpha+\beta}^n, \quad i + |\alpha| = j + |\beta| = n.$$

For instance in dimension $d = 2$, and for $n = 1$:

$$\mathbf{M}_1(\phi^n) = \begin{bmatrix} \phi_{00}^n & \phi_{10}^n & \phi_{01}^n \\ \phi_{10}^n & \phi_{20}^n & \phi_{11}^n \\ \phi_{01}^n & \phi_{11}^n & \phi_{02}^n \end{bmatrix} = \widetilde{\mathbf{M}}_1(\tilde{\phi}^n) = \begin{bmatrix} \tilde{\phi}_{2,00}^n & \tilde{\phi}_{1,10}^n & \tilde{\phi}_{1,01}^n \\ \tilde{\phi}_{1,10}^n & \tilde{\phi}_{0,20}^n & \tilde{\phi}_{0,11}^n \\ \tilde{\phi}_{1,01}^n & \tilde{\phi}_{0,11}^n & \tilde{\phi}_{0,02}^n \end{bmatrix}$$

Theorem 2.1. (*Blekherman [1, Theorem 2.3]*) Let ϕ^n and $\tilde{\phi}^n$ be such that $\widetilde{\mathbf{M}}_n(\tilde{\phi}^n) \succeq 0$. If $s := \text{rank } \widetilde{\mathbf{M}}_n(\tilde{\phi}^n) \leq 3n - 3$ when $n \geq 3$ (or $s \leq 6$ when $n = 2$) then $\tilde{\phi}^n$ has an atomic representing measure on \mathbb{R}^{d+1} supported on s atoms.

For every n and $\phi^n = (\phi_\alpha)_n$, define $\phi^{\hat{n}} := (\phi_\alpha^n)_{\alpha \in \mathbb{N}_{2n-1}^d}$. That is, $\phi^{\hat{n}}$ is the restriction of the vector ϕ^n to its “moments” up to degree $2n - 1$.

Corollary 2.2. (*Fialkow [3]*) Let ϕ^n and $\tilde{\phi}^n$ be such that $\widetilde{\mathbf{M}}_n(\tilde{\phi}^n) \succeq 0$ and $s := \text{rank } \widetilde{\mathbf{M}}_n(\tilde{\phi}^n) \leq 3n - 3$ (or ≤ 6 if $n = 2$). Then $\phi^{\hat{n}} = (\phi_\alpha)_n$ has an atomic representing measure on \mathbb{R}^d .

See also the comment after Theorem 1.3 in [4, p. 948]. We will also need the following consequence:

Corollary 2.3. Let ϕ^n and $\tilde{\phi}^n$ be such that $\widetilde{\mathbf{M}}_n(\tilde{\phi}^n) \succeq 0$ and $\text{rank } \widetilde{\mathbf{M}}_n(\tilde{\phi}^n) \leq 3n - 3$ (or ≤ 6 if $n = 2$). Then $\phi^{\hat{n}} = (\phi_\alpha)_n$ has an atomic representing measure on \mathbb{R}^d supported on at most $r \leq s$ points, and with mass $\phi^n(1)$. If $r = s$ then the whole sequence ϕ^n has a representing measure supported on s points of \mathbb{R}^d .

Proof. Let $\{(x_0(1), \mathbf{x}(1)), \dots, (x_0(s), \mathbf{x}(s))\} \subset \mathbb{R}^{d+1}$, be the support of $\tilde{\phi}^n$ with associated (strictly) positive weights $\lambda_1, \dots, \lambda_s$. Let $\Delta := \{i : x_0(i) \neq 0\}$. Then for every $\alpha \in \mathbb{N}_{2n}^d$ with $|\alpha| < 2n$,

$$\begin{aligned} \tilde{\phi}_{2n-|\alpha|,\alpha}^n &= \phi_\alpha^n = \sum_{i=1}^s \lambda_i x_0(i)^{2n-|\alpha|} \prod_{j=1}^d x_j(i)^{\alpha_j} \\ &= \sum_{i \in \Delta} \lambda_i x_0(i)^{2n} \prod_{j=1}^d \left(\frac{x_j(i)}{x_0(i)} \right)^{\alpha_j} = \int_{\mathbb{R}^d} \mathbf{x}^\alpha d\mu, \end{aligned}$$

with $\mu = \sum_{i \in \Delta} \lambda_i x_0(i)^{2n} \delta_{(\frac{x_1(i)}{x_0(i)}, \dots, \frac{x_d(i)}{x_0(i)})}$. That is, μ is a representing measure for $\phi^{\hat{n}}$. In addition, $\mu(1) = \phi^n(1)$ because $\tilde{\phi}^n(x_0^{2n}) = \phi^n(1) = \mu(1)$. Next, for every α

with $|\alpha| = 2n$,

$$\begin{aligned}\tilde{\phi}_{0,\alpha}^n = \phi_\alpha^n &= \sum_{i=1}^s \lambda_i x_0(i)^{2n-|\alpha|} \prod_{j=1}^d x_j(i)^{\alpha_j} \\ &= \sum_{i \in \Delta} \lambda_i x_0(i)^{2n} \prod_{j=1}^d \left(\frac{x_j(i)}{x_0(i)} \right)^{\alpha_j} + \sum_{i \notin \Delta} \lambda_i \prod_{j=1}^d x_j(i)^{\alpha_j},\end{aligned}$$

and therefore, if $|\Delta| = s$ we may and will conclude that $\phi_\alpha^n = \int_{\mathbb{R}^d} \mathbf{x}^\alpha d\mu$ for all $\alpha \in \mathbb{N}_{2n}^d$, i.e., μ is a representing measure of the whole sequence ϕ^n . \square

3. MAIN RESULT

With \mathbf{K} as in (1.2), let $v := \max_{1 \leq j \leq m} \lceil \deg(g_j)/2 \rceil$. Given sequence $\phi^n = (\phi_\alpha^n)_{\alpha \in \mathbb{N}_{2n}^d}$ recall the notation $\phi^{\hat{n}} = (\phi_\alpha^n)_{\alpha \in \mathbb{N}_{2n-1}^d}$, i.e., $\phi^{\hat{n}}$ is the restriction of ϕ^n to moments up to degree $2n-1$. Our first result provides a sufficient condition to solve the \mathbf{K} -moment problem.

Theorem 3.1. *With $n \geq v$, let $\phi^n = (\phi_\alpha^n)_{\alpha \in \mathbb{N}_{2n}^d}$ be such that*

$$(3.1) \quad \phi(1) = 1; \quad \mathbf{M}_n(\phi^n) \succeq 0; \quad \mathbf{M}_{n-d_j}(g_j \phi^n) \succeq 0, \quad j = 1, \dots, m.$$

If $s := \text{rank}(\mathbf{M}_n(\phi^n)) \leq n - v + 1$ then $\phi^{\hat{n}}$ has a representing measure supported on at most $r \leq s$ points of \mathbf{K} . In addition, if $r = s$ then ϕ^n has a representing measure supported on s points of \mathbf{K} .

A detailed proof is postponed to §6.1. The case $n = 1$ is only meaningful when $v = 1$. In this case $s \leq 1$ implies that the whole sequence ϕ^n has a representing measure, the Dirac at some point $\mathbf{x} \in \mathbb{R}^d$.

We next investigate the consequence of Theorem 3.1 for polynomial optimization. With f as in (1.1), define $d_f := \lceil \deg(f)/2 \rceil$ and recall that $d_j := \lceil \deg(g_j)/2 \rceil$, for all $j = 1, \dots, m$, and $v := \max_{j \leq m} d_j$.

Theorem 3.2. *Let \mathbf{P} be as in (1.1). For every n with $n \geq v$ and $2n-1 \geq \deg(f)$, let $\phi^n = (\phi_\alpha^n)_{\alpha \in \mathbb{N}_{2n}^d}$ be an optimal solution of the semidefinite relaxation \mathbf{Q}_n :*

(i) If $s = \text{rank}(\mathbf{M}_n(\phi^n)) \leq n - v + 1$ then $\phi^n(f) = f^$, that is, $\rho_n = f^*$ and the relaxation \mathbf{Q}_n is exact. Moreover, $\phi^{\hat{n}}$ has a representing measure supported on at most s global minimizers of \mathbf{P} .*

(ii) Next assume that $\deg(f) \leq 2$ and $\deg(g_j) \leq 2$ for all j , so that $v = 1$ (and \mathbf{P} is a QCQP). Then one may exhibit a probability measure supported on global minimizers of \mathbf{P} .

A detailed proof is postponed to §6.2. So at an optimal solution ϕ^n of \mathbf{Q}_n , Theorem 3.2 provide a sufficient rank condition on $\mathbf{M}_n(\phi^n)$ to ensure that the semidefinite relaxation \mathbf{Q}_n is exact. In addition, for QCQPs one may extract global minimizers by looking at moment matrices (submatrices of the moment matrix $\mathbf{M}_n(\phi^n)$), of degree lower than n .

Remark 3.3. *The rank condition in Theorem 3.1 and Theorem 3.2 more severe than in Theorem 2.1. However this is quite natural as the rank conditions must also guarantee an important additional feature of a representing measure of ϕ^n , namely*

its support should be contained in \mathbf{K} , whereas in Corollary 2.2, $\phi^{\hat{n}}$ being in now way related to any set $\mathbf{K} \subset \mathbb{R}^d$, one cannot expect any localization property of its representing measure whenever the latter exists.

4. THE UNCONSTRAINED CASE

In [1, p. 72] the author claims: “Theorem 2.1 also leads to an interesting stopping criterion for sum of squares relaxations. Sum of squares methods lead to a hierarchy of relaxations indexed by degree.” and later still in p. 72:

“Stopping criterion for SOS relaxations.

Suppose that the sum of squares relaxation truncated in degree $2d$ with $d \geq 3$, returns an optimal linear functional with moment matrix of rank at most $3d - 3$. Then the relaxation is exact.”

The above statement is incorrect because for constrained POPs in (1.1), even if in an optimal solution ϕ^* of the relaxation \mathbf{Q}_n , $\text{rank}(\mathbf{M}_n(\phi^*)) = 3n - 3$ (in [1] d is the degree whereas for us n is the degree) then a representing measure for ϕ is *not* guaranteed to be supported on \mathbf{K} . Precisely, Theorem 3.1 provides more restrictive rank conditions to ensure that the support of ϕ^* is indeed in \mathbf{K} ; see Remark 3.3.

However, we next show that indeed Blekherman’s result is useful for *unconstrained* polynomial optimization, that is, so solve:

$$(4.1) \quad \mathbf{P} : \quad f^* = \inf_{\mathbf{x} \in \mathbb{R}^d} f(\mathbf{x})$$

where $f \in \mathbb{R}[\mathbf{x}]_{2n}$ with $n \geq 2$ (the case $n = 1$ being trivial). With \mathbf{P} is associated the *single* semidefinite relaxation:

$$(4.2) \quad \rho = \inf_{\phi \in \mathbb{R}^{s(n)}} \{ \phi(f) : \phi(1) = 1; \mathbf{M}_n(\phi) \succeq 0 \}.$$

Indeed there is *no* hierarchy to consider. Either $f - f^*$ is SOS and $\rho = f^*$, or $f - f^*$ is not an SOS and then $\rho < f^*$ (with possibly $\rho = -\infty$). The dual of (4.2) reads:

$$(4.3) \quad \rho^* = \sup_{\lambda} \{ \lambda : f - \lambda \in \Sigma[\mathbf{x}]_n \},$$

where $\Sigma[\mathbf{x}]_n \subset \mathbb{R}[\mathbf{x}]_{2n}$ is the convex cone of SOS polynomials of degree at most $2n$.

Theorem 4.1. *Consider the unconstrained POP in (4.1) and its associated (single) semidefinite relaxation (4.2). Let $\rho > -\infty$ and let ϕ^* be an optimal solution of (4.2). Suppose that $r := \text{rank}(\mathbf{M}_n(\phi^*)) \leq 3n - 3$ if $n \geq 3$ or $r := \text{rank}(\phi^*) \leq 6$ if $n = 2$. Then $\rho = f^*$ and there exist some $k (\leq r)$ global minimizers.*

Proof. We prove the result when $n \geq 3$ while the arguments are similar for the case $n = 2$. Let \tilde{f} (or $\text{hom}(f)$) be the homogenization of f , that is, $\tilde{f} \in \mathbb{R}[x_0, \mathbf{x}]_{2n}$ with

$$(4.4) \quad \tilde{f}(x_0, \mathbf{x}) = \begin{cases} x_0^{2n} f(\mathbf{x}/x_0) & \text{if } x_0 \neq 0 \\ f_{2n}(\mathbf{x}) & \text{if } x_0 = 0 \end{cases},$$

where f_{2n} is the degree- $2n$ homogeneous part of f . So $\tilde{f}(1, \mathbf{x}) = f(\mathbf{x})$ and $\tilde{f}(0, \mathbf{x}) = f_{2n}(\mathbf{x})$ for all $\mathbf{x} \in \mathbb{R}^d$. In particular if a polynomial is nonnegative or SOS then so is its homogenization and the converse is true as well.

Slater’s condition holds for (4.2). Indeed let μ be the Gaussian measure $\mathcal{N}(0, \mathbf{I})$ on \mathbb{R}^d (with identity matrix \mathbf{I} as covariance matrix). Then $\mathbf{M}_n(\mu) \succ 0$ and $\mu(f)$ is finite. Therefore $\rho = \rho^*$. Next, by optimality of ϕ^* and the necessary KKT optimality conditions, there exists $\mathbf{X}^* \succeq 0$ such that

$$f(\mathbf{x}) - \rho = \mathbf{v}_n(\mathbf{x})^T \mathbf{X}^* \mathbf{v}_n(\mathbf{x}) \quad \forall \mathbf{x} \in \mathbb{R}^d; \quad \langle \mathbf{M}_n(\phi^*), \mathbf{X}^* \rangle = 0.$$

Recall the homogenization $\tilde{\phi}^* = (\tilde{\phi}_{2n-|\alpha|,\alpha}^*)$, $\alpha \in \mathbb{N}_{2n}^d$ of ϕ which reads $\tilde{\phi}_{2n-|\alpha|,\alpha}^* = \phi_\alpha^*$ for every $\alpha \in \mathbb{N}_{2n}^d$, and observe that the homogeneous moment matrix $\widetilde{\mathbf{M}}_n(\tilde{\phi}^*) (= \mathbf{M}_n(\phi^*))$ of $\tilde{\phi}^*$ satisfies

$$\mathbf{w}_n(x_0, \mathbf{x})^T \widetilde{\mathbf{M}}(\tilde{\phi}^*) \mathbf{w}_n(x_0, \mathbf{x}) = \text{hom}(\mathbf{v}_n(\mathbf{x})^T \mathbf{M}(\phi^*) \mathbf{v}_n(\mathbf{x})),$$

where $\mathbf{w}_n(x_0, \mathbf{x}) = \text{hom}(\mathbf{v}_n(\mathbf{x}))$. Next, if $r := \text{rank}(\mathbf{M}_n(\phi^*)) \leq 3n - 3$ then by Theorem 2.1, $\tilde{\phi}^*$ has a representing measure $\tilde{\phi}^*$ on \mathbb{R}^{d+1} supported on at most r atoms $\{(x_0(1), \mathbf{x}(1)), \dots, (x_0(r), \mathbf{x}(r))\} \subset \mathbb{R}^{d+1}$, and

$$\phi_\alpha^* = \int_{\mathbb{R}^{d+1}} x_0^{2n-|\alpha|} \mathbf{x}^\alpha d\tilde{\phi}^*(x_0, \mathbf{x}) = \sum_{j=1}^r \lambda_j (x_0(j)^{2n-|\alpha|} \mathbf{x}(j)^\alpha), \quad \forall \alpha \in \mathbb{N}_{2n}^d.$$

In particular, let

$$(x_0, \mathbf{x}) \mapsto \text{hom}(f - \rho) = \tilde{f}(\mathbf{x}) - \rho x_0^{2n}$$

be the homogenization of $f - \rho$. Then

$$\mathbf{x} \mapsto f(\mathbf{x}) - \rho \text{ is SOS} \Rightarrow (x_0, \mathbf{x}) \mapsto \text{hom}(f - \rho) = \tilde{f}(x_0, \mathbf{x}) - \rho x_0^{2n} \text{ is SOS.}$$

Moreover,

$$\int \underbrace{\tilde{f}(x_0, \mathbf{x}) - \rho x_0^{2n}}_{\text{SOS}} d\tilde{\phi}^*(x_0, \mathbf{x}) = \phi^*(f - \lambda) = \langle \mathbf{X}^*, \mathbf{M}_n(\phi^*) \rangle = 0,$$

and therefore $\tilde{f}(x_0(j), \mathbf{x}(j)) - \rho x_0(j)^{2n} = 0$ for all $j = 1, \dots, r$. Next, let $\Gamma := \{j : x_0(j) \neq 0\}$, and assume that $\Gamma \neq \emptyset$. Then invoking (4.4), one obtains

$$x_0(j)^{2n} (f(\mathbf{x}(j)/x_0(j)) - \rho) = 0, \quad \forall j \in \Gamma,$$

which implies $f(\mathbf{x}(j)/x_0(j)) = \rho$ for all $j \in \Gamma$, and therefore for every $j \in \Gamma$, $\mathbf{x}(j)/x_0(j)$ is a global minimizer of f and $f^* = \rho$. It remains to prove that $\Gamma \neq \emptyset$. But this follows from $1 = \phi^*(1) = \int x_0^{2n} d\tilde{\phi}^*$. \square

In the atomic support of $\tilde{\phi}^*$, the above proof needs to treat separately points with $x_0(j) \neq 0$ from points with $x_0(j) = 0$. Indeed if $\tilde{\phi}^*$ has a representing measure $\tilde{\phi}^*$ on \mathbb{R}^{d+1} , only the subsequence $(\phi_\alpha^*)_{|\alpha| \leq 2n-1}$ has a representing measure ϕ^* on \mathbb{R}^d . Then

$$0 = \int_{\mathbb{R}^{d+1}} \tilde{f}(x_0, \mathbf{x}) - \rho x_0^{2n} d\tilde{\phi}^*(x_0, \mathbf{x}) \not\equiv 0 = \int_{\mathbb{R}^d} (f - \rho) d\phi^*(\mathbf{x}),$$

because as $\deg(f) = 2n > 2n - 1$, $\phi^*(f - \rho) \neq \int_{\mathbb{R}^d} (f - \rho) d\phi^*$ in general.

5. CONCLUSION

We have provided a rank condition on the moment matrix of an optimal solution of the degree- n semidefinite relaxation of the Moment Hierarchy applied to POP. When satisfied, the corresponding semidefinite relaxation is exact, i.e., the Moment-SOS hierarchy has finite converge (and for QCQPs, global minimizers can be extracted). These conditions are in the spirit of Blekherman's condition, i.e., are concerned with a single moment matrix in contrast to Curto & Fialkow's flat extension condition. While they are more restrictive, they apply to constrained POPs whereas Blekherman's condition only helps for unconstrained POPs.

6. APPENDIX

6.1. Proof of Theorem 3.1.

Proof. Observe that if $s \leq n - v + 1$ then $s \leq 3n - 3$ if $n \geq 3$ and $s \leq 6$ if $n = 2$. Then by Corollary 2.2, $\phi^{\hat{n}}$ has an atomic representing measure ϕ supported on at most r points $\mathbf{x}(1), \dots, \mathbf{x}(r) \in \mathbb{R}^d$ with $r \leq s$. That is, there exist $\lambda_i > 0$, $i = 1, \dots, r$, such that

$$\phi = \sum_{i=1}^r \lambda_i \delta_{\mathbf{x}(i)} \quad \text{and} \quad \int \mathbf{x}^\alpha d\phi = \phi_\alpha^n, \quad \forall \alpha \in \mathbb{N}_{2n-1}^d.$$

Let $1 \leq j \leq m$ be fixed arbitrary.

Case $r = s$. Then by Corollary 2.3, the *whole* sequence ϕ^n has a representing measure ϕ supported on s points $\mathbf{x}(1), \dots, \mathbf{x}(s) \in \mathbb{R}^d$. Next, since $2(n - d_j) + \deg(g_j) \leq 2n$, the localizing matrix $\mathbf{M}_{n-d_j}(g_j \phi)$ contains only moments of degree at most $2n$ and therefore, $\phi^n(q g_j) = \int q g_j d\phi$ for all $q \in \mathbb{R}[\mathbf{x}]_{2(n-d_j)}$. Next, by (2.4),

$$(6.1) \quad 0 \preceq \mathbf{M}_{n-d_j}(g_j \phi^n) \Rightarrow \phi^n(q g_j) = \int q g_j d\phi \geq 0,$$

for every SOS polynomial $q \in \mathbb{R}[\mathbf{x}]_{2(n-d_j)}$. From this we deduce that at least one point $\mathbf{x}(i)$ satisfies $g_j(\mathbf{x}(i)) \geq 0$, i.e., the cardinality $|\Gamma|$ of the set $\Gamma := \{i : g_j(\mathbf{x}(i)) \geq 0\}$ is at least 1. Next, suppose that $|\Gamma| < s$. Then consider the polynomial $p \in \mathbb{R}[\mathbf{x}]_{2|\Gamma|}$ defined by:

$$(6.2) \quad \mathbf{x} \mapsto p(\mathbf{x}) := \prod_{i \in \Gamma} \left(\sum_{k=1}^d (x_k - x_k(i))^2 \right) = \prod_{i \in \Gamma} \|\mathbf{x} - \mathbf{x}(i)\|^2.$$

Then $p(\mathbf{x}(i)) = 0$ for every $i \in \Gamma$ and $p(\mathbf{x}(i)) > 0$ for every $i \notin \Gamma$. Moreover, p is an SOS of degree $2|\Gamma| < 2s$, and as $s \leq n - v + 1$, $2|\Gamma| \leq 2n - 2v \leq 2(n - d_j)$. Hence by (6.1), $0 \leq \phi^n(p g_j) = \int p g_j d\phi$, and one obtains the contradiction

$$0 \leq \int p g_j d\phi = \sum_{i \in \Gamma} \lambda_i \underbrace{p(\mathbf{x}(i))}_{=0} g_j(\mathbf{x}(i)) + \sum_{i \notin \Gamma} \lambda_i \underbrace{p(\mathbf{x}(i))}_{>0} \underbrace{g_j(\mathbf{x}(i))}_{<0}.$$

Therefore $|\Gamma| = s$ which implies $g_j(\mathbf{x}(i)) \geq 0$ for all $i = 1, \dots, s$. As j was arbitrary, $\mathbf{x}(i) \in \mathbf{K}$ for all $i = 1, \dots, s$, i.e., ϕ is supported on \mathbf{K} .

Case $r < s$. Then by Corollary 2.3, $\phi^{\hat{n}}$ has a representing measure ϕ supported on r points $\mathbf{x}(1), \dots, \mathbf{x}(r) \in \mathbb{R}^d$. Next, since $2(n - 1 - d_j) + \deg(g_j) \leq 2n - 1$, the localizing matrix $\mathbf{M}_{n-1-d_j}(g_j \phi)$ contains only moments of degree at most $2n - 1$, and therefore $\phi^n(q g_j) = \phi^{\hat{n}}(q g_j) = \int q g_j d\phi$ for all $q \in \mathbb{R}[\mathbf{x}]_{2(n-1-d_j)}$. Moreover $\mathbf{M}_{n-d_j}(g_j \phi) \Rightarrow \mathbf{M}_{n-1-d_j}(g_j \phi) \succeq 0$, and therefore, by (2.4),

$$(6.3) \quad 0 \preceq \mathbf{M}_{n-1-d_j}(g_j \phi^n) \Rightarrow \phi^n(q g_j) = \phi^{\hat{n}}(q g_j) = \int q g_j d\phi \geq 0,$$

for every SOS polynomial $q \in \mathbb{R}[\mathbf{x}]_{2(n-d_j-1)}$. Again we deduce that at least one point $\mathbf{x}(i)$ satisfies $g_j(\mathbf{x}(i)) \geq 0$. So the cardinality of the set $\Gamma := \{i : g_j(\mathbf{x}(i)) \geq 0\}$ is at least 1. Next, suppose that $|\Gamma| < r$. Let $p \in \mathbb{R}[\mathbf{x}]_{2|\Gamma|}$ be the SOS polynomial

defined in (6.2). As $r < n - v + 1$ and $|\Gamma| < r$, $|\Gamma| \leq n - v - 1$, and so p is an SOS of degree at most $2(n - d_j - 1)$. Moreover, as

$$\deg(g_j p) \leq 2(n - d_j - 1) + \deg(g_j) \leq 2n - 2,$$

$\phi^n(p g_j) = \phi^{\hat{n}}(p g_j) = \int p g_j d\phi$. By (6.3), one obtains the contradiction

$$\begin{aligned} 0 \leq \phi^{\hat{n}}(p g_j) &= \int p g_j d\phi \\ &= \sum_{i \in \Gamma} \lambda_i \underbrace{p(\mathbf{x}(i))}_{=0} g_j(\mathbf{x}(i)) + \sum_{i \notin \Gamma} \lambda_i \underbrace{p(\mathbf{x}(i))}_{>0} \underbrace{g_j(\mathbf{x}(i))}_{<0}. \end{aligned}$$

Therefore $|\Gamma| = r$, which implies $g_j(\mathbf{x}(i)) \geq 0$ for all $i = 1, \dots, r$, and as j was arbitrary, $\mathbf{x}(i) \in \mathbf{K}$ for all $i = 1, \dots, r$, i.e., ϕ is supported on \mathbf{K} . \square

6.2. Proof of Theorem 3.2.

Proof. (i) By Theorem 3.1, $\phi^{\hat{n}}$ has an atomic representing measure ϕ supported on at most r ($\leq s$) points $\mathbf{x}(1), \dots, \mathbf{x}(r) \in \mathbf{K}$. Moreover, as $\deg(f) \leq 2n - 1$,

$$f^* \geq \rho_n = \phi^n(f) = \phi^{\hat{n}}(f) = \int_{\mathbf{K}} f d\phi = \sum_{i=1}^r \underbrace{\lambda_i}_{\in(0,1]} \underbrace{f(\mathbf{x}(i))}_{\geq f^*} \geq f^*,$$

(with $\sum_i \lambda_i = 1$) which proves that \mathbf{Q}_n is exact, and $f(\mathbf{x}(i)) = f^*$ for all $i = 1, \dots, r$.

(ii) For every $k \leq n$, let $\mu^k := (\phi_{\alpha}^n)_{|\alpha| \leq 2k}$. Suppose that $r = \text{rank}(\mathbf{M}_{n-1}(\mu^{n-1})) = \text{rank}(\mathbf{M}_n(\phi^n))$. As $v = 1$ and $\mathbf{M}_{n-1}(g_j \phi^n) \succeq 0$ for all $j = 1, \dots, m$, then by the flat extension theorem of Curto and Fialkow [2, 3], ϕ^n has an atomic representing measure μ supported on r atoms of \mathbf{K} , and the r atoms can be extracted; see e.g. [6]. hence the result follows.

On the other hand, if $\text{rank}(\mathbf{M}_{n-1}(\mu^{n-1})) < \text{rank}(\mathbf{M}_n(\phi^n))$ ($\leq n$), then

$$(6.4) \quad \text{rank}(\mathbf{M}_{n-1}(\mu^{n-1})) \leq n-1 = (n-1-v+1) \quad \text{and} \quad \mu^{n-1}(f) = \phi^n(f) = f^*.$$

Observe that μ^{n-1} satisfies the condition in Theorem 3.1 (with $n-1$ in lieu of n).

– If $n-1 = 1$ then $\text{rank}(\mathbf{M}_{n-1}(\mu^{n-1})) \leq 1$ and in fact $= 1$ (because $\mu^{n-1}(1) = 1$), which in turn implies that μ^{n-1} has a representing measure on \mathbf{K} which is the Dirac at some point $\mathbf{x}^* \in \mathbf{K}$, and the result follows.

– If $n-1 > 1$, then $2n-2 > 2$ and so $\mu^{n-2}(f) = \phi^n(f) = f^*$. Moreover as $\text{rank}(\mathbf{M}_{n-1}(\mu^{n-1})) \leq n-1 = (n-1-v+1)$, then by Theorem 3.1, $\widehat{\mu^{n-1}}$ has a representing measure on \mathbf{K} supported on $r \leq n-1$ points of \mathbf{K} and as $\widehat{\mu^{n-1}}(f) = \phi^n(f) = f^*$ (because $2n-3 \geq 2$), the r points are global minimizers of \mathbf{P} .

We may and will repeat the argument with μ^{n-1} in lieu of ϕ^n . Suppose that $\text{rank}(\mathbf{M}_{n-2}(\mu^{n-2})) = \text{rank}(\mathbf{M}_{n-1}(\mu^{n-1}))$. Then as $\mathbf{M}_{n-2}(g_j \mu^{n-1}) \succeq 0$ for all j , again by the flat extension theorem of Curto & Fialkow, μ^{n-1} has an atomic representing measure supported on $r = \text{rank}(\mathbf{M}_{n-1}(\mu^{n-1}))$ ($\leq n-1$) points of \mathbf{K} that can be extracted; see [6]. As $\mu^{n-1}(f) = \phi^n(f) = f^*$ and $f \geq f^*$ on \mathbf{K} , then all points are global minimizers and the result follows.

On the other hand, if $\text{rank}(\mathbf{M}_{n-2}(\mu^{n-2})) < \text{rank}(\mathbf{M}_{n-1}(\mu^{n-1}))$ ($\leq n-1$) then

$$\text{rank}(\mathbf{M}_{n-2}(\mu^{n-2})) \leq n-2 \quad \text{and} \quad \mu^{n-2}(f) = \phi^n(f) = f^*,$$

because $n - 2 \geq 1$. That is, we are back to the case (6.4) but now with $n - 2$ instead of $n - 1$. By iterating, we stop with a measure μ^{n-k} and either:

- $n - k > 1$, $\text{rank}(\mathbf{M}_{n-k}(\mu^{n-k})) = \text{rank}(\mathbf{M}_{n-k-1}(\mu^{n-k-1}))$, and in addition, $\mu^{n-k}(f) = \phi^n(f) = f^*$, $\mathbf{M}_{n-k-1}(g_j \mu^{n-k}) \succeq 0$, $j = 1, \dots, m$. Hence by Flat extension theorem of Curto & Fialkow, μ^{n-k} has a representing measure supported on $\text{rank}(\mathbf{M}_{n-k}(\mu^{n-k}))$ global minimizers of \mathbf{P} , and the minimizers can be extracted [6]; hence the result follows.
- or $n - k = 1$ in which case $\text{rank}(\mathbf{M}_{n-k}(\mu^{n-k})) = 1$ because $\mu^{n-k}(1) = \phi^n(1) = 1$. Then μ^{n-k} is represented by the Dirac measure at the point $\mathbf{x}^* = (\mu^{n-k}(x_1), \dots, \mu^{n-k}(x_d))$, and as $v = 1$, $\mathbf{M}_{n-k-1}(g_j \mu^{n-k}) = g_j(\mathbf{x}^*) \geq 0$ for every $j = 1, \dots, m$. Hence $\mathbf{x}^* \in \mathbf{K}$, $\mu^{n-k}(f) = \phi^n(f) = f^*$, and so the result follows.

□

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