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► **To cite this version:**

Joao Manoel Gomes da Silva, Sophie Tarbouriech. Saturating Sampled-Data Control Design for Lur'e Systems. 2024 IEEE 63rd Conference on Decision and Control (CDC), Dec 2024, Milan, Italy. pp.1613-1618, <10.1109/CDC56724.2024.10886514>. <hal-05241456>

HAL Id: hal-05241456

<https://laas.hal.science/hal-05241456v1>

Submitted on 5 Sep 2025

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Saturating Sampled-Data Control Design for Lur'e Systems

João Manoel Gomes da Silva Jr. and Sophie Tarbouriech

Abstract—This paper addresses the problem of designing saturating sampled-data control laws for Lur'e type systems. The closed-loop system is modeled as an hybrid system to formally take into account the aperiodic sampling phenomenon and a control law that depends on the time elapsed between two consecutive sampling instants. This control law considers a nonlinear feedback of the sampled state and the previous value of the control signal. Under this setup and the use of timer-dependent quadratic Lyapunov functions, regional stabilizing conditions are proposed as timer-dependent linear matrix inequalities (LMIs). Considering an affine timer dependency, an optimization problem based on semi-definite programming is proposed to compute the control law gains aiming at enlarging an estimate of the region of attraction of the closed-loop system. A numerical example illustrates the application of the proposed results.

Index Terms—Lur'e systems; sampled-data systems; control saturation; hybrid systems.

I. INTRODUCTION

Many nonlinear systems can be modeled as a feedback connection between a linear system and a sector bounded nonlinearity in a configuration known as a Lur'e System. For such systems, several methods to certify stability have been proposed in the literature, considering frequential methods, based on an IQC framework and the use of appropriate multipliers [1], and Lyapunov methods, considering quadratic and Lur'e type functions [2], [3], [4], [5], [6]. Also, control design methods for stabilizing the origin of continuous-time Lur'e systems have been proposed, for instance, in [7] and [8].

Differently from linear systems, in the case of periodic sampled-data control of nonlinear system an exact discretized model is in general not possible to obtain. On the other hand, it has been shown in [9] that a simple numerical discretization of the plant and the computation of a discrete-time controller may not guarantee the stability of the actual sampled-data system [9], [10]. The problem is even more critical when an aperiodic sampled-data policy is considered, which represents a realistic scenario in the context of networked control systems [11]. These facts motivated the study of stability certification taking into account the inherent hybrid behavior of Lur'e systems under sampled-data control, where at each sampling instant the control value is updated and the system evolves in continuous-time with a fixed constant control input value between two

sampling instants. In this context we can cite works using Lyapunov-Krasovskii [12], [13] and looped-functionals [14], IQC methods [15] and an hybrid system framework [16]. Regarding the design of the control law taking explicitly into account the sampling behavior, some works have also been recently proposed. In [17], considering periodic sampling, a method based on differential LMIs (DLMIs) have been derived to design a state feedback to minimize a guaranteed quadratic cost. In [18], using a looped functional approach, a method to design stabilizing control laws under aperiodic sampling is proposed. To tackle the fact that the derived matrix inequalities conditions in that paper are nonlinear, a particle swarm optimization algorithm to compute the sampled-data state feedback control law is considered. It should be noticed that these methods do not consider the physical limitations of the actuators and the effects of control saturation.

In this work we address the design of aperiodic sampled-data control laws for Lur'e type systems taking also into account the possible actuator saturation. Differently from [17] and [18], a more generic control law, in the sense that it presents a nonlinear state feedback term and the gains depend on the time elapsed between two consecutive sampling instants, is considered. An hybrid system framework is therefore applied to formally model the closed-loop system under aperiodic sampling. From this framework, considering timer-dependent quadratic Lyapunov functions, conditions to design a regionally stabilizing control law are formulated as timer-dependent linear matrix inequalities (LMIs). Therefore, considering an affine timer dependency, an optimization problem based on semi-definite programming is proposed to compute the control law gains aiming at enlarging an estimate of the region of attraction of the origin of the closed-loop system under saturating sampled-data control. A numerical example illustrates the application of the proposed results.

Notation: \mathbb{N} is the set of natural numbers, \mathbb{R} is the set of real numbers, and $\mathbb{R}_{\geq 0}$ is the set of nonnegative real numbers. The i th element of a vector $v \in \mathbb{R}^n$ is denoted by $v_{(i)}$. The vector v has Euclidean norm given by $|v| = \sqrt{v_{(1)}^2 + \dots + v_{(n)}^2}$. The distance of a vector v to a closed set \mathcal{A} is denoted $|v|_{\mathcal{A}}$ and it is defined as $|v|_{\mathcal{A}} = \inf_{y \in \mathcal{A}} |v - y|$. \mathbb{S}^n is the set of symmetric matrices of order n , and for a symmetric matrix $S \in \mathbb{S}^n$, $S \succ 0$ means that S is positive definite. \mathbb{D}^n and $\mathbb{D}_{\geq 0}^n$ are the sets of diagonal and positive semi-definite diagonal matrices of order n , respectively. M^{\top} denotes the transpose of M , and $\text{He}\{M\} = M^{\top} + M$. The matrices I_n and $0_{n \times m}$ denote an identity matrix of order n

This work was supported by Coordenação de Aperfeiçoamento de Pessoal de Nível Superior (CAPES) - Finance Code 001 and Conselho Nacional de Desenvolvimento Científico e Tecnológico (CNPq) (grant PQ 307449/2019-0) from Brazil. It was also supported by Project Stic-AmSud NetConHyb-SDP, code 22-STIC-09, funded by CAPES (Brazil) and CNRS (France). J.M. Gomes da Silva Jr. is with UFRGS, Porto Alegre, Brazil and S. Tarbouriech is with LAAS-CNRS, Université de Toulouse, CNRS, Toulouse, France

and a $n \times m$ matrix of zeros, respectively.

II. PROBLEM STATEMENT

This paper focuses on the following Lur'e system:

$$\begin{cases} \dot{x}(t) = Ax(t) + B_u u(t) + B_\phi \phi(q(t)) \\ q(t) = Cx(t) \end{cases} \quad (1)$$

with $A \in \mathbb{R}^{n \times n}$, $B_u \in \mathbb{R}^{n \times p}$, $B_\phi \in \mathbb{R}^{n \times m}$, and $C \in \mathbb{R}^{m \times n}$. The vector $x \in \mathbb{R}^n$ represents the plant state, $u \in \mathbb{R}^p$ is the control input and $q \in \mathbb{R}^m$ is the argument of the function ϕ , which depends linearly on the state.

The nonlinear Lur'e function $\phi : \mathbb{R}^m \rightarrow \mathbb{R}^m$ is a decentralized and memoryless vector-valued function, i.e

$$\phi(q) \triangleq [\phi_{(1)}(q_{(1)}) \ \dots \ \phi_{(m)}(q_{(m)})]^\top, \quad (2)$$

that satisfies the following assumption.

Assumption 1: The function ϕ is sector bounded, that is, for all $q \in \mathbb{R}^m$ it satisfies

$$\phi_{(i)}(0) = 0, \quad (3)$$

$$\frac{\phi_{(i)}(q)}{q_{(i)}} \in [0, \lambda_{(i)}], \quad (4)$$

for all $i = 1, 2, \dots, m$, where $\lambda \in \mathbb{R}^m$ is the vector defining the upper sector bounds.

Defining $\Lambda = \text{diag}(\lambda_1, \dots, \lambda_m)$, from properties (3) and (4) the nonlinearity ϕ satisfies the following lemma (see also [19], [7]).

Lemma 1: If ϕ satisfies (3)-(4), then the relation

$$\phi(q)^\top S^{-1}(\Lambda q - \phi(q)) \geq 0 \quad (5)$$

is satisfied for all $q \in \mathbb{R}^m$, with any matrix $S \in \mathbb{D}_{\geq 0}^m$.

For control purposes we consider an aperiodic sampled-data policy, where the control is updated at discrete-time instants (i.e. the sampling instants), denoted by $t = t_k$, $k \in \mathbb{N}$, from the measured data and kept constant between two successive sampling instants, i.e. between t_k and t_{k+1} by means of a zero order hold. The intersampling time is denoted by $\delta_k = t_{k+1} - t_k$ and we assume that it is lower and upper bounded as follows:

$$0 < T_1 \leq \delta_k \leq T_2. \quad (6)$$

In this work we assume that the state is measurable. Thus our aim is to design a stabilizing sampled-data feedback control law composed by terms depending on the state, on the previous value of the control signal, and on the nonlinearity ϕ . Moreover, the actuator limits and the possibility of control saturation should be taken into account. In this context we consider the following control law:

$$u(t) = \begin{cases} \text{sat}(K_x(\delta_{k-1})x(t_k) + K_u(\delta_{k-1})u(t_{k-1}) \\ \quad + K_\phi(\delta_{k-1})\phi(Cx(t_k))) \text{ if } t = t_k \\ u(t_{k-1}) \text{ if } t \in [t_{k-1}, t_k) \end{cases} \quad (7)$$

where $K_x \in \mathbb{R}^{p \times n}$, $K_u \in \mathbb{R}^{p \times p}$ and $K_\phi \in \mathbb{R}^{p \times m}$ are the controller gains that depend on the length of the last sampling

interval (i.e. δ_{k-1}) and $\text{sat}(\cdot) : \mathbb{R}^m \rightarrow \mathbb{R}^m$ is a vector-valued saturation function defined as follows:

$$\text{sat}(v)_{(i)} = \text{sign}(v_{(i)}) \min(|v_{(i)}|, 1) \quad \forall i = 1, \dots, m \quad (8)$$

Since the system is nonlinear and the control signal is bounded, in general it is not possible to compute the gains K_x , K_u and K_ϕ to achieve the global asymptotic stability of the origin. In this case, the convergence to the origin will be ensured only for initial conditions in the so-called *region of attraction* or *basin of attraction* of the origin, which is defined as follows:

$$R_{a,0} = \{\xi \in \mathbb{R}^n : \lim_{t \rightarrow \infty} x(t) = 0 \text{ for } x(0) = \xi\}$$

As the geometry of $R_{a,0}$ depends on the control law gains, in general, its direct maximization in a design problem is quite complicated. Instead, the idea is to compute the controller gains that lead to large estimates of the $R_{a,0}$. This estimates are in fact subsets of $R_{a,0}$, which are, in general, generated from sublevel sets of Lyapunov functions [19], [20], [21].

Hence, the control design problem we intend to solve in this paper can be summarized as follows.

Problem 1: Design the controller gains $K_x \in \mathbb{R}^{p \times n}$, $K_u \in \mathbb{R}^{p \times p}$ and $K_\phi \in \mathbb{R}^{p \times m}$ in order to guarantee the asymptotic stability of the origin of the closed-loop system (1)-(7) and to maximize an estimate of $R_{a,0}$.

III. HYBRID SYSTEM REPRESENTATION

Due to the considered sampled-data policy (7), the control signal $u(t)$ experiences a jump (i.e. its value is instantaneously updated) at each sampling time and it is held constant between two successive sampling times. The input then follows a jump dynamics, whereas the state x follows a continuous-time evolution. It is then natural to address Problem 1, by taking advantage of the hybrid dynamical system framework proposed in [22].

By defining $z = [x^\top \ u^\top]^\top$, we consider the augmented state $\eta = [z^\top \ \tau]^\top$, where τ is a timer that counts the time elapsed since the last sampling instant. As a consequence, a jump is triggered when $\tau \in [T_1, T_2]$. After the jump, the value of τ is reset to 0. Hence, the closed-loop dynamics can be represented by the following hybrid model:

$$\begin{cases} \dot{\eta} = \begin{bmatrix} \dot{z} \\ \dot{\tau} \end{bmatrix} = f_F(\eta), & \forall \eta \in \mathcal{C} \\ \eta^+ = \begin{bmatrix} z^+ \\ \tau^+ \end{bmatrix} = f_J(\eta), & \forall \eta \in \mathcal{D} \end{cases} \quad (9)$$

with the flow set $\mathcal{C} = \mathbb{R}^{n+p} \times [0, T_2]$ and the jump set $\mathcal{D} = \mathbb{R}^{n+p} \times [T_1, T_2]$. The flow and jump maps $f_F : \mathbb{R}^h \rightarrow \mathbb{R}^h$ and $f_J : \mathbb{R}^h \rightarrow \mathbb{R}^h$, with $h = n + p + 1$, are defined as follows:

$$f_F(\eta) = \begin{bmatrix} A_F z + B_F \phi(C_z z) \\ 1 \end{bmatrix} \quad (10)$$

$$f_J(\eta) = \begin{bmatrix} (A_J + B_J K_z(\tau))z + B_J K_\phi(\tau)\phi(C_z z) \\ -B_J \psi(K_z(\tau)z + K_\phi(\tau)\phi(C_z z)) \\ 0 \end{bmatrix} \quad (11)$$

with:

$$A_F = \begin{bmatrix} A & B_u \\ 0 & 0 \end{bmatrix}, B_F = \begin{bmatrix} B_\phi \\ 0 \end{bmatrix}, C_z = [C \ 0],$$

$$A_J = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}, B_J = \begin{bmatrix} 0 \\ I \end{bmatrix}, K_z(\tau) = [K_x(\tau) \ K_u(\tau)].$$

and the function $\psi : \mathbb{R}^p \rightarrow \mathbb{R}^p$ defined as

$$\psi(v) = v - \text{sat}(v) \text{ for } v \in \mathbb{R}^p. \quad (12)$$

The solutions to system (9) are given in terms of an hybrid arc $\eta(t, k)$, which is parameterized in function of both flow and jump times defined in the domain $\text{dom } \eta = \cup_{k=0}^{\infty} ([t_k, t_{k+1}], k)$. Furthermore, system (9) satisfies the basic conditions for well-posedness (i.e closure of \mathcal{C} and \mathcal{D} , and continuity of f_F and f_J) and the solutions are complete [22]. Due to the fact that $T_1 > 0$, there is no Zeno behavior.

The stability problem of system (9) consists in studying the stability of a compact set \mathcal{A} , containing the origin and the domain of the timer, i.e. $\mathcal{A} = \{0\} \times [0, T_2]$. Once the set \mathcal{A} is asymptotically stable for the system (9), the region (or basin) of attraction of \mathcal{A} is defined as [22, p.141] (see also [23] and [24] using similar arguments in different contexts):

$$R_{a,\mathcal{A}} = \left\{ \zeta \in \mathbb{R}^h : \lim_{(t+k) \rightarrow \infty} |z(t, k)| = 0 \text{ for } z(0, 0) = \zeta \right\} \times [0, T_2]$$

To take into account the presence of the control saturation in the stability conditions to be developed, the following result, which is an extension of the generalized sector condition for deadzone nonlinearities [20], can be stated.

Lemma 2: Let $w(\eta) = K_z(\tau)z + K_\phi(\tau)\phi(C_z z)$ Consider a vector function $g : \mathbb{R}^h \rightarrow \mathbb{R}^p$. If

$$|w(\eta) - g(\eta)| \leq 1, \quad i = 1, \dots, m \quad (13)$$

then relation

$$r(R(\eta), g(\eta)) = 2\psi(w(\eta))^\top R(\eta)^{-1}(\psi(w(\eta)) - g(\eta)) \leq 0 \quad (14)$$

is verified for any vector-dependent diagonal positive semi-definite matrix $R : \mathbb{R}^h \rightarrow \mathbb{D}_{\geq 0}^{p \times p}$.

The next theorem provides sufficient conditions to ensure that \mathcal{A} is locally asymptotically stable for the system (9), along with a characterization of an estimate of its region of attraction. It corresponds to an adapted version of Theorem 3.18 in [22].

Theorem 1: If there exist a function $V : \mathbb{R}^h \rightarrow \mathbb{R}_{\geq 0}$, a vector function $g : \mathbb{R}^h \rightarrow \mathbb{R}^p$, a vector dependent diagonal matrix $R : \mathbb{R}^h \rightarrow \mathbb{D}_{\geq 0}^{p \times p}$, class \mathcal{K}_∞ functions α_1 and α_2 , and a positive definite function ρ such that

$$\alpha_1(|\eta|_{\mathcal{A}}) \leq V(\eta) \leq \alpha_2(|\eta|_{\mathcal{A}}), \quad \forall \eta \in (C \cup D) \quad (15)$$

$$\langle \nabla V(\eta), f_F(\eta) \rangle \leq -\rho(|\eta|_{\mathcal{A}}), \quad \forall \eta \in C \setminus \mathcal{A} \quad (16)$$

$$V(f_J(\eta)) - V(\eta) - r(R(\eta), g(\eta)) \leq -\rho(|\eta|_{\mathcal{A}}), \quad \forall \eta \in D \setminus \mathcal{A} \quad (17)$$

$$|w(\eta)_{(i)} - g(\eta)_{(i)}|^2 \leq \frac{1}{\mu} V(\eta), \quad \forall \eta \in D \setminus \mathcal{A}, \quad i = 1, \dots, m \quad (18)$$

then, considering system (9):

1. the set \mathcal{A} is locally asymptotically stable;
2. for any initial condition

$$\eta(0, 0) \in \mathcal{L}_V(\mu) = \{\eta \in \mathbb{R}^h : V(\eta) \leq \mu\},$$

it follows that $\eta(t, k) \rightarrow \mathcal{A}$ as $(t+k) \rightarrow \infty$, that is $\mathcal{L}_V(\mu) \subset R_{a,\mathcal{A}}$.

Proof: It follows the same steps of Theorem 1 in [25]. \blacksquare

It is important to emphasize that if \mathcal{A} is asymptotically stable (AS) for the hybrid system (9) with flow and jump maps given by (10) and (11), it follows that 0 (the origin) is AS for the closed-loop system given by (1) and (7). Note that in the present case $|\eta|_{\mathcal{A}} = |z|$ and thus if $\eta(t, k) \rightarrow \mathcal{A}$ as $(t+k) \rightarrow \infty$ it follows that $z(t, k) \rightarrow 0$ as $(t+k) \rightarrow \infty$.

IV. MAIN RESULTS

A. Stability conditions

Consider as Lyapunov function candidate a timer-dependent quadratic function given by:

$$V(\eta) \triangleq V(z, \tau) = z^\top P(\tau)z = z^\top W(\tau)^{-1}z \quad (19)$$

where $P : [0, T_2] \rightarrow \mathbb{S}^{q+m}$ with each element being a continuous Lipschitz function of τ .

Theorem 2: If there exist matrix functions $W : [0, T_2] \rightarrow \mathbb{S}^n$, $Y_z : [0, T_2] \rightarrow \mathbb{R}^{p \times (n+p)}$, $Y_\phi : [0, T_2] \rightarrow \mathbb{R}^{p \times m}$, $X_z : [0, T_2] \rightarrow \mathbb{R}^{p \times (n+p)}$, $S_1 : [0, T_2] \rightarrow \mathbb{D}^m$, $S_2 : [0, T_2] \rightarrow \mathbb{D}^m$, $R : [0, T_2] \rightarrow \mathbb{D}^p$ that satisfy the following linear matrix inequalities:

$$W(\tau) > 0, \quad \forall \tau \in [0, T_2] \quad (20)$$

$$\left[\begin{array}{c} \text{He}\{A_F W(\tau) - \frac{1}{2} \dot{W}(\tau)\} \\ * \\ \forall \tau \in [0, T_2] \end{array} \quad B_F S_1(\tau) + W(\tau) C_z^\top \Lambda \right] \prec 0, \quad (21)$$

$$\left[\begin{array}{cccc} W(\tau) & -W(\tau) C_z^\top \Lambda & -X_z(\tau)^\top & \Gamma(\tau)^\top \\ * & 2S_2(\tau) & -Y_\phi(\tau)^\top & Y_\phi^\top(\tau) B_J^\top \\ * & * & 2R(\tau) & -R(\tau) B_J^\top \\ * & * & * & W(0) \end{array} \right] \succ 0, \quad (22)$$

$\forall \tau \in [T_1, T_2]$, with $\Gamma(\tau)^\top = W(\tau) A_J^\top + Y_z(\tau)^\top B_J^\top$

$$\left[\begin{array}{ccc} W(\tau) & Y_{z(i)}(\tau)^\top - X_{z(i)}(\tau)^\top \\ * & 1 \end{array} \right] \succeq 0, \quad \forall \tau \in [T_1, T_2], \quad i = 1, \dots, p \quad (23)$$

then, the gains $K_z(\tau) = Y_z(\tau)W(\tau)^{-1}$ and $K_\phi(\tau) = Y_\phi(\tau)S_2(\tau)^{-1}$ ensure that for all initial conditions $\eta(0, 0) = [z_0^\top \ \tau_0^\top]^\top$, with $\tau_0 \in [T_1, T_2]$, belonging to the the set

$$\mathcal{E}(W(\tau_0)) = \{z \in \mathbb{R}^{n+p} : z^\top W(\tau_0)^{-1}z \leq 1\} \quad (24)$$

the trajectories of the system (9) converge asymptotically to \mathcal{A} and thus the trajectories of the closed-loop system (1)-(7) converge asymptotically to the origin.

Proof: From (20), it follows that $W(\tau)$ is nonsingular $\forall \tau \in [0, T_2]$. Consider $P(\tau) = W^{-1}(\tau)$. Taking into account that $\frac{dP(\tau)W(\tau)}{d\tau} = P(\tau)\frac{dW(\tau)}{d\tau} + \frac{dP(\tau)}{d\tau}W(\tau) = 0$,

$\dot{W}(\tau) \triangleq \frac{dW(\tau)}{d\tau} \dot{\tau}$ and the fact that $\dot{\tau} = 1$ for $\eta \in C$, it follows that:

$$\dot{W}(\tau) = -W(\tau)\dot{P}(\tau)W(\tau). \quad (25)$$

Taking into account this, left and right multiplying (21) by $\text{diag}(P(\tau), S_1(\tau)^{-1})$ leads to the following equivalent matrix inequality:

$$\begin{bmatrix} A_F^\top P(\tau) + P(\tau)A_F + \dot{P}(\tau) & P(\tau)B_F + C_z^\top \Lambda S_1(\tau)^{-1} \\ \star & -2S_1(\tau)^{-1} \end{bmatrix} < 0, \quad \forall \tau \in [0, T_2] \quad (26)$$

Now left and right multiplying (26) by vector $[z^\top \ \phi(C_z z)^\top]$ and its transpose, respectively, it follows that

$$\langle \nabla V(\eta), f_F(\eta) \rangle + 2\phi(C_z z)^\top S_1(\tau)^{-1}(\Lambda C_z z - \phi(C_z z)) < 0 \quad (27)$$

$\forall \eta \in C \setminus \mathcal{A}$ and thus, using Lemma 1, we can conclude that (21) implies that condition (16) of Theorem 1 is satisfied with $V(\eta) = z^\top P(\tau)z$.

Now, left and right multiplying (22) by the matrix $\text{diag}(P(\tau), S_2(\tau)^{-1}, R(\tau)^{-1}, I)$, it follows that (22) is equivalent to

$$\begin{bmatrix} P(\tau) & -C_z^\top \Lambda S_2(\tau)^{-1} & -G_z(\tau)^\top R(\tau)^{-1} \\ \star & 2S_2(\tau)^{-1} & -K_\phi(\tau)^\top R(\tau)^{-1} \\ \star & \star & 2R(\tau)^{-1} \\ \star & \star & \star \\ A_J^\top + K_z(\tau)^\top B_J^\top \\ K_\phi^\top(\tau) B_J^\top \\ -B_J^\top \\ W(0) \end{bmatrix} \succ 0, \quad \forall \tau \in [T_1, T_2]. \quad (28)$$

where $G_z(\tau) = X_z(\tau)P(\tau)$. Applying now the Schur's complement to (28) one gets that

$$\begin{bmatrix} P(\tau) & -C_z^\top \Lambda S_2(\tau)^{-1} & -G_z(\tau)^\top R(\tau)^{-1} \\ \star & 2S_2(\tau)^{-1} & -K_\phi(\tau)^\top R(\tau)^{-1} \\ \star & \star & 2R(\tau)^{-1} \end{bmatrix} - \begin{bmatrix} A_J^\top + K_z(\tau)^\top B_J^\top \\ K_\phi^\top(\tau) B_J^\top \\ -B_J^\top \end{bmatrix} P(0) \begin{bmatrix} A_J^\top + K_z(\tau)^\top B_J^\top \\ K_\phi^\top(\tau) B_J^\top \\ -B_J^\top \end{bmatrix}^\top \succ 0 \quad (29)$$

Then, left and right multiplying (29) by $[z^\top \ \phi(C_z z)^\top \ \psi(w(\eta))^\top]$ and its transpose, respectively, and considering $g(\eta) = G_z(\tau)z + K_\phi \phi(C_z z)$, it follows that

$$V(\eta) - V(f_J(\eta)) + r(R(\eta), g(\eta)) - 2\phi(C_z z)^\top S_2(\tau)^{-1}(\Lambda C_z z - \phi(C_z z)) > 0, \quad \forall \eta \in D \setminus \mathcal{A}$$

Thus, from Lemmas 1 and 2, we conclude that that condition (17) of Theorem 1 is satisfied.

Finally, left and right multiplying (23) by $\text{diag}(P(\tau), I)$ and applying Schur's complement leads to the following equivalent matrix inequality:

$$P(\tau) - (K_z(\tau)_{(i)} - G_z(\tau)_{(i)})^\top (K_z(\tau)_{(i)} - G_z(\tau)_{(i)}) \succeq 0, \quad (30)$$

$\forall \tau \in [T_1, T_2]$. Left and right multiplying (30) by z^\top and z , it follows that condition (18) of Theorem 1 is verified with $\mu = 1$.

Hence, the satisfaction of conditions (20), (21), (22) and (23) imply the satisfaction of the conditions of Theorem 1¹ with $V(\eta) = z^\top P(\tau)z$, which concludes the proof. ■

B. Affine dependence on τ

It is important to note that the conditions of Theorem 2 turns the control design and stability analysis problems of the closed-loop system into the feasibility problem of a collection of LMIs that should be verified over continuous intervals of τ . Thus, the search for feasible solutions to (20), (21), (22), and (23) is quite challenging since such conditions need to be checked over infinite values of τ . A way to overcome this is to consider some particular structure for the time-dependent matrices.

In this paper we consider that the matrices present an affine dependency on τ . Hence, letting $M(\tau)$ be a generic τ -dependent matrix that represents the variables of Theorem 2, we consider that:

$$M(\tau) = M_0 + M_1 \tau \quad (31)$$

where M_0 and M_1 are constant matrices. In this case, the conditions presented in Theorem 2 can be cast as a finite set of LMIs. Indeed, by means of convexity arguments, it suffices to verify the conditions at the boundaries of the respective intervals, that is, to verify (20) and (21) for $\tau \in \{0, T_2\}$ and to verify (22) and (23) for $\tau \in \{T_1, T_2\}$.

V. OPTIMIZATION PROBLEM

The idea is to use the conditions in Theorem 2 considering matrices with affine dependence to solve Problem 1.

Recall that the region $\mathcal{E}(W(\tau_0))$ is defined in the space \mathbb{R}^{n+p} , corresponding to vector $z = [x^\top \ u^\top]^\top$. By considering $W(\tau)$ with the structure given in (31), we assume the following partitions for $P(\tau_0) = W(\tau_0)^{-1} = (W_0 + \tau_0 W_1)^{-1}$ and $W(\tau_0)$:

$$P(\tau_0) = \begin{bmatrix} P_{11} & P_{12} \\ P_{12}^\top & P_{22} \end{bmatrix}, \quad W(\tau_0) = \begin{bmatrix} W_{11} & W_{12} \\ W_{12}^\top & W_{22} \end{bmatrix} \quad (32)$$

where $P_{11}, W_{11} \in \mathbb{R}^{n \times n}$ and $P_{22}, W_{22} \in \mathbb{R}^{p \times p}$.

Let now x_0 be the plant initial condition and choose:

$$u_0 = -P_{22}^{-1} P_{12}^\top x_0 \quad (33)$$

Hence, it follows that

$$\begin{aligned} z_0^\top P(\tau_0) z_0 &= x_0^\top P_{11} x_0 + 2x_0^\top P_{12} u_0 + u_0^\top P_{22} u_0 \\ &= x_0^\top P_{11} x_0 - x_0^\top P_{12} P_{22}^{-1} P_{12}^\top x_0 \end{aligned} \quad (34)$$

Furthermore, it can be shown that $P_{11} - P_{12} P_{22}^{-1} P_{12}^\top = W_{11}^{-1}$ [26]. Hence, we can conclude that if $x_0 \in \mathcal{E}(W_{11}^{-1}) = \{x \in \mathbb{R}^n : x^\top W_{11}^{-1} x \leq 1\}$, it follows that $z_0 \in \mathcal{E}(W(\tau_0))$ provided that u_0 is computed as in (33).

¹Note that $|\eta|_{\mathcal{A}} = |z|$ and since the LMIs (21) and (22) are strict, it is always possible to determine a scalar $\tilde{\rho}$ such that $\rho(|\eta|_{\mathcal{A}}) = \tilde{\rho}|z|^2$.

Thus, the idea is to maximize this “safe” set of plant initial states given by $\mathcal{E}(W_{11}^{-1})$, considering some size criterion. For instance, assuming the affine dependency on τ for the decision variables, we can consider the minimization of $tr(W_{11}^{-1})$ through the following convex optimization problem:

$$\text{OP} \sim \begin{cases} \min tr(M) & \text{s.t.} \\ (20), (21) & \text{for } \tau \in \{0, T_2\} \\ (22), (23) & \text{for } \tau \in \{T_1, T_2\} \\ \begin{bmatrix} M & I \\ I & W_{11} \end{bmatrix} \succeq 0 \end{cases} \quad (35)$$

Note that the results of Theorem 1 and the reasoning above hold for any choice of $\tau_0 \in [T_1, T_2]$, since we consider that $\eta(0, 0)$ belongs to the jump set. In particular, in the numerical example of the next section, we consider $\tau_0 = T_1$.

VI. NUMERICAL EXAMPLES

Consider system (1) with the following matrices:

$$A = \begin{bmatrix} -2 & 1 \\ 1 & -3 \end{bmatrix}, \quad B_u = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad B_\phi = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad \text{and} \quad C = [0 \quad 1].$$

Considering $\tau_o = T_1$ and $W(\tau_0) = W_0 + T_1 W_1$, Tables I and II shows the values of $tr(W_{11}^{-1})$ obtained from the solution of the optimization problem (35). It should be noted that as the value of Λ increases (i.e. large is the sector for the admissible nonlinearities), $tr(W_{11}^{-1})$ also increases indicating that the size of the achievable estimate of region of attraction is smaller. This also happens when T_2 increases, which is quite expected since in this case the control is potentially updated less frequently. It can also be observed that if the sector where the nonlinearity ϕ lies is sufficiently small the estimate increases considerably. Actually, for $\lambda = 0.99$ it can be shown that the origin is globally stabilizable. Figure 1 compares the estimates of the region of attraction for $T_2 = 0.5$ considering $\Lambda = 1.24$ and $\Lambda = 1.3$.

TABLE I

$tr(W_{11}^{-1})$ FOR $T_1 = 0.01$, $T_2 = 0.5$ AND DIFFERENT VALUES OF Λ

Λ	1	1.24	1.3	1.5
$tr(W_{11}^{-1})$	3×10^{-4}	0.2844	0.5150	NF

TABLE II

$tr(W_{11}^{-1})$ FOR $T_1 = 0.01$, $T_2 = 0.7$ AND DIFFERENT VALUES OF Λ

Λ	1	1.1	1.24	1.3
$tr(W_{11}^{-1})$	3.1×10^{-4}	0.0448	0.4737	NF

For $\Lambda = 1.24$, $T_1 = 0.01$ and $T_2 = 0.5$, optimization problem (35) results in the following matrices:

$$W_0 = \begin{bmatrix} 33.3603 & -10.4340 & -2.3270 \\ -10.4340 & 7.5869 & -1.1764 \\ -2.3270 & -1.1764 & 1.0015 \end{bmatrix},$$

$$W_1 = \begin{bmatrix} -43.6788 & 17.9104 & 1.6487 \\ 17.9104 & -7.3448 & -0.4932 \\ 1.6487 & -0.4932 & 1.1675 \end{bmatrix},$$

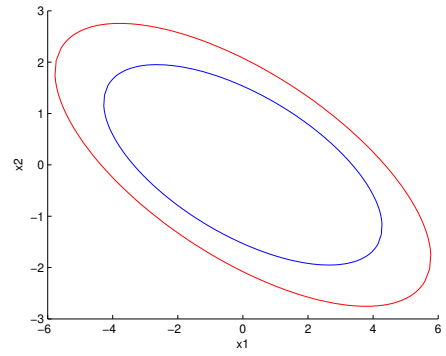


Fig. 1. Estimates for $\Lambda = 1.24$ (red) and $\Lambda = 1.3$ (blue)

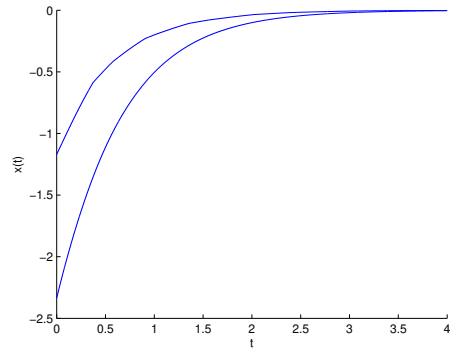


Fig. 2. States evolution

$$Y_{z0} = [-2.3270 \quad -1.1764 \quad 1.0010],$$

$$Y_{z1} = [1.1753 \quad -0.4816 \quad -0.1249],$$

$$Y_{\phi 0} = -0.0015, \quad Y_{\phi 1} = 0.0019$$

$$S_{20} = 1.3017 \times 10^6, \quad S_{21} = -2.5068 \times 10^6.$$

With the gains $K_z(\tau) = (Y_{z0} + \tau Y_{z1})(W_0 + \tau W_1)^{-1}$ and $K_\phi(\tau) = (Y_{\phi 0} + \tau Y_{\phi 1})(S_{20} + \tau S_{21})^{-1}$ computed from the above matrices, we simulate now the closed-loop system considering $T_1 = 0.01$, $T_2 = 0.5$, the nonlinearity $\phi(q) = q - \sin(q) = x_2 - \sin(x_2)$, which belongs to the sector defined with $\Lambda = 1.24$, and the initial condition

$$x_0 = [-2.3368 \quad -1.1703]^\top,$$

which is at the border of the ellipsoidal estimate. Figures 2 and 3 show the states and the control responses. Note that the control saturates at the first instants of the simulation. Figure 4 shows the sampling instants and the value of δ_k corresponding to the time between two consecutive samplings.

VII. CONCLUSION

This paper addressed the problem of designing saturating sampled-data control laws for Lur’e type systems. Leveraging the hybrid dynamical system framework, the aperiodic sampling phenomenon and a control law with gains depending on the time elapsed between two consecutive

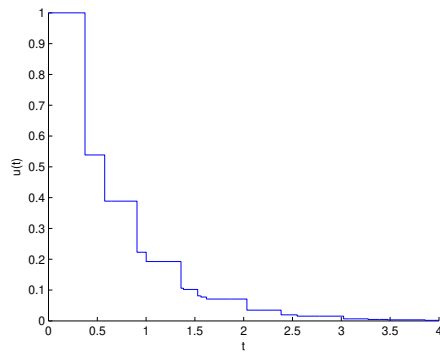


Fig. 3. Control evolution

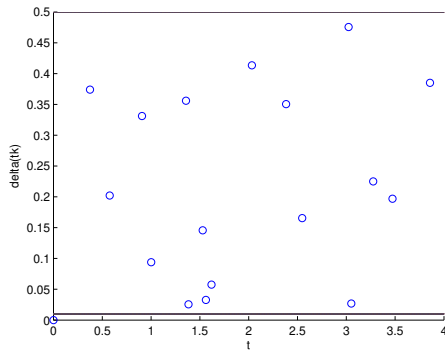


Fig. 4. Sampling instants

time instants are taking into account. The proposed control law corresponds to a nonlinear feedback of the sampled state and the value of the previous value of the control signal. Regional stability conditions are formulated through LMI-based conditions thanks to the use of timer-dependent quadratic Lyapunov functions. By considering an affine timer dependency, an optimization scheme is proposed with the aim of enlarging an estimate of the region of attraction of the origin of the closed-loop system.

The proposed results pave the way for future works. In particular it could be interesting to ensure some level of performance in presence of disturbances or uncertainties affecting the system. The conservatism reduction by considering polynomial timer dependency on the matrices should also be investigated.

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