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Setpoint Tracking for a Class of Lur'e Discrete-Time Systems

F. Ferrante, M. Leomanni, M. L. Fravolini, and S. Tarbouriech

Abstract—Setpoint tracking for a class of discrete-time nonlinear plants is addressed. The plant under consideration is a feedback interconnection of a square linear system (with the same number of inputs and outputs) and a nonlinearity obeying to an incremental quadratic constraint. Within this setting, we design a dynamic feedback controller ensuring, for the closed-loop system, internal exponential stability and output tracking of constant setpoints with tunable performance guarantees. A proportional-integral feedback controller is proposed. Combining Lyapunov theory and the use of a quadratic Lyapunov function, sufficient conditions for the design of the controller are stated in terms of linear matrix inequalities. An optimal controller design algorithm based on semidefinite programming is proposed to design the controller gains to improve transient performance and limit the control effort. Two numerical examples illustrate the application of the proposed methodology.

I. INTRODUCTION

The setpoint tracking problem has been a central problem in control theory due its practical relevance. This problem consists of controlling a given plant to ensure its output tracks a given constant reference. For linear plants, the underlying theory is well established and rooted in the so-called output regulation theory, which revolves on the notion of internal model introduced in [1]. Whenever the plant is nonlinear, the problem becomes notably harder and other tools are needed. In this regard, an extension of the linear internal model theory to nonlinear plants is presented in [2], yet a general design methodology is still lacking. For the specific case of setpoint tracking, output regulation theory prescribes the use of controllers embedding an integral action processing the tracking error. This led researchers to address the problem of integral control of nonlinear plants. Fundamental results about this problem have been established in [3], [4] for the case of continuous-time input-affine nonlinear systems. More recently, the problem of integral control for the same class of nonlinear systems has been revisited via contraction theory in [5].

In several cases of practical relevance, the plant can be modeled as the feedback interconnection of a linear system and a sector bounded nonlinearity. This class of systems is generally denoted as Lur'e systems. In the last two decades, there has been a growing research interest in the analysis, control, and observation of Lur'e systems, both in continuous-time and discrete-time. A prominent advantage of this class of systems is the possibility of representing

the system nonlinearities via quadratic constraints. Taking advantage of this feature, sufficient conditions guaranteeing asymptotic stability can be expressed in terms of linear matrix inequalities (LMIs) [6]. These conditions can be efficiently solved through via interior point methods readily available as off-the-shelf software.

Over the years, LMI-based controller design methodologies for Lur'e systems have been extensively used in addressing stability analysis, stabilization, synchronization, and observer design. Relevant results in these settings can be found in [7], [8], [9], [10], [11], [12], [13], just to mention a few. However, the setpoint tracking problem has received comparatively less attention. One of the challenges in addressing this problem is that the extension from stabilization to setpoint tracking is nontrivial, owing to the nonlinearity of the plant. The problem of setpoint tracking for Lipschitz nonlinear systems (a particular class of Lur'e systems) is tackled in [14]. A similar problem is addressed in [15] and [16] for linear systems in the presence of saturating input. More recently, set point tracking for Lipschitz discrete-time nonlinear systems has been addressed [17]. As opposed to [14], in [17] the authors assume, similarly as in [8], that the system nonlinearity depends both on the control and the plant state. The approach [17] results into an optimization problem involving bilinear matrix inequality (BMI) constraints for the design of the controller. An iterative convex relaxation scheme is proposed to solve the mentioned BMI optimization problem. This, however requires an initial feasible solution to the underlying optimization problem to be exploited.

Inspired by the setup in [17], in this paper we address the problem of setpoint tracking for a class of discrete-time Lur'e plants. In particular, we consider plants that are modeled as a feedback interconnection of a linear system and a nonlinearity obeying to an incremental quadratic constraint as those considered in [18] and to some extent in [10]. The feedback interconnection we consider is rather general as it includes a feedthrough term depending on the control input. Within this setting, we provide sufficient conditions in the form of LMIs for the design of a state controller involving a discrete-time integral action that ensure global exponential internal stability and global exponential tracking of constant setpoints. In addition, we formulate a suitable accompanying optimal control problem that allows for balancing transient performance against control effort. This problem is later cast in terms of a semidefinite program, i.e, a convex optimization problem over LMI constraints, that can be efficiently exploited for the design of the proposed controller.

The remainder of the paper is organized as follows. Section II illustrates and formalizes the problem we solve.

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Section III contains the main results of the paper. Section IV is devoted to optimal controller design. Finally, Section V illustrates the applicability of our methodology in two numerical examples.

A. Notation

The symbol $\mathbb{N}_{\geq 0}$ denotes the set of nonnegative integers and \mathbb{R} is the set of real numbers. The Euclidean space of dimension n is represented by \mathbb{R}^n , $\mathbb{R}^{n \times m}$ is the set of $n \times m$ real matrices, with $n, m \in \mathbb{N}$. Given $A \in \mathbb{R}^{n \times m}$, A^\top denotes the transpose of A . When $n = m$, $\text{He}(A) := A + A^\top$ and $\text{specr}(A)$ indicates the spectral radius of A . The symbol $\ker A$ indicates the kernel of the matrix A . The $n \times n$ identity matrix is denoted by I_n , while I indicates a suitably sized identity matrix. The symbol \mathbb{S}_+^n (\mathbb{S}_-^n) represents the set of $n \times n$ symmetric positive (negative) definite matrices. For a symmetric matrix A , $A \succ 0$ ($A \prec 0$) means that A is positive (negative) definite. The symbols $\lambda_{\min}(A)$ and $\lambda_{\max}(A)$ indicate, respectively, the smallest and largest eigenvalue of the symmetric matrix A . The notation $A_1 \oplus A_2 \oplus \dots \oplus A_n$ indicates the block diagonal matrix with A_1, A_2, \dots, A_n on the diagonal. The shorthand notation $A^\top B A = (\bullet)^\top B A$ is used throughout the paper. In partitioned symmetric matrices, the symbol \star represents a symmetric block. For a vector $x \in \mathbb{R}^n$, $\|x\|$ denotes its Euclidean norm. Given two vectors x and y , we use the equivalent notation $(x, y) = [x^\top y^\top]^\top$.

B. Preliminaries

In this paper, we consider nonlinear discrete-time control systems of the form

$$\begin{aligned} x^+ &= g(x, r) \\ y &= h(x) \end{aligned} \quad (1)$$

where $x \in \mathbb{R}^n$ is the system state, $y \in \mathbb{R}^q$ is a regulated output and $r \in \mathbb{R}^q$ is a constant setpoint. In particular, given $r \in \mathbb{R}^q$ a solution to (1) is any function $x: \mathbb{N}_{\geq 0} \rightarrow \mathbb{R}^n$ such that¹

$$x(j+1) = g(x(j), r), \quad \forall j \in \mathbb{N}_{\geq 0}.$$

The following notion of stability is used throughout the paper.

Definition 1 (0-input global exponential stability): The origin of (1) is 0-input globally exponentially stable if there exist positive scalars κ and λ such that, for $r = 0$, the unique solution x to (1) satisfies

$$|x(j)| \leq \kappa e^{-\lambda j} |x(0)|, \quad \forall j \in \mathbb{N}_{\geq 0}.$$

II. PROBLEM SETTING

In this work, we consider the following class of nonlinear square plants

$$\begin{cases} x_p^+ = A_p x_p + B_p u + M_p \varphi(z) \\ z = G_p x_p + N_p u \\ y = C_p x_p \end{cases} \quad (2)$$

¹By definition, solutions to (1) are meant to be complete. This is not restrictive since any maximal solution to (1) is complete.

where $x_p \in \mathbb{R}^n$, $u \in \mathbb{R}^q$, $y \in \mathbb{R}^q$ are, respectively, the plant state, control input, and regulated output. Matrices $A_p \in \mathbb{R}^{n \times n}$, $B_p \in \mathbb{R}^{n \times q}$, $M_p \in \mathbb{R}^{n \times n}$, $G_p \in \mathbb{R}^{s \times n}$, $N_p \in \mathbb{R}^{s \times q}$, $C_p \in \mathbb{R}^{q \times n}$ are assumed to be known. For the mapping $\varphi: \mathbb{R}^s \rightarrow \mathbb{R}^p$ a milder assumption is considered later (see Assumption 2). Within this setting, our goal consists of designing a feedback controller ensuring, for the closed-loop system, internal exponential stability and output tracking of constant setpoints with tunable performance guarantees. To address this problem, we consider the following assumptions.

Assumption 1: For all $r \in \mathbb{R}^q$, there exists a unique pair $(\bar{x}_p, \bar{u}) \in \mathbb{R}^{n+q}$ such that

$$\begin{aligned} \bar{x}_p &= A_p \bar{x}_p + B_p \bar{u} + M_p \varphi(G_p \bar{x}_p + N_p \bar{u}) \\ C_p \bar{x}_p &= r. \end{aligned} \quad (3)$$

Assumption 1 ensures, for any value of the setpoint r , the existence of a unique equilibrium pair (\bar{x}_p, \bar{u}) ensuring zero tracking error.

Assumption 2: There exist $S \in \mathbb{S}_+^s$, $T \in \mathbb{R}^{s \times p}$, and $R \in \mathbb{S}_-^p$ such that, for all $z_1, z_2 \in \mathbb{R}^s$,

$$\begin{bmatrix} z_1 - z_2 \\ \varphi(z_1) - \varphi(z_2) \end{bmatrix}^\top \begin{bmatrix} S & T \\ \star & R \end{bmatrix} \begin{bmatrix} z_1 - z_2 \\ \varphi(z_1) - \varphi(z_2) \end{bmatrix} \geq 0. \quad (4)$$

Moreover

$$\varphi(0) = 0. \quad (5)$$

Assumption 2 is somewhat related to the notion of incremental sector boundedness considered in [10], [19] and it enables to address the considered output regulation problem via the use of LMIs. Assuming that φ vanishes at zero ensures that the origin is a zero-input equilibrium for the plant.

Remark 1: Assumption 2 captures several nonlinearities of interest such as Lipschitz and slope restricted functions; see [18] for more details.

To solve the considered output regulation problem, in this section we propose a feedback controller that asymptotically stabilizes the *unknown* equilibrium point \bar{x}_p characterized in Assumption 1. To this end, we rely on the following proportional-integral feedback controller:

$$\begin{cases} \xi^+ = \xi + r - y \\ u = K_P x_p + K_I \xi \end{cases} \quad (6)$$

where $\xi \in \mathbb{R}^q$ is the controller state (integral of the tracking error) and $K_P \in \mathbb{R}^{n \times n}$ and $K_I \in \mathbb{R}^{q \times q}$ are the controller gains to be designed. The interconnection of (2) and (6) yields the following closed-loop system with state $x := (x_p, \xi) \in \mathbb{R}^{n+q}$

$$\begin{cases} x_p^+ = (A_p + B_p K_P) x_p + B_p K_I \xi + M_p \varphi(z) \\ \xi^+ = \xi - C_p x_p + r \\ z = G_p x_p + N_p (K_P x_p + K_I \xi) \\ y = C_p x_p \end{cases} \quad (7)$$

In particular, by defining

$$\begin{aligned} A &:= \begin{bmatrix} A_p & 0 \\ -C_p & I \end{bmatrix}, B := \begin{bmatrix} B_p \\ 0 \end{bmatrix}, K := [K_P \quad K_I] \\ M &:= \begin{bmatrix} M_p \\ 0 \end{bmatrix}, Q := \underbrace{\begin{bmatrix} G_p & 0 \end{bmatrix}}_G + N_p K, J := \begin{bmatrix} 0 \\ I \end{bmatrix} \\ C &:= [C_p \quad 0], \end{aligned} \quad (8)$$

closed-loop system (7) can be rewritten more compactly as follows

$$\begin{aligned} x^+ &= (A + BK)x + M\varphi(Qx) + Jr \\ y &= Cx. \end{aligned} \quad (9)$$

We are now in a position to formally state the problem we solve in this paper.

Problem 1: Design K_P and K_I such that the following properties hold:

- (i) the origin of the closed-loop system (9) is 0-input GES;
- (ii) for all $r \in \mathbb{R}^q$, any solution x to (9) is such that

$$\lim_{j \rightarrow \infty} Cx(j) = r.$$

III. MAIN RESULTS

A. Sufficient Conditions

By leveraging Assumption 1, the following result can be established.

Proposition 1: Assume that K_I is nonsingular. Let $r \in \mathbb{R}^q$. Then, there exists a unique $\bar{x} \in \mathbb{R}^{n+q}$ such that

$$\begin{aligned} \bar{x} &= (A + BK)\bar{x} + M\varphi(Q\bar{x}) + Jr \\ r &= C\bar{x}. \end{aligned} \quad (10)$$

Proof: From Assumption 1, there exists a unique pair $(\bar{x}_p, \bar{u}) \in \mathbb{R}^{n+q}$ such that

$$\begin{aligned} \bar{x}_p &= A_p \bar{x}_p + B_p \bar{u} + M\varphi(G_p \bar{x}_p + N_p \bar{u}) \\ C_p \bar{x}_p &= r. \end{aligned}$$

From the nonsingularity of K_I , there exists a unique $\bar{\xi} \in \mathbb{R}^q$ such that $\bar{u} = K_P \bar{x}_p + K_I \bar{\xi}$. In particular, $\bar{\xi} = K_I^{-1}(\bar{u} - K_P \bar{x}_p)$. Then, (10) follows with $\bar{x} = (\bar{x}_p, \bar{\xi})$. ■

Proposition 1 shows that under Assumption 1 the closed-loop system admits a unique equilibrium ensuring zero output tracking error. Building upon this property, next we provide sufficient conditions to solve Problem 1.

Proposition 2: If there exist matrices $P \in \mathbb{S}_+^{n+q}$, $K \in \mathbb{R}^{q \times (n+q)}$, and a scalar $\tau \geq 0$ such that

$$\Upsilon \prec 0 \quad (11)$$

where the matrix Υ is defined in (12) (at the top of the next page), then

$$[K_P \quad K_I] = K \quad (13)$$

solves Problem 1, with K_I being nonsingular.

Proof: The proof of the result relies on Proposition 1. To this end, first we show that K_I defined in (13) is nonsingular. By contradiction, assume that K_I is singular and that (12) holds. From the satisfaction of (11), since $S \in \mathbb{S}_+^s$, one has that

$$(A + BK)^\top P(A + BK) - P \prec 0,$$

which in turn ensures

$$\text{specr}(A + BK) < 1. \quad (14)$$

Let $v \in \ker(K_I) \setminus \{0\}$. This selection is possible due to K_I being singular. Then

$$(A + BK) \begin{bmatrix} 0 \\ v \end{bmatrix} = \begin{bmatrix} A_p + B_p K_P & B_p K_I \\ -C_p & I \end{bmatrix} \begin{bmatrix} 0 \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ v \end{bmatrix}$$

which shows that $1 \in \text{spec}(A + BK)$. This contradicts (14), thereby showing that K_I is nonsingular. We are now in a position to use Proposition 1 to conclude the proof.

Preliminary steps. Let \tilde{x} be defined as in (10). Define

$$\tilde{x} := x - \bar{x}. \quad (15)$$

Then, from (10)

$$\tilde{x}^+ = \underbrace{(A + BK)\tilde{x} + M\delta_\varphi(Q\tilde{x}, Q\bar{x})}_{g(\tilde{x})} \quad (16)$$

where, for all $h_1, h_2 \in \mathbb{R}^s$, $\delta_\varphi(h_1, h_2) := \varphi(h_1 + h_2) - \varphi(h_2)$. In particular, observe that from Assumption 2, it follows that

$$\begin{bmatrix} Q\tilde{x} \\ \delta_\varphi(Q\tilde{x}, Q\bar{x}) \end{bmatrix}^\top \begin{bmatrix} S & T \\ \star & R \end{bmatrix} \begin{bmatrix} Q\tilde{x} \\ \delta_\varphi(Q\tilde{x}, Q\bar{x}) \end{bmatrix} \geq 0 \quad \forall \tilde{x} \in \mathbb{R}^{n+q}. \quad (17)$$

The latter, can be equivalently rewritten, for all $\tilde{x} \in \mathbb{R}^{n+q}$, as

$$\begin{bmatrix} \tilde{x} \\ \delta_\varphi(Q\tilde{x}, Q\bar{x}) \end{bmatrix}^\top \underbrace{\begin{bmatrix} Q^\top S Q & Q^\top T \\ \star & R \end{bmatrix}}_\Phi \begin{bmatrix} \tilde{x} \\ \delta_\varphi(Q\tilde{x}, Q\bar{x}) \end{bmatrix} \geq 0 \quad (18)$$

Notice that, under the nonsingularity of K_I , from Proposition 1, for $r = 0$, $\bar{x} = 0$. The satisfaction of items (i) and (ii) in Problem 1 is equivalent to the GES of the origin of (16). Next we show that this property holds. Let, for all $\tilde{x} \in \mathbb{R}^{n+q}$, $V(\tilde{x}) := \tilde{x}^\top P \tilde{x}$, with $P = P^\top \succ 0$. In particular, since $P \in \mathbb{S}_+^{n+q}$, V is positive definite and radially unbounded. Pick $\tilde{x} \in \mathbb{R}^{n+q}$ and define $\chi := (\tilde{x}, \delta_\varphi(Q\tilde{x}, Q\bar{x}))$. Then, from the definition of Υ in (12), simple calculations yield

$$\chi^\top \Upsilon \chi = V(g(\tilde{x})) - V(\tilde{x}) + \tau \chi^\top \Phi \chi. \quad (19)$$

Hence, from (11), the latter gives $V(g(\tilde{x})) - V(\tilde{x}) + \tau \chi^\top \Phi \chi \leq -|\lambda_{\max}(\Upsilon)| \tilde{x}^\top \tilde{x}$ which, due to (18), gives

$$V(g(\tilde{x})) - V(\tilde{x}) \leq -|\lambda_{\max}(\Upsilon)| \tilde{x}^\top \tilde{x}. \quad (20)$$

Since the above inequality does not depend on the specific selection of \tilde{x} , one can conclude that it holds for any $\tilde{x} \in \mathbb{R}^{n+q}$. Hence, by invoking [20, Theorem 28] one has that the origin of (16) is GES. This concludes the proof. ■

B. Controller Design

Proposition 2 provides sufficient condition for the solution to Problem 1. However, it is worth to notice that (11) is nonlinear in the decision variables P , K , and τ . Therefore, this condition is hard to exploit from a numerical standpoint. To overcome this drawback, next we provide an equivalent LMI condition.

$$\Upsilon := \begin{bmatrix} (\bullet)^\top P(A+BK) - P + \tau(\bullet)^\top S(G+N_p K) & (A+BK)^\top PM + \tau(G+N_p K)^\top T \\ \star & (\bullet)^\top PM + \tau R \end{bmatrix} \quad (12)$$

$$\Pi := \begin{bmatrix} -W & (WG^\top + Y^\top N^\top)T & WA^\top + Y^\top B^\top & WG^\top + Y^\top N^\top \\ \star & \rho R & \rho M^\top & 0 \\ \star & \star & -W & 0 \\ \star & \star & \star & -\rho S^{-1} \end{bmatrix} \quad (21)$$

Proposition 3: If there exist matrices $W \in \mathbb{S}_+^{n+q}$ and $Y \in \mathbb{R}^{q \times (n+q)}$ and a positive scalar ρ such that

$$\Pi \prec 0 \quad (22)$$

where the matrix Π is defined in (21) (at the top of the page), then,

$$\begin{bmatrix} K_P & K_I \end{bmatrix} = YW^{-1} \quad (23)$$

solves Problem 1, with K_I being nonsingular.

Proof: Let $\Lambda := W^{-1} \oplus \frac{1}{\rho}I \oplus I \oplus I$. Then, from (22), by congruence, one has

$$\Lambda \Pi \Lambda = \begin{bmatrix} -W^{-1} \frac{(G+N_p K)^\top T}{\rho} & (A+BK)^\top (G+N_p K)^\top \\ \star & \frac{\rho}{\rho} M^\top & 0 \\ \star & \star & -W & 0 \\ \star & \star & \star & -\rho S^{-1} \end{bmatrix} \prec 0.$$

From Schur complement, the latter inequality gives

$$\begin{bmatrix} -W^{-1} & \frac{1}{\rho} (G+N_p K)^\top T \\ \star & \frac{1}{\rho} R \end{bmatrix} + (\bullet)^\top \begin{bmatrix} W^{-1} & 0 \\ \star & \frac{1}{\rho} S \end{bmatrix} \begin{bmatrix} A+BK & M \\ G+N_p K & 0 \end{bmatrix} \prec 0.$$

Straightforward computations show that the matrix in the left-hand side of the above inequality corresponds to the matrix Υ defined in (12) with $P = W^{-1}$ and $\tau = \rho^{-1}$ and K as defined in (23). Therefore, by invoking Proposition 2 the result is established. ■

IV. OPTIMAL CONTROLLER DESIGN

To ensure some level of performance, we introduce the following quadratic cost

$$\mathcal{J}(x(\cdot), r) := \sum_{j=0}^{\infty} (x(j) - \bar{x})^\top C_\omega^\top LC_\omega (x(j) - \bar{x}) + (Kx(j) - \bar{u})^\top H(Kx(j) - \bar{u}) \quad (24)$$

where $C_\omega \in \mathbb{R}^{\omega \times (n+q)}$, $L \in \mathbb{S}_+^\omega$, $H \in \mathbb{S}_+^q$, \bar{u} and \bar{x} are the values of u and x at the equilibrium point defined, respectively, in Assumption 1 and Proposition 1. These values depend on the selection of r , and impose that for any r and any (maximal) solution x to (7), $\mathcal{J}(x(\cdot), r)$ is upper bounded. This leads to the following problem.

Problem 2: Let $\mu > 0$ be given. Design K_P and K_I nonsingular such that the following properties hold:

(i) the origin of the closed-loop system (9) is 0-input GES;

(ii) for any $r \in \mathbb{R}^q$, any solution x to (9) is such that

$$\lim_{j \rightarrow \infty} Cx(j) = r$$

(iii) for any $r \in \mathbb{R}^q$, any solution x to (9) is such that

$$\mathcal{J}(x(\cdot), r) \leq \mu(x(0) - \bar{x})^\top (x(0) - \bar{x}).$$

where \bar{x} is defined in (10).

Proposition 4: Let $L \in \mathbb{S}_+^\omega$, $H \in \mathbb{S}_+^q$, and $\mu > 0$. If there exist matrices $P \in \mathbb{S}_+^{n+q}$, $K \in \mathbb{R}^{q \times (n+q)}$, and a scalar $\tau \geq 0$ such that

$$\Upsilon + \begin{bmatrix} C_\omega^\top LC_\omega + K^\top HK & 0 \\ 0 & 0 \end{bmatrix} \prec 0 \quad (25)$$

$$P - \mu I \preceq 0$$

where the matrix Υ is defined in (12) (at the top of the page). Then, K solves Problem 2, with K_I being nonsingular.

Proof: Since the satisfaction of (25) implies (11), by Proposition 2 items (i) and (ii) in Problem 2 are fulfilled. Hence, to conclude the proof, we show that item (iii) in Problem 2 with $\mu = \lambda_{\max}(P)$. To this end, let $V(\tilde{x}) := \tilde{x}^\top P \tilde{x}$, for all $\tilde{x} \in \mathbb{R}^n$. Then, by following similar steps as in the proof of Proposition 2, (25) yields, for all $\tilde{x} \in \mathbb{R}^n$

$$V(g(\tilde{x})) - V(\tilde{x}) \leq \tilde{x}^\top (L + K^\top HK) \tilde{x} \quad (26)$$

with g being defined as in (16). Now observe that from the proof of Proposition 2, it follows that the satisfaction of (25) ensures that K_I is nonsingular. Therefore, by building upon Proposition 1, given any $r \in \mathbb{R}^q$ and any trajectory x to (9), the cost index defined in (24) can be rewritten as

$$\mathcal{J}(x(\cdot), r) = \sum_{j=0}^{\infty} \tilde{x}(j)^\top (C_\omega^\top LC_\omega + K^\top HK) \tilde{x}(j) \quad (27)$$

where \tilde{x} is the unique trajectory of the error system (16) such that $\tilde{x}(0) = x(0) - \bar{x}$. Notice that, as shown in the proof of Proposition 2, one has that

$$\lim_{j \rightarrow \infty} \tilde{x}(j) = 0. \quad (28)$$

From, (26) it holds, for all $j \in \mathbb{N}_{\geq 0}$ that

$$V(\tilde{x}(j+1)) - V(\tilde{x}(j)) + \tilde{x}(j)^\top (C_\omega^\top LC_\omega + K^\top HK) \tilde{x}(j) \leq 0$$

which in turn, by simple manipulations, yields, for all $j \in \mathbb{N}_{\geq 0}$,

$$V(\tilde{x}(j)) - V(\tilde{x}(0)) + \sum_{k=1}^j \tilde{x}(k)^\top (C_\omega^\top LC_\omega + K^\top HK) \tilde{x}(k) \leq 0.$$

By taking the limit for j that goes to infinity from the latter, by using (28) and the fact that V is continuous and positive definite, one gets

$$\sum_{k=1}^{\infty} \tilde{x}(k)^\top (C_\omega^\top LC_\omega + K^\top HK) \tilde{x}(k) \leq V(\tilde{x}(0)).$$

Hence, thanks to (27) and the definition of V , the proof is completed. ■

Similarly to Proposition 2, Proposition 4 cannot be exploited directly to design the controller gains K_P and K_I , condition (25) being nonlinear in the decision variables. To overcome this drawback, next we provide an equivalent condition to (25) that is linear in the decision variables.

Proposition 5: Let $L \in \mathbb{S}_{++}^\omega$ and $H \in \mathbb{S}_{++}^q$. If there exist matrices $W \in \mathbb{S}_{++}^{n+q}$ and $Y \in \mathbb{R}^{q \times (n+q)}$ and a positive scalar ρ such that

$$\left[\begin{array}{c|cc} \Pi & WC_\omega^\top & Y^\top \\ \hline * & * & * \\ * & * & * \end{array} \right] \prec 0 \quad (29)$$

$$\begin{bmatrix} -\mu I & I \\ * & -W \end{bmatrix} \preceq 0. \quad (30)$$

where the matrix Π is defined in (21) (at the top of the previous page). Then,

$$[K_P \quad K_I] = YW^{-1}$$

solves Problem 1, with K_I being nonsingular. □

The proof of Proposition 5 is wholly similar to the one of Proposition 3, therefore it is omitted.

Although the value of the bound μ introduced in Problem 2 is assumed to be given, in practice one may be interested in designing the controller gains with the objective of minimizing this value. This can be achieved by consider the following optimization problem.

$$\begin{aligned} & \underset{W, Y, \rho, \mu}{\text{minimize}} && \mu \\ & \text{subject to} && (29), (30). \end{aligned} \quad (31)$$

It is worth to mention that (31) is a semidefinite program, i.e., a convex optimization problem over LMI constraints. Therefore, it can efficiently solved by relying on off-the-shelf software.

V. NUMERICAL EXAMPLES

In this section, we illustrate the effectiveness of the proposed approach in two numerical examples. Example 1 is an academic example and is used to illustrate the results in Section III.B. Example 2 is a more practically relevant example and is used as a test-bed for the optimal design presented in Section IV.

Example 1: We consider the plant in [17, Example 1] that can be written as (2) with

$$A_p = \begin{bmatrix} 1 - 2t_s & 3t_s \\ 3t_s & 1 + t_s \end{bmatrix}, B_p = \begin{bmatrix} 0 \\ t_s \end{bmatrix}, C_p = [1 \quad 0]$$

$$M_p = \begin{bmatrix} 0 \\ t_s \end{bmatrix}, N_p = -0.1, G_p = [1 \quad 0], \varphi(z) = \cos(z).$$

The scalar t_s represents the sampling period introduced by discretization of the original continuous-time plant and, as in [17, Example 1], we select $t_s = 0.01$. Concerning the nonlinearity φ , notice that since for all $z \in \mathbb{R}$, $\frac{d}{dz} \varphi(z) \in [-1, 1]$, from Cauchy's mean value theorem one has that for all $z_1 \neq z_2$

$$-1 \leq \frac{\varphi(z_2) - \varphi(z_1)}{z_2 - z_1} \leq 1.$$

Namely for all $z_1, z_2 \in \mathbb{R}$

$$\begin{aligned} \mathcal{L}(z_1, z_2) := & (-\varphi(z_2) + \varphi(z_1) + z_2 - z_1) \times \\ & (\varphi(z_2) + \varphi(z_1) + z_2 - z_1) \geq 0. \end{aligned}$$

Simple calculations show that

$$\mathcal{L}(z_1, z_2) = (\bullet)^\top \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} z_2 - z_1 \\ \varphi(z_2) - \varphi(z_1) \end{bmatrix} \geq 0$$

which reads as (4) with $S = 1, T = 0$, and $R = -1$. By solving the conditions in Proposition 3, one gets the following controller gains $K_P = [-100 \quad -33]$, $K_I = 6.4$. Fig. 1 shows the evolution of the regulated output and of the control input with $x_p(0) = 0, \xi(0) = 0$. Simulations clearly show that the proposed control design ensures asymptotic tracking of the setpoint.

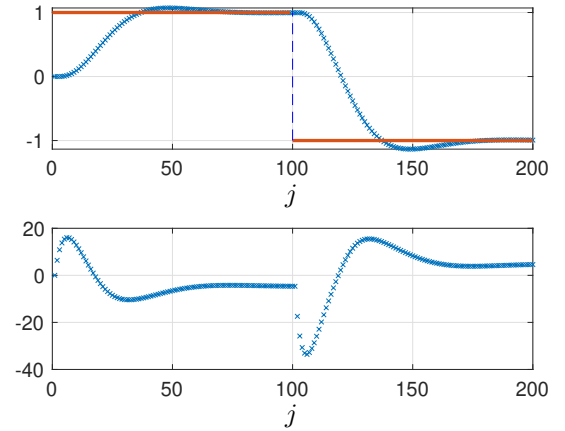


Fig. 1: Example 1. Top picture: evolution of the regulated output y_p (blue x) and of the setpoint (orange solid-line). Bottom picture: evolution of the control input.

Example 2: We consider the discretized version of the flexible link model in [21]. In particular, by using Euler discretization with a sampling time 0.1 the model writes as system (2) with

$$A_p = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -48.6 & -1.25 & 48.6 & 0 \\ 0 & 0 & 0 & 1 \\ 19.5 & 0 & -19.5 & 0 \end{bmatrix}, B_p = 0.1 \begin{bmatrix} 0 \\ 21.6 \\ 0 \\ 0 \end{bmatrix},$$

$$C_p = [0 \quad 0 \quad 1 \quad 0], M_p = 0.1 \begin{bmatrix} 0 \\ 0 \\ 0 \\ -3.33 \end{bmatrix}, N_p = 0$$

$$G_p = [0 \quad 0 \quad 1 \quad 0], \varphi(z) = \sin(z).$$

Similarly as in Example 1, φ satisfies (4) with $S = 1$, $R = -1$, and $T = 0$. For this example, we design the controller gains by solving optimization problem (31) with $C_\omega = \begin{bmatrix} 0 & 0 & 1 & 0 & 1 \end{bmatrix}$, $L = 10I$, and $H \in \{0.01, 0.1\}$. The evolution of the tracking error and control signal for the two designs are reported, respectively, in Fig. 2 and Fig. 3. In these simulations, $x_p(0) = 0$, $\xi(0) = 0$, and the setpoint is selected as $r = 3$ for $j \in \{0, 1, \dots, 44\}$ and $r = -3$ for $j \in \{45, 46, \dots, 90\}$. As it appears from the simulations, the proposed optimization is effective in reducing the control effort without impairing tracking performance.

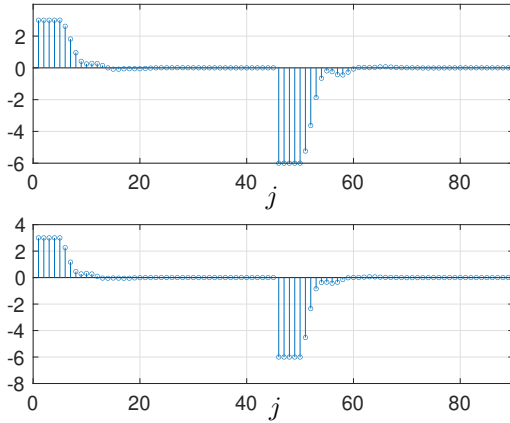


Fig. 2: Example 2. Evolution of the tracking error for $H = 0.1$ (top) and $H = 0.01$ (bottom).

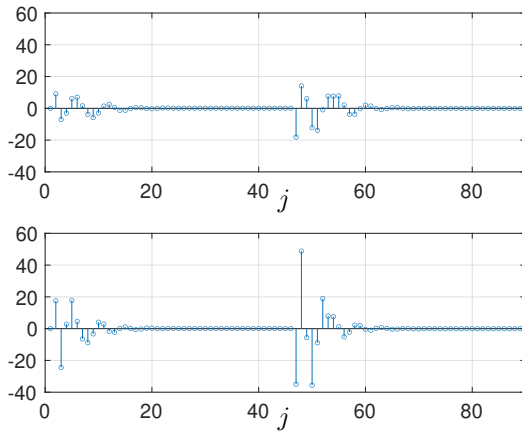


Fig. 3: Example 2. Evolution of the control input for $H = 0.1$ (top) and $H = 0.01$ (bottom).

VI. CONCLUSION

The paper addressed the problem of setpoint tracking for a class of discrete-time nonlinear plants. Considering a plant resulting from a feedback interconnection of a linear system and a nonlinearity obeying to an incremental quadratic constraint, the design of a proportional-integral feedback controller was proposed. A suitable optimal control problem focusing on the satisfaction of a performance level with

respect to a quadratic cost was formulated and recast into a semidefinite program. The results obtained open the door for forthcoming works. In particular, the extension to non-square, and possibly saturated, plants is currently part of our research.

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