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Matteo Della Rossa

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En vue de l'obtention du

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Présentée et soutenue le 09/10/2020 par :

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### Non-Smooth Lyapunov Functions for Stability Analysis of Hybrid Systems

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# Abstract

Modeling of many phenomena in nature escape the rather common frameworks of continuous-time and discrete-time models. In fact, for many systems encountered in practice, these two paradigms need to be intrinsically related and connected, in order to reach a satisfactory level of description in modeling the considered physical/engineering process. These systems are often referred to as *hybrid systems*, and various possible formalisms have appeared in the literature over the past years.

The aim of this thesis is to analyze the stability of particular classes of hybrid systems, by providing Lyapunov-based sufficient conditions for (asymptotic) stability. In particular, we will focus on non-differentiable locally Lipschitz candidate Lyapunov functions. The first chapters of this manuscript can be considered as a general introduction of this topic and the related concepts from non-smooth analysis.

This will allow us to study a class of piecewise smooth maps as candidate Lyapunov functions, with particular attention to the continuity properties of the constrained differential inclusion comprising the studied hybrid systems. We propose “relaxed” Lyapunov conditions which require to be checked only on a *dense set* and discuss connections to other classes of locally Lipschitz or piecewise regular functions.

Relaxing the continuity assumptions, we then investigate the notion of generalized derivatives when considering functions obtained as *max-min* combinations of smooth functions. This structure turns out to be particularly fruitful when considering the stability problem for differential inclusions arising from regularization of *state-dependent switched systems*. When the studied switched systems are composed of *linear* sub-dynamics, we refine our results, in order to propose algorithmically verifiable conditions.

We further explore the utility of set-valued derivatives in establishing *input-to-state* stability results, in the context of perturbed differential inclusions/switched systems, using locally Lipschitz candidate Lyapunov functions. These developments are then used in analyzing the stability problem for interconnections of differential inclusion, with an application in designing an observer-based controller for state-dependent switched systems.

*Keywords:* Stability analysis, locally Lipschitz Lyapunov functions, generalized derivatives, hybrid systems, switched systems, input-to-state stability.



# Résumé

La Nature, dans ses multiples manifestations, nous fournit un grand nombre d'exemples pour lesquels il est nécessaire d'aller au-delà de la distinction classique entre modèles où le temps est décrit comme une entité continue et modèles où le temps est discret/discrétisé. En particulier, pour une multitude de systèmes en physique/ingénierie, ces deux aspects temporels sont fondamentalement liés, et nécessitent donc que ces deux paradigmes soient connectés et mis en relation, pour une meilleure précision et fidélité dans la représentation du phénomène. Cette famille de systèmes est souvent appelée "systèmes hybrides", et différentes formalisations mathématiques ont été proposées.

L'objectif de cette thèse est l'analyse et l'étude de la stabilité (asymptotique) pour certaines classes de systèmes hybrides, en proposant des conditions suffisantes à la Lyapunov. Plus spécifiquement, nous nous concentrerons sur des fonctions de Lyapunov non-lisses ; pour cette raison, les premiers chapitres de cette thèse peuvent être considérés comme une introduction générale de ce sujet, proposant les instruments nécessaires issus de l'analyse non-lisse. Tout d'abord, grâce à ces outils, nous pourrons étudier une classe de fonctions de Lyapunov construites par morceaux, avec une attention particulière aux propriétés de continuité des inclusions différentielles qui composent le système hybride considéré. Nous proposons des conditions qui doivent être vérifiées seulement sur un sous-ensemble dense, et donc allant au-delà de résultats existants.

En négligeant les hypothèses de continuité, nous étudions ensuite comment les notions de dérivées généralisées se spécialisent en considérant des fonctions construites comme combinaisons de maximum/ minimum de fonctions lisses. Cette structure devient particulièrement fructueuse quand on regarde la classe des systèmes à commutation dépendant de l'état du système. Dans le cas où les sous-dynamiques sont linéaires, nous étudions comment les conditions proposées peuvent être vérifiées algorithmiquement.

L'utilité des notions de dérivées généralisées est finalement explorée dans le contexte de la stabilité entrée-état (ISS) pour inclusions différentielles avec perturbations extérieures. Ces résultats nous permettent de proposer des critères de stabilité pour systèmes interconnectés, et notamment une application du design de contrôleurs pour systèmes à commutation dépendant de l'état.

*Mots clés* : Analyse de stabilité, Fonctions de Lyapunov localement Lipschitz, dérivées généralisées, systèmes hybrides, systèmes à commutation, stabilité entrée-état (ISS).





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# Notation

This section provides the notations used all along the thesis.

- $\mathbb{N}$ : the set of positive integers,
- $\mathbb{R}$ : the set of real numbers,
- $\mathbb{R}^n$ : the set of  $n$ -dimensional real-vectors
- $\mathbb{R}_{>0}$ , ( $\mathbb{R}_{\geq 0}$ ): set of strictly positive (non-negative) real numbers.
- *Comparison Functions Classes*: a function  $\alpha : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}$  is *positive definite* ( $\alpha \in \mathcal{PD}$ ) if it is continuous,  $\alpha(0) = 0$ , and  $\alpha(s) > 0$  if  $s \neq 0$ . A function  $\alpha : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$  is of *class  $\mathcal{K}$*  ( $\alpha \in \mathcal{K}$ ) if it is continuous,  $\alpha(0) = 0$ , and strictly increasing; it is of *class  $\mathcal{K}_\infty$*  if, in addition, it is unbounded. A continuous function  $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$  is of *class  $\mathcal{KL}$*  if  $\beta(\cdot, s)$  is of class  $\mathcal{K}$  for all  $s$ , and  $\beta(r, \cdot)$  is decreasing and  $\beta(r, s) \rightarrow 0$  as  $s \rightarrow \infty$ , for all  $r$ .
- $\langle x, y \rangle$ , (or  $x^\top y$ ): *Euclidean scalar product* in  $\mathbb{R}^n$  between  $x$  and  $y \in \mathbb{R}^n$ .
- $|x| := \sqrt{\langle x, x \rangle}$ : *Euclidean norm* (or 2-norm) of vectors in  $\mathbb{R}^n$ ,
- $|x|_{\mathcal{A}}$ : given  $\mathcal{A} \subset \mathbb{R}^n$  and  $x \in \mathbb{R}^n$ , it denotes the Euclidean distance between  $x$  and  $\mathcal{A}$ , that is  $|x|_{\mathcal{A}} := \inf\{|y - x| \mid y \in \mathcal{A}\}$
- *Basic Topology Notation*: given  $\mathcal{A} \subset \mathbb{R}^n$  we denote by  $\text{int}(\mathcal{A})$  the *interior* of  $\mathcal{A}$ , by  $\overline{\mathcal{A}}$  its *closure*, by  $\text{bd}(\mathcal{A})$  its *boundary*, by  $\text{co}(\mathcal{A})$  its *convex hull* and by  $\overline{\text{co}}(\mathcal{A})$  the closure of its convex hull.
- $\overline{\mathbb{B}}(x, r)$  and  $\mathbb{B}(x, r)$ : closed and open ball (respectively) centered in  $x \in \mathbb{R}^n$  of radius  $r > 0$ .
- $\mu_L(\mathcal{A})$ : the Lebesgue measure of a Lebesgue-measurable set  $\mathcal{A} \subset \mathbb{R}^n$ .
- $f : \text{dom}(f) \subset \mathbb{R}^n \rightarrow \mathbb{R}^m$  denotes a function  $f$  defined on a domain  $\text{dom}(f)$  (if not stated otherwise, an open subset of  $\mathbb{R}^n$ ), returning values in  $\mathbb{R}^m$ .
- $F : \text{dom}(F) \subset \mathbb{R}^n \rightrightarrows \mathbb{R}^m$  denotes a map  $F$  whose values are *subsets* of  $\mathbb{R}^m$ , and we define  $\text{dom}(F) := \{x \in \mathbb{R}^n \mid F(x) \neq \emptyset\}$ .
- $\nabla f(x)$ : gradient of a function  $f : \mathcal{O} \rightarrow \mathbb{R}$  at  $x \in \mathcal{O}$  (if it exists).
- $\mathcal{C}^p(\mathcal{O}, \mathbb{R}^m)$ : given an open set  $\mathcal{O} \subset \mathbb{R}^n$  and an integer  $p \geq 0$ , the set of functions  $f : \mathcal{O} \rightarrow \mathbb{R}^m$  such that each component  $f_1, \dots, f_m : \mathcal{O} \rightarrow \mathbb{R}$  is continuously differentiable  $p$ -times (all the derivatives of order smaller than or equal to  $p$  are continuous at every point of  $\mathcal{O}$ ).
- $\text{Lip}_{\text{loc}}(\mathcal{O}, \mathbb{R}^m)$ : given an open set  $\mathcal{O} \subset \mathbb{R}^n$ , the set of *locally Lipschitz functions* on  $\mathcal{O}$ .

- *Absolute Continuity:* Given a compact interval  $I : [a, b] \subset \mathbb{R}$ ,  $\phi : I \rightarrow \mathbb{R}^n$  is *absolutely continuous on  $I$*  if  $\phi$  is differentiable almost everywhere in  $I$ , its derivative  $\phi'$  is Lebesgue integrable and

$$\phi(x) = \phi(a) + \int_a^x \phi'(s) ds, \quad \forall x \in I.$$

When considering a right-open interval  $I := [0, b)$  (and possibly  $b = +\infty$ ),  $\phi : I \rightarrow \mathbb{R}^n$  is locally absolutely continuous, if it is absolutely continuous on any compact subinterval of  $I$ . In the presentation, for sake of readability, we will avoid the term “locally”.

- *Linear Algebra Notations:*  $\mathbb{R}^{n \times m}$  denotes the set of real matrices with  $n$  rows and  $m$  columns; for any  $A \in \mathbb{R}^{n \times m}$ ,  $A^\top$  denotes the *transpose* matrix of  $A$ ,  $\text{Rank}(A)$  denotes the *rank* of  $A$ ,  $\det(A)$  the *determinant* of  $A$ .  $I$  denotes the identity matrix (of appropriate dimension). The symbol  $\text{Sym}(\mathbb{R}^n) := \{A \in \mathbb{R}^{n \times n} \mid A^\top = A\}$  denotes the set of *symmetric* matrices in  $\mathbb{R}^{n \times n}$ . Given  $A \in \text{Sym}(\mathbb{R}^n)$ ,  $A > 0$  ( $A \geq 0$ ) stands for  $A$  positive (semi-) definite, while  $A < 0$  ( $A \leq 0$ ) stands for  $A$  negative (semi-) definite.

# 1

## Introduction

In this introductory chapter, we provide an overview of the concepts and problems that we will study in this thesis. In Section 1.1 we introduce the main classes of dynamical systems which have been studied, notably hybrid systems and switching systems. The main stability notions are recalled in Section 1.2, together with the classical Lyapunov theorem in the context of differential inclusions. In Section 1.3, we provide a review of recent results for stability of differential inclusions, hybrid systems and switching systems via Lyapunov functions. Finally, in Section 1.4, we present the main directions this thesis will explore, providing an overview of the chapters to follow.

### 1.1 Systems Class

Many phenomena in nature or in engineering can be properly modeled using a coupling or interaction between continuous-time dynamics and discrete events. Systems in which these two kinds of behavior are intrinsically connected are called *hybrid systems*, and constitute an active area of research over the past several years.

Many mathematical models for various kind of hybrid systems (i.e. models which explicitly distinguish between “continuous” and “discrete” evolutions) have been proposed in the past. For a survey of “historical” models of hybrid system we refer the doctoral dissertation [21], where a deep comparison is provided. Among other examples, the so-called model of hybrid-automata is introduced in [65], while for impulsive differential equation we refer to the book [12]. In [125, Chapter 2] the interested reader can find several examples of hybrid systems models arising from several different areas of application.

In this thesis, we will focus on two more recent frameworks, notably switching systems, as studied formally in [84] and hybrid systems in the formalism of [58]. The former formalism, which describes the so called *switching systems*, considers a subclass of hybrid systems, emphasizing the continuous behavior and seeing the discrete events as *switching* among a prescribed countable family of vector fields. Putting the emphasis on the class of switching rules/signals and exploiting the properties of the individual subsystems have been fruitful paths of research. This model is presented in Section 1.1.2, and we refer to [84] and references therein for the interested reader.

The latter setting provides a paradigm for hybrid systems that is remarkable for its generality and mathematical rigor, which allows us to derive and adapt many concepts and results from classical continuous time/discrete time systems theory.



This model is presented in Section 1.1.3, and for further details we refer to [58].

Both formalisms make use, partially or in some specific cases, of differential inclusions as modeling tool. For that reason we recall in what follows, the basic results and solution concepts for this kind of dynamical systems.

### 1.1.1 Differential Inclusions

This subsection is devoted to introducing the basic notions of differential inclusions, following the formalism of [5], [45] and [114].

A *set-valued map*  $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$  is a function that at each point  $x \in \mathbb{R}^n$  associates a *set*  $F(x) \subset \mathbb{R}^m$ . The set

$$\text{dom}(F) := \{x \in \mathbb{R}^n \mid F(x) \neq \emptyset\}.$$

is the domain of  $F$ . In the subsequent Section 2.1 we will provide the main results of set-valued analysis, defining the necessary properties of set-valued maps.

We are interested in dynamical systems whose right-hand side is described by a set-valued map. Intuitively, this idea allows us to consider systems for which the velocity of a generic solution is not *uniquely* determined by state position, but instead it lies in a *set* that depends on that position.

More formally, given a set  $\mathcal{C} \subset \mathbb{R}^n$  and a set-valued map  $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$  with  $\text{dom}(F) \supset \mathcal{C}$ , we consider the *differential inclusion*

$$\dot{x}(t) \in F(x(t)), \quad x(t) \in \mathcal{C}. \quad (1.1)$$

A *solution* to (1.1) is an absolutely continuous function  $\phi : \text{dom } \phi \rightarrow \mathcal{C}$ , with  $\text{dom } \phi := [0, T_\phi)$  for some  $T_\phi > 0$  (and possibly  $T_\phi = +\infty$ ), such that

$$\dot{\phi}(t) \in F(\phi(t)), \quad \text{for almost all } t \text{ in } \text{dom } \phi.$$

We say that a solution  $\phi : \text{dom}(\phi) \rightarrow \mathbb{R}^n$  of (1.1) is *maximal* if it cannot be extended forward in time, that is, if there does not exist any  $\tilde{\phi} : \text{dom}(\tilde{\phi}) \rightarrow \mathbb{R}^n$  solution of (1.1), with  $\text{dom}(\phi) \subsetneq \text{dom}(\tilde{\phi})$  and  $\phi(t) = \tilde{\phi}(t)$  for all  $t \in \text{dom}(\phi)$ . The differential inclusion (1.1) is said to be *forward complete* if, for any maximal solution  $\phi$ , we have  $\text{dom}(\phi) = [0, \infty)$ .

Given a point  $x \in \mathcal{C}$ , we denote the set of all solutions of (1.1) with  $\phi(0) = x$  by  $\mathcal{S}_{F,\mathcal{C}}(x)$ . We define the set of solutions of (1.1) as  $\mathcal{S}_{F,\mathcal{C}} := \cup_{x \in \mathcal{C}} \mathcal{S}_{F,\mathcal{C}}(x)$ .

Formalism (1.1) is in particular suitable to model a large class of physical systems, for which the differential equations setting is not rich enough. As a simple example, one can consider the case of differential equations for which the right-hand side is known to be *approximately* equal to a function  $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$  with a margin of error in norm, say  $\varepsilon > 0$ . Then any solution of the “real” differential equation will be a solution of the differential inclusion

$$\dot{x} \in F(x) := \{f(x)\} + \overline{\mathbb{B}}(0, \varepsilon), \quad x \in \mathbb{R}^n.$$

Further examples will be provided in what follows, for an overview we refer the reader to [114, Chapter 1].

### 1.1.2 Switching Systems

As studied in [84], in this section, we present the formalism of switching systems. Intuitively, given a family of vector fields, the setting of switching systems allows describing the evolution of the state guided by a switching rule/signal that somehow “chooses” the vector field the state must follow.

Consider a discrete set  $\mathcal{I} \subset \mathbb{N}$ , the so called *index set*, and for each  $i \in \mathcal{I}$ , we associate a continuous-time dynamical subsystem

$$\dot{x} = f_i(x),$$

where  $f_i : \mathbb{R}^n \rightarrow \mathbb{R}^n$  is locally Lipschitz, for any  $i \in \mathcal{I}$ . If the state is free to switch between subsystems  $\dot{x} = f_i(x)$  for any  $i \in \mathcal{I}$  without any constraint, we can define the switched system under *arbitrary switching rule*, which can be modeled by the following differential inclusion

$$\dot{x} \in \text{co}\{f_i(x) \mid i \in \mathcal{I}\}. \quad (1.2)$$

Intuitively, solutions of (1.2) are free to evolve following any convex combination of the subsystems  $f_i : \mathbb{R}^n \rightarrow \mathbb{R}^n$  for any  $i \in \mathcal{I}$ , see [113, Section 2] for the technical details.

In many situations, further information is available, and the system is known to follow a prescribed *switching rule* that encodes when and how a discrete event, i.e. a switching between individual modes, labeled by  $i \in \mathcal{I}$ , can occur. As a first case, we consider switching rules that depend only on the time variable, as formalized in the following definition.

#### Definition 1.1: Time-Dependent Switching Signals

Given a discrete set  $\mathcal{I} \subset \mathbb{N}$ , define the set of *time-dependent switching signals* as

$$\mathcal{S} := \{\sigma : \mathbb{R}_+ \rightarrow \mathcal{I} \mid \sigma \text{ is piecewise constant and locally finite}\}, \quad (1.3)$$

where, by *locally finite*, we mean that any  $\sigma \in \mathcal{S}$  has a finite number of discontinuities on any bounded subset of  $\mathbb{R}_+$ .

Without loss of generality, we suppose that signals  $\sigma \in \mathcal{S}$  are right-continuous. Given a signal  $\sigma \in \mathcal{S}$ , we define the set of *switching instants*, that is the points at which  $\sigma$  is discontinuous, and we denote it by  $\{t_k^\sigma\} \subset \mathbb{R}_+$ . The collection  $\{t_k^\sigma\} \subset \mathbb{R}_+$  may be infinite, or finite, possibly reduced to the initial instant  $t_0^\sigma := 0$ . If it is infinite, then, by local finiteness, it is divergent, i.e.  $t_k^\sigma \rightarrow \infty$  for  $k \rightarrow \infty$ . Thus, a switched system with *time-dependent switching rule*  $\sigma \in \mathcal{S}$  can be described by

$$\dot{x}(t) = f_{\sigma(t)}(x(t)). \quad (1.4)$$

We note that the class of switching systems with prescribed time-dependent switching rule is a subclass of non-autonomous differential equations. Indeed, given  $\sigma \in \mathcal{S}$ , and defining  $\tilde{f}(t, x) := f_{\sigma(t)}(x)$ , we have that (1.4) can be represented as

$$\dot{x}(t) = \tilde{f}(t, x(t)).$$

In many situations, given a discrete set  $\mathcal{I} \subset \mathbb{N}$ , and a set of vector fields  $\mathcal{F} = \{f_i \in Lip_{loc}(\mathbb{R}^n, \mathbb{R}^n) \mid i \in \mathcal{I}\}$ , we are interested in studying the properties of system (1.4) not simply for *one* prescribed  $\sigma \in \mathcal{S}$ , but for a *subclass* of signals. Commonly considered subclasses are the class of *dwell-time switching signals*, given by

$$\mathcal{S}_{dw}(\tau) := \left\{ \sigma \in \mathcal{S} \mid t_k^\sigma - t_{k-1}^\sigma \geq \tau, \forall t_k^\sigma > 0 \right\}, \quad (1.5)$$

or the class of *average dwell-time switching signals*, defined as

$$\mathcal{S}_{adw}(\tau, N_0) := \left\{ \sigma \in \mathcal{S} \mid N_\sigma(T, t) \leq N_0 + \frac{T-t}{\tau}, \forall T > t \geq 0 \right\}, \quad (1.6)$$

where  $N_\sigma(T, t)$  denotes the number of discontinuity points of  $\sigma$  on the interval  $(t, T)$ .

These classes have attracted attention in the switching systems community in the past decades and [84, Chapter 3] provides a thorough overview of stability analysis for switched systems with such switching signals.

In this thesis, we will mostly focus on another constrained-switching paradigm, considering switching signals which depend only on the position of the state  $x$  with respect to a pre-defined partition of the state space  $\mathbb{R}^n$  rather than on time variable. More formally, given a discrete set  $\mathcal{I}$ , and a set of vector fields  $f_i \in Lip_{loc}(\mathbb{R}^n, \mathbb{R}^n)$ , for any  $i \in \mathcal{I}$ , we suppose there exists a set of so-called *operating regions*  $X_i$ ,  $i \in \mathcal{I}$ , such that  $\mathbb{R}^n = \cup_{i \in \mathcal{I}} X_i$ . While the state is in a specific operating region  $X_i$ , solutions will follow the dynamical system  $\dot{x} = f_i(x)$ . Whenever the system solution crosses the boundary of an operating region, its behavior can “switch” according to the region it will lie in. At this stage, the notions of state-space partition, crossing solution, boundaries of operating regions are still not precise. We postpone these technical discussions to Section 2.2, where we will specify what a proper partition of the state-space is and how this definition ensures the well-posedness of state-dependent switched systems.

To make a parallel with time-dependent switched systems defined in (1.4), given a discrete set  $\mathcal{I}$ , a set of vector fields  $f_i \in Lip_{loc}(\mathbb{R}^n, \mathbb{R}^n)$  and a *state-dependent switching signal*  $\sigma : \mathbb{R}^n \rightarrow \mathcal{I}$ , constant on some pre-defined operating regions  $X_i \subset \mathbb{R}^n$ ,  $i \in \mathcal{I}$ , which cover the state space, we can define a state-dependent switched system as the differential equation

$$\dot{x}(t) = f_{\sigma(x(t))}(x(t)). \quad (1.7)$$

Due to the discontinuous behavior of  $\sigma : \mathbb{R}^n \rightarrow \mathcal{I}$ , it is clear that the right-hand side of (1.7) is in general discontinuous with respect to the state  $x \in \mathbb{R}^n$ . Not surprisingly, the differential inclusions framework will allow us to “enlarge” (1.7), ensuring well-posedness, and this is the content of Section 2.2.

### 1.1.3 Hybrid Systems

As presented in [58], in what follows, we formally define the hybrid systems model. This general paradigm is defined by considering two regions of the state space: in the first one the system evolves accordingly to a differential inclusion, while in the second one the system behavior is guided by a *difference* inclusion, that is a discrete-time dynamics whose right-hand side is set-valued.

More formally, given  $\mathcal{C}, \mathcal{D} \subset \mathbb{R}^n$  and  $F : \text{dom } F \rightrightarrows \mathbb{R}^n$ ,  $G : \text{dom } G \rightrightarrows \mathbb{R}^n$  set-valued maps such that  $\mathcal{C} \subset \text{dom } F$  and  $\mathcal{D} \subset \text{dom } G$ , a hybrid dynamical system  $\mathcal{H} = (\mathcal{C}, \mathcal{D}, F, G)$  is defined by

$$\mathcal{H} : \begin{cases} \dot{x} \in F(x), & x \in \mathcal{C}, \\ x^+ \in G(x), & x \in \mathcal{D}. \end{cases} \quad (1.8)$$

We will refer to  $\mathcal{C}$  and  $\mathcal{D}$  as the *flow* and *jump sets* respectively, while  $F : \text{dom } F \rightrightarrows \mathbb{R}^n$ ,  $G : \text{dom } G \rightrightarrows \mathbb{R}^n$  are called *flow* and *jump maps*, respectively. The symbol  $x^+$  denotes, as for discrete-time systems, the value of the state  $x$  after an instantaneous change. We do not require any non-overlapping property of  $\mathcal{C}$  and  $\mathcal{D}$ , possibly allowing solutions starting from point  $x \in \mathcal{C} \cap \mathcal{D}$  to evolve either according to  $F$  or  $G$ . We will formally define, in Section 2.3, the notion of solution for hybrid systems (1.8). We note here that the framework (1.8) is flexible and can model many “heterogeneous” systems arising in control theory. Among other examples, hybrid automata, impulsive dynamical systems and reset systems can be defined and studied exploiting this formalism, as noted in [58, Chapter 1]. Time-dependent switched systems as defined in (1.4) can as well be written in the form of hybrid systems as in (1.8), as shown in [58, Section 1.4.4]. However, the formalism that we introduced in the previous section allows us to emphasize the particular properties of switching systems, providing more tailored results depending explicitly on the class of switching signal.

## 1.2 Stability Theory

In this section we consider several stability concepts for differential inclusions, as defined in Section 1.1.1, providing a first illustration of the Lyapunov direct method. The extension in the hybrid setting is postponed to Section 2.3.

### 1.2.1 Stability Notions

Since differential inclusions, in general, have multiple solutions from a given initial point, all the stability concepts have a *weak* and a *strong* counterpart. Typically, given any initial point (close enough to the origin), weak notions require that a property is satisfied by *at least one* solution, while strong notions require that the same property is satisfied by *all* the solutions. Since we are interested in differential inclusions arising from state-dependent switched systems or defined as flow-map of a given hybrid system, we will focus on the *strong* concepts only. Weak notions are important, for example, when the differential inclusion is seen as model for control systems, that is, defining

$$\dot{x} \in F(x) := \{f(x, u) \mid u \in \mathcal{U}\}, \quad (1.9)$$

where  $f : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$  defines a control system and  $\mathcal{U} \subset \mathbb{R}^m$  is the admissible-controls set. In this context weak asymptotic stability corresponds to the existence, for any initial solution  $x_0$ , of a control  $u : \mathbb{R}_+ \rightarrow \mathcal{U}$  which guides the state to the origin, (also called global asymptotic controllability to 0) see [38].

Moreover, classical stability theory deals with a single equilibrium point, typically the origin. Due to the hybrid behavior of the systems that we will study, we introduce, in general, stability for closed attractors  $\mathcal{A} \subset \mathbb{R}^n$ ; the standard case will be recovered considering  $\mathcal{A} = \{0\}$ .

### Definition 1.2: Stability for Differential Inclusions

Consider  $\mathcal{C} \subset \mathbb{R}^n$ ,  $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$  with  $\text{dom}(F) \supset \mathcal{C}$  and the differential inclusion (1.1). Given a closed set  $\mathcal{A} \subset \mathbb{R}^n$  we say that

1. The set  $\mathcal{A}$  is *uniformly globally stable* (UGS) if there exists  $\alpha \in \mathcal{K}_\infty$  such that any solution  $\phi$  of (1.1) satisfies  $|\phi(t)|_{\mathcal{A}} \leq \alpha(|\phi(0)|_{\mathcal{A}})$  for all  $t \in \text{dom } \phi$ .
2. The set  $\mathcal{A}$  is *uniformly globally attractive* for (1.1) if, for any  $\varepsilon > 0$  and any  $r > 0$  there exists  $T > 0$  such that, for any solution  $\phi$  of (1.1) with  $|\phi(0)|_{\mathcal{A}} \leq r$ , we have that  $|\phi(t)|_{\mathcal{A}} \leq \varepsilon$ , for any  $t \in \text{dom}(\phi)$  such that  $t \geq T$ .
3. The set  $\mathcal{A}$  is *uniformly globally asymptotically stable* (UGAS) for (1.1) if it is uniformly globally stable and uniformly globally attractive.

Given an open set  $\mathcal{O} \subset \mathbb{R}^n$  such that  $\mathcal{O} \subset \mathcal{C}$ , and  $\mathcal{A} \subset \mathcal{O}$ , if the properties 1., 2. and 3. in Definition 1.2 are satisfied only by solutions with  $\phi(0) \in \mathcal{O}$ , we say that the set  $\mathcal{A}$  is *uniformly locally stable* (ULS), *uniformly locally attractive* (ULA) and *uniformly locally asymptotically stable* (ULAS), respectively.

We underline that, in Definition 1.2, we are not assuming a priori existence and/or completeness of solutions, since this is not the case, in general, for constrained differential inclusions. In particular a system which exhibits finite explosion of solution in finite time it would be uniformly globally attractive, satisfying point 2. in Definition 1.2. This kind of “counterintuitive” phenomenon is partially justified by the fact that this definition will be then easily adapted in the context of hybrid systems, where solutions typically leave the flow set  $\mathcal{C}$  recurrently, and in finite time. We also note that the uniform global attractivity introduced in Definition 1.2 is sometimes called uniform global *pre*-attractivity (UGpA), for example in [58], and UGAS is then called uniform global *pre*-asymptotic stability, to underline this “lack of completeness”. For notational simplicity, we choose to avoid the prefix *-pre*. In what follows existence and completeness will be indeed proved/assumed case-by-case, but we find it useful to have a stability theory as general as possible.

Global uniform stability of course implies the common notion of (*uniform*) *Lya-punov stability*, that is:

For any  $\varepsilon > 0$ , there exists  $\delta(\varepsilon) > 0$  such that any solution  $\phi$  of (1.1) with  $|\phi(0)|_{\mathcal{A}} \leq \delta(\varepsilon)$  satisfies  $|\phi(t)|_{\mathcal{A}} \leq \varepsilon$  for all  $t \in \text{dom } \phi$ .

The two properties are equivalent asking that  $\lim_{\varepsilon \rightarrow \infty} \delta(\varepsilon) = +\infty$ . The term “uniform” in these two properties underlines the fact the bound functions ( $\alpha \in \mathcal{K}_\infty$  and  $\delta : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ ) do not depend *on the specific solution* we are looking at, (since

in general we do not have uniqueness), and historically comes from the framework described in (1.9).

If forward completeness is assured, then uniform global attractivity implies that, for every solution  $\phi$  of (1.1), we have  $\lim_{t \rightarrow \infty} |\phi(t)|_{\mathcal{A}} = 0$ . This property is commonly called *global attractivity* and it is weaker than item 2. in Definition 1.2, since uniformity requires that  $T > 0$  depend only on the margin  $\varepsilon > 0$  and on  $r$ , the distance of the initial conditions from  $\mathcal{A}$ ;  $T$  *does not* depend on the particular solution or initial point. Under some assumptions on the map  $F$  and on the attractor  $\mathcal{A}$ , Lyapunov stability+global attractivity imply UGAS, as proved for example in [36, Proposition 2.2].

The concept of UGAS in Definition 1.2 can be equivalently defined using the formalism of  $\mathcal{KL}$  functions, in fact we have that

- $\mathcal{A}$  is uniformly globally asymptotically stable (UGAS), if and only if there exists  $\beta \in \mathcal{KL}$  such that any solution  $\phi$  of (1.1) satisfies  $|\phi(t)|_{\mathcal{A}} \leq \beta(|\phi(0)|_{\mathcal{A}}, t)$  for all  $t \in \text{dom } \phi$ ,

see [122, Proposition 1] for the proof and for further discussions about  $\mathcal{KL}$ -stability for differential inclusions.

### 1.2.2 Lyapunov Direct Method: Smooth Case

A common and powerful tool in stability theory is given by the Lyapunov direct method. Definition 1.2 involves properties that are required to hold for *all* the solutions starting from *any* initial point; and thus it is hard to check these conditions directly. In fact this would require the possibility of analytically compute all the solutions of a given system, and this is a challenging problem, even in the simpler case of autonomous differential equations. Roughly speaking, in order to bypass this issue, the Lyapunov method idea is to find a positive definite function, a so-called Lyapunov function, which is strictly decreasing (or non-increasing for the UGS case) along the solutions of system (1.1). We formalize this idea in the following statement.

#### Theorem 1.3: Smooth Lyapunov Theorem

Consider  $\mathcal{C} \subset \mathbb{R}^n$ ,  $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$  with  $\text{dom}(F) \supset \mathcal{C}$  and a closed set  $\mathcal{A} \subset \mathbb{R}^n$ . We say that a function  $V : \text{dom } V \rightarrow \mathbb{R}$  is a *smooth Lyapunov function* for system (1.1) with respect to  $\mathcal{A}$  if  $\mathcal{C} \subset \text{dom } V \subset \mathbb{R}^n$ ,  $V \in \mathcal{C}^1(\text{dom } V, \mathbb{R})$ , there exist  $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$  such that

$$\alpha_1(|x|_{\mathcal{A}}) \leq V(x) \leq \alpha_2(|x|_{\mathcal{A}}), \quad \forall x \in \mathcal{C}, \quad (1.10)$$

and there exists  $\gamma \in \mathcal{PD}$  such that

$$\langle \nabla V(x), f \rangle \leq -\gamma(|x|_{\mathcal{A}}), \quad \forall x \in \mathcal{C}, \forall f \in F(x). \quad (1.11)$$

If there exists a smooth Lyapunov function for system (1.1) with respect to  $\mathcal{A}$ , then the set  $\mathcal{A}$  is UGAS for (1.1). If there exists  $V$  satisfying (1.10) and (1.11) with  $\gamma \equiv 0$ , then the set  $\mathcal{A}$  is UGS for (1.1).

The following proof is fairly standard, and provides the ideas commonly used in direct Lyapunov results. However, we find it useful to propose it in this thesis, since it could be taken as our “foundation stone”. Indeed most of the results that we will propose are obtained by adapting/relaxing the following ideas in various contexts.

*Proof :* Let us suppose  $V : \text{dom } V \rightarrow \mathbb{R}$  is a smooth Lyapunov function, and consider any  $\phi : \text{dom}(\phi) \rightarrow \mathcal{C}$ , a solution of system (1.1) and let us denote  $\text{dom}(\phi) = [0, T_\phi)$ , with  $T_\phi > 0$ , possibly  $T_\phi = \infty$ . By definition, since  $\phi$  is absolutely continuous and  $V$  is continuously differentiable, the function  $V \circ \phi : [0, T_\phi) \rightarrow \mathbb{R}$  is absolutely continuous, and thus its derivative exists almost everywhere, and for almost every  $t \in [0, T_\phi)$  there exists  $f \in F(\phi(t))$  such that

$$\frac{d}{dt}V(\phi(t)) = \langle \nabla V(\phi(t)), f \rangle. \quad (1.12)$$

Using the fundamental theorem of calculus, given any  $\tau \in [0, T_\phi)$ ,

$$\begin{aligned} \alpha_1(|\phi(\tau)|_{\mathcal{A}}) &\leq V(\phi(\tau)) = V(\phi(0)) + \int_0^\tau \frac{d}{dt}V(\phi(t))dt \\ &\leq V(\phi(0)) - \int_0^\tau \gamma(|\phi(t)|_{\mathcal{A}})dt \leq V(\phi(0)) \leq \alpha_2(|\phi(0)|_{\mathcal{A}}) \end{aligned} \quad (1.13)$$

where we made use of (1.10), (1.11) and (1.12). We have thus proved (UGS) since, defining  $\tilde{\alpha} := \alpha_1^{-1} \circ \alpha_2 \in \mathcal{K}_\infty$  we have

$$|\phi(\tau)|_{\mathcal{A}} \leq \tilde{\alpha}(|\phi(0)|_{\mathcal{A}}), \quad \text{for every } \tau \in \text{dom}(\phi). \quad (1.14)$$

Note that we can follow exactly the same steps if  $\gamma \equiv 0$ , thus proving (UGS) also when  $V$  satisfies (1.11) with  $\gamma \equiv 0$ .

In order to prove uniform global attractivity, considering  $\varepsilon > 0$  and  $r > 0$ , it suffices to define

$$\delta := \tilde{\alpha}^{-1}(\varepsilon), \quad R := \tilde{\alpha}(r), \quad m := \min_{s \in [\delta, R]} \gamma(s) \quad \text{and} \quad T' := \frac{\alpha_2(r) - \alpha_1(\delta)}{m}.$$

By definition of  $\delta$  and  $R$ , and recalling (1.14) we have the following implications

$$\begin{aligned} |\phi(0)|_{\mathcal{A}} \leq \delta &\Rightarrow |\phi(t)|_{\mathcal{A}} \leq \varepsilon, \quad \text{for every } t \in \text{dom}(\phi), \\ |\phi(0)|_{\mathcal{A}} \leq r &\Rightarrow |\phi(t)|_{\mathcal{A}} \leq R, \quad \text{for every } t \in \text{dom}(\phi), \end{aligned}$$

for every solution  $\phi$  of (1.1). The attractivity argument now follows by contradiction. Consider any  $T > T'$  and suppose that there exists  $\phi : \text{dom}(\phi) \rightarrow \mathbb{R}^n$  solution of (1.1) with  $|\phi(0)|_{\mathcal{A}} \leq r$  such that  $|\phi(t)|_{\mathcal{A}} \geq \delta$ , for all  $t \in \text{dom}(\phi) \cap [0, T]$ . This implies  $\gamma(|\phi(t)|_{\mathcal{A}}) \geq m$ , for all  $t \in \text{dom}(\phi) \cap [0, T]$ . Consider now a  $\tau \in \text{dom}(\phi) \cap (T', T]$ , with the same steps as in (1.13), we have

$$\alpha_1(|\phi(\tau)|_{\mathcal{A}}) \leq \alpha_2(|\phi(0)|_{\mathcal{A}}) - \tau m < \alpha_2(|\phi(0)|_{\mathcal{A}}) - T' m \leq \alpha_2(r) - \alpha_2(r) + \alpha_1(\delta),$$

and thus  $|\phi(\tau)|_{\mathcal{A}} < \delta$ , a contradiction. Thus, for any solution  $\phi : \text{dom}(\phi) \rightarrow \mathbb{R}^n$  of (1.1) with  $|\phi(0)|_{\mathcal{A}} \leq r$ , there exists a  $\tau \in \text{dom}(\phi) \cap [0, T]$  such that  $|\phi(\tau)|_{\mathcal{A}} \leq \delta$ , implying that  $|\phi(t)|_{\mathcal{A}} \leq \varepsilon$ , for all  $t \in \text{dom}(\phi) \cap [T, \infty)$ , as we wanted to prove. We

want to stress again that the argument is vacuously satisfied for non-complete solutions such that  $T_\phi < T$ .  $\diamond$

Theorem 1.3 is an important result, and provides *sufficient* conditions: if a smooth Lyapunov function exists then the systems is UGAS. Under some further assumptions on  $\mathcal{C} \subset \mathbb{R}^n$ ,  $\mathcal{A} \subset \mathbb{R}^n$  and  $F : \mathcal{C} \rightrightarrows \mathbb{R}^n$  the converse is also true: if the attractor  $\mathcal{A}$  is UGAS then a smooth Lyapunov function exists, see [122], [36], [86] for the case  $\mathcal{C} = \mathbb{R}^n$ , or [58, Theorem 7.31] in the more general context of hybrid systems, which includes constrained differential inclusions as special case.

### 1.3 Lyapunov Direct Results: Literature Review

As discussed in Section 1.2, the Lyapunov direct method is a powerful tool for establishing (various kinds of) stability properties for a large class of systems. This section is devoted to recalling the main results for differential inclusions, hybrid/switching systems which have appeared over the past years. This will allow discussing the motivations behind the work of this thesis, correctly placing the results presented in subsequent chapters in the contemporary context. This section *should not* be regarded as an exhaustive survey on Lyapunov theory, since we will only recall and refer to results that are somehow related with our work. For an overview of Lyapunov theory in control systems we refer to [10]; while for a general discussion in the context of ODE's, a good reference could be the book [82].

Considering differential inclusions as in (1.1), the study of smooth Lyapunov functions has attracted some attention. For a statement similar to Theorem 1.3 we refer to [114, Theorem 8.3] in the unconstrained case ( $\mathcal{C} = \mathbb{R}^n$ ), while in [5, Chapter 6] the authors present various Lyapunov-like results, in the context of constrained differential inclusions. Among other examples, a Lyapunov theorem for differential equations coupled with monotone set-valued operators is presented in [26, Theorem 6.3], while in [22] (smooth) Lyapunov functions are used to study instability of differential inclusions as in (1.1).

The adaptation of the Lyapunov Theorem in the context of hybrid systems is introduced in [58, Theorem 3.18], and it is formally presented in Section 2.3; possible extensions can be found in [120], or [95].

As we already said, various converse Lyapunov theorems have been proved for (1.1), see [122], [36] or [58, Theorem 7.31] for the formal statement and [81] for a thorough review. More precisely, under some assumptions on  $\mathcal{C}$  and  $F$ , we have the following result: If the origin of the differential inclusion (1.1) is globally asymptotically stable (GAS), then there exists a *smooth* Lyapunov function. From a theoretical point of view, there are many advantages in having a *smooth* Lyapunov function: in particular the existence of such a function also gives us information about the *robustness* of the stability for the considered dynamical system. Indeed, for a general hybrid systems, it is shown in [29] that if a smooth Lyapunov function exists, then the asymptotic stability is robust to small perturbations of the data.

On the other hand, many stability criteria for (1.1) relying on non-smooth Lyapunov functions have been proposed in the past years. Among other examples we refer to [34, Chapter 4] and references therein for a general treatment; more detailed



discussions will be provided in what follows. Indeed, despite the cited *smooth* converse Lyapunov results, in many situations the construction of less regular Lyapunov functions can be easier from a computational point of view, or can be even suggested by the structure of the considered problem, as we discuss in what follows.

The Lyapunov direct method applied to switching systems as defined in Section 1.1.2 deserves a special discussion. In fact, this setting can be taken as a first and intuitive framework in which results relying on *non-smooth Lyapunov functions* can effectively provide less demanding stability criteria with respect to the smooth Lyapunov functions case sketched in Theorem 1.3 for differential inclusions.

One possible approach to proving stability for switched system (1.2) under arbitrary switching is to design a *common Lyapunov function*, that is a positive definite function which satisfies a decrease property as in (1.11) for all  $f_i : \mathbb{R}^n \rightarrow \mathbb{R}^n$ ,  $i \in \mathcal{I}$ . Of course a necessary condition for the existence of such a function is the fact that subsystems  $\dot{x} = f_i(x)$  are themselves globally asymptotically stable (with respect to the origin), but this is in general not sufficient. Interestingly, the existence of such common Lyapunov functions can be related with the commutativity properties of the functions  $f_i : \mathbb{R}^n \rightarrow \mathbb{R}^n$ , for  $i \in \mathcal{I}$ . In the linear case, i.e.  $f_i(x) = A_i x$ , for some  $A_i \in \mathbb{R}^{n \times n}$ , it is proven in [94] that if all  $\{A_i\}_{i \in \mathcal{I}}$  are Hurwitz and commute, then a common quadratic Lyapunov function exists, while in [89] this commutativity property is used to show the possible extensions to the nonlinear subsystems case. These ideas can be generalized, giving stability conditions for switched systems that depend on the properties of the *Lie algebra* generated by the set of matrices  $\{A_i\}_{i \in \mathcal{I}}$ , as presented in [2]. These results can be adapted, via linearization, to prove local asymptotic stability in the nonlinear case, while for the study of robustness of these conditions, see [1]. For a thorough discussion of this subject we refer to [84, Section 2.2] and references therein.

When one is interested in switched systems as in (1.3) driven by a particular time-dependent signal (or a signal that is known to belong to a particular subclass as for example (1.5) or (1.6)), the existence of a common Lyapunov function with respect to all the subsystems is a very restrictive (and not necessary) condition. In this context, stability certificates can be obtained relying on a *multiple Lyapunov functions* approach. More precisely, the idea is that, if all the subsystems  $\dot{x} = f(x)$ ,  $i \in \mathcal{I}$  are globally asymptotically stable, we can find, for each  $i \in \mathcal{I}$ , a smooth Lyapunov function  $V_i : \mathbb{R}^n \rightarrow \mathbb{R}$  for  $f_i$  with respect to the origin. Then, one needs to combine these functions  $\{V_i\}_{i \in \mathcal{I}}$  to obtain a time-dependent (or switching signal-dependent) Lyapunov function. To prove that this time-varying function is decreasing, one needs to impose further conditions at the switching instants, since the function is in general discontinuous at that instants.

For example one can ask that the sequence of values of the Lyapunov function at switching instants be decreasing as in [21], that is

$$V_{\sigma(t_k)}(\phi(t_k)) < V_{\sigma(t_{k-1})}(\phi(t_{k-1})), \quad \text{for any } t_k > 0,$$

and any solution  $\phi : \mathbb{R}_+ \rightarrow \mathbb{R}^n$  of (1.4). Another possibility is to ensure that the value of the function decreases at each time the switching signal activates/deactivates the same subsystem. More precisely, considering  $\sigma \in \mathcal{S}$ , for the decreasing property

at enter-switching times we ask

$$V_{\sigma(t_{k'})}(\phi(t_{k'})) < V_{\sigma(t_k)}(\phi(t_k)),$$

for any  $t_k > 0$  and any  $k' > k + 1$  such that  $\sigma(t_k) = \sigma(t_{k'})$  and any solution  $\phi$  of (1.4) see [84, Theorem 3.1] or [21]. We refer the interested reader to [84, Chapter 3].

Another remarkable example of the benefits of the multiple Lyapunov functions approach in switched systems is the *minimal dwell time problem*. More precisely, in the linear and finite-index set case (i.e.  $f_i(x) \equiv A_i x$ , with  $A_i \in \mathbb{R}^{n \times n}$  for all  $i \in \mathcal{I} := \{1, \dots, K\}$ ), it is well known that, if all the matrices  $A_1, \dots, A_K$  are Hurwitz, there exists a (large enough) dwell-time  $\tau \geq 0$  for which the switched system (1.4) is asymptotically stable with respect to the class  $\mathcal{S}_{\text{dw}}(\tau)$  defined in (1.5), see [93]. For this reason, a widely studied problem is the numerical approximation of the minimal dwell-time  $\tau_{\text{dw}} \in \mathbb{R}_+$  for which this holds. To deal with this problem, a multiple Lyapunov functions approach has commonly been considered: in [56] a family of quadratics is used to give lower-bounds on  $\tau_{\text{dw}}$ , constructing Lyapunov functions decreasing between switching but possibly increasing at the switching instants, while in [4], a different construction is proposed, with the peculiarity that the resulting Lyapunov functions are non-increasing at every switching instant. Similar results can be found in [25], [32], [15]. Regarding the converse Lyapunov problem in this setting, see the result presented in [127, Corollary 6.5].

This multiple Lyapunov functions approach can be somehow adapted in studying the stability of state-dependent switched systems, as defined in (1.7). Given  $\{f_i\}_{i \in \mathcal{I}}$  and a partition of the state space  $\{X_i\}_{i \in \mathcal{I}}$  such that  $\mathbb{R}^n = \bigcup_{i \in \mathcal{I}} X_i$  we suppose that there exists a set of positive definite functions  $\{V_i\}_{i \in \mathcal{I}}$  such that the Lyapunov condition

$$\langle \nabla V_i(x), f_i(x) \rangle \leq -\rho(|x|), \quad \text{for any } x \in X_i,$$

is satisfied, for any  $i \in \mathcal{I}$ . Then, defining an “overall” function  $V : \mathbb{R}^n \rightarrow \mathbb{R}$  as

$$V(x) := V_i(x), \quad \text{if } x \in X_i,$$

the main issue is to ensure proper continuity/decreasing properties of this “piecewise” constructed  $V$  at the boundaries of the operating regions  $X_i$ . This idea was firstly introduced in [77] and has been used more recently in [11] and [63], or also in [70] in the context of piecewise affine systems. In [61] the authors studied conditions under which a state-dependent switched system with linear subsystems admits a common quadratic Lyapunov function. Since state-dependent switched systems are in general a subclass of discontinuous dynamical systems, an overview of Lyapunov techniques is provided in [39].

## 1.4 Contribution and Organization of the Thesis

This thesis focuses on analyzing stability of state-dependent switched and hybrid systems using multiple Lyapunov functions. The constructions we provide render the overall Lyapunov function locally Lipschitz, and thus, in general, non-smooth. More precisely, the main contribution lies in proposing various relaxations of inequality (1.11) in Theorem 1.3, when considering candidate Lyapunov functions

that are merely locally Lipschitz. The main tools used in the analysis arise from the non-smooth analysis literature, mainly [34], [37], [105]. In the context of differential inclusions with outer-semicontinuous dynamics, we follow the ideas presented in [8] and [30] where tighter conditions are proposed if the Lyapunov functions satisfy some further assumptions. More specifically, in this thesis we apply the non-smooth Lyapunov framework in two particular settings:

Firstly, imposing (rather strong) continuity assumptions on the considered set-valued maps, we study constrained differential inclusions as in (1.1), proposing stability conditions relying on a subclass of locally Lipschitz functions. We show that, in this context, using generalized differentiation techniques may be difficult or conservative or, in fact, unclear when the differential inclusion (1.1) is constrained to a closed set  $\mathcal{C} \subset \mathbb{R}^n$ , as it naturally is the case in hybrid systems. We illustrate these issues and we show that for a wide class of functions with piecewise structure, it suffices to check Lyapunov inequalities on a *dense subset* of the set  $\mathcal{C}$  and only at points where the functions are differentiable. This avoids several non-smooth technicalities and appears less conservative than the alternatives, for examples the ones introduced in [88] in the context of hybrid systems.

Secondly, we apply the (non-smooth) Lyapunov construction firstly proposed in [7] to the *state-dependent switched systems* framework. Since the “set-valued” regularization of state-dependent switched systems would not satisfy the continuity assumption imposed in the preceding chapter, we will need to follow a different approach. In particular we study the class of functions obtained by iterating pointwise maximum and minimum operators on a finite class of  $\mathcal{C}^1$  functions. This allow us to propose relaxed Lyapunov conditions, and to analyze how these conditions relate with the *sliding motion* phenomenon, which commonly arises in switching systems. Exploiting the combinatory structure of these max-min functions, we provide algorithmic procedures to check the proposed conditions, in the case where the subsystems are linear. This idea of using non-smooth Lyapunov functions is then generalized to studying how similar constructions can establish also *input-to-state stability* (ISS) (see [115]) in the context of perturbed state-dependent switched systems. This will permits us to propose an observer-based controller for certain nonlinear two-mode state-dependent switched systems.

More specifically, the structure of this thesis is the following one:

- The non-smooth analysis preliminaries will be presented in **Chapter 2**, together with a thorough discussion about solution concepts and well-posedness for the systems introduced in this introduction.
- In **Chapter 3**, we propose a class of locally Lipschitz functions with piecewise structure as candidate Lyapunov functions for differential inclusions. Subject to some regularity of the dynamics, we show that Lyapunov inequalities can be checked only on a dense subset, and thus avoiding points of nondifferentiability of the Lyapunov function. We also relate the proposed class to *piecewise continuously differentiable functions*, introduced in [31] and [110] for optimization purposes. Finally we present applications to hybrid dynamical systems. The content of this chapter follows the material presented in the articles [46] and [47].

- In **Chapter 4** we study a particular class of locally Lipschitz functions: starting from a finite family of continuously differentiable positive definite functions, we study conditions under which a function obtained by *max-min* combinations over this family is a Lyapunov function, in the context of state dependent switched systems. Such *max-min* type of Lyapunov functions were recently proposed in the context of discrete-time switching systems [3], [101]. We specify generalized notions of directional derivatives for these max-min functions, and use them in deriving stability conditions. For the case of state-dependent switched systems with linear sub-dynamics, using the S-Procedure, our conditions result in bilinear matrix inequalities. The articles [49] and [51] are the main sources of this chapter.
- In **Chapter 5** we study robustness analysis for interconnections of differential inclusions. We first provide sufficient conditions for input-to-state stability (ISS) for differential inclusions, using piecewise  $\mathcal{C}^1$  candidate Lyapunov functions. The structure of these functions, formally introduced in [110], fits particularly well the *piecewise-smooth structure* of the state-dependent switched system (1.7). We apply our approach to study ISS of an interconnection of two differential inclusions. In this context, we extend the *small-gain theorem* presented in [75], and if the interconnection is in the so-called *cascade* form, we adapt the construction proposed in [119]. Also, for differential inclusions, the construction proposed in [85] allows for Lipschitz continuous functions by studying the Clarke gradient, while our results rely on a weaker notion of derivative, see the subsequent Section 2.4 for the details. The technical content of this chapter is mainly taken from [52].
- In **Chapter 6** we present some final discussion, underlying the possible future paths of research.



# 2

## Technical Preliminaries

In this chapter, we provide the necessary preliminaries for studying the system classes presented in Chapter 1. In Section 2.1, we recall the main continuity properties for set-valued maps. This will allow studying the problem of existence and completeness of solutions for differential inclusions. In Section 2.2, we show how state-dependent switched systems can be modeled by “well-posed” differential inclusions, while in Section 2.3, we define the notions of solutions and stability for hybrid systems. Finally, in Section 2.4 we discuss some possible extensions of the Lyapunov Theorem 1.3 when considering locally Lipschitz Lyapunov functions.

### 2.1 Set-Valued Analysis and Differential Inclusions

This section is devoted to presenting the basic properties of differential inclusions, following the formalism of [105], [5], [6], [45], [114]. Consider a set valued map  $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ . Since its values are subsets of  $\mathbb{R}^m$  we have to clarify the continuity concepts in this context. The main reference for the following definitions is [105, Chapter 5]. (Below,  $x_k \rightarrow^{\mathcal{C}} x$  stands for  $x_k \in \mathcal{C}$  and  $x_k \rightarrow x$  as  $k \rightarrow \infty$ ).

#### Definition 2.1: (Semi-)Continuity of Set-Valued Maps

Consider a set  $\mathcal{C} \subset \mathbb{R}^n$  and a set valued map  $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ , with  $\text{dom } F \supset \mathcal{C}$ .

- $F$  is *outer semicontinuous relative to  $\mathcal{C}$*  at  $x \in \mathcal{C}$  if

$$F(x) \supset \limsup_{y \rightarrow^{\mathcal{C}} x} F(y) := \{f \in \mathbb{R}^m \mid \exists x_k \rightarrow^{\mathcal{C}} x, \exists f_k \in F(x_k) \text{ s.t. } f_k \rightarrow f\}.$$

The map  $F$  is *outer semicontinuous* (on  $\mathcal{C}$ ) if it is outer semicontinuous relative to  $\mathcal{C}$  at each  $x \in \mathcal{C}$ .

- $F$  is *inner semicontinuous relative to  $\mathcal{C}$*  at  $x \in \mathcal{C}$  if

$$F(x) \subset \liminf_{y \rightarrow^{\mathcal{C}} x} F(y) := \{f \in \mathbb{R}^m \mid \forall x_k \rightarrow^{\mathcal{C}} x, \exists f_k \in F(x_k) \text{ s.t. } f_k \rightarrow f\}.$$

The map  $F$  is *inner semicontinuous* (on  $\mathcal{C}$ ) if it is inner semicontinuous relative to  $\mathcal{C}$  at each  $x \in \mathcal{C}$ .

The  $\limsup$ - $\liminf$  notation is motivated by considering the so-called *Painlevé-Kuratowski set convergence* concept, for which we refer to [105, Chapter 4]. Moreover we say that

- $F$  is *continuous relative to  $\mathcal{C}$*  at  $x \in \mathcal{C}$  if it is inner and outer semicontinuous relative to  $\mathcal{C}$  at  $x$ . It is *continuous* (on  $\mathcal{C}$ ) if it is continuous relative to  $\mathcal{C}$  at each  $x \in \mathcal{C}$ .
- $F$  is *locally bounded relative to  $\mathcal{C}$*  at  $x \in \mathcal{C}$  if for some neighborhood  $\mathcal{U}$  of  $x$  the set  $F(\mathcal{U} \cap \mathcal{C}) \subset \mathbb{R}^n$  is bounded. It is *locally bounded* (on  $\mathcal{C}$ ) if this holds at every  $x \in \mathcal{C}$ .

With an abuse of notation, considering maps  $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$  with  $\text{dom}(F) = \mathbb{R}^n$  and considering continuity/boundedness properties relative to  $\mathbb{R}^n$ , we will drop the “relative to”- prefix.

Given  $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ , it is important to underline that if  $F$  is outer semicontinuous at  $x \in \text{dom}(F)$ , then  $F(x)$  is a closed set. In fact it is clear that, in general,  $F(x) \subset \limsup_{y \rightarrow x} F(y)$ : it suffices, given  $f \in F(x)$ , to consider the trivial convergent sequences  $x_k \equiv x$  and  $f_k \equiv f$ . Thus, outer semicontinuity at  $x$  is equivalent to the property  $F(x) = \limsup_{y \rightarrow x} F(y)$ , and since the right-hand side is closed, we conclude.

In the set-valued analysis/differential inclusions literature, as for example in [5], [6], [114], the reader can find the notions of *upper* and *lower* semicontinuity for set-valued maps. The formal definitions of these concepts are the following:  $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$  is upper semicontinuous at  $x$  if for each open set  $\mathcal{V}$  with  $F(x) \subset \mathcal{V}$  there exists an open neighborhood  $\mathcal{U}$  of  $x$  such that  $F(z) \subset \mathcal{V}$ , for all  $z \in \mathcal{U}$ , while  $F$  is lower semicontinuous at  $x$  if for each open set  $\mathcal{V}$  with  $F(x) \cap \mathcal{V} \neq \emptyset$  there exists a neighborhood  $\mathcal{U}$  of  $x$  such that  $F(z) \cap \mathcal{V} \neq \emptyset$ , for all  $z \in \mathcal{U}$ . While inner semicontinuity and lower semicontinuity are equivalent, we have the following implications (see [58, Lemma 5.15] or [105, Theorem 5.19] for the proof):

- If  $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$  is outer semicontinuous and locally bounded at  $x$ , then  $F$  is upper semicontinuous at  $x$ ;
- If  $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$  is upper semicontinuous at  $x$  and  $F(x)$  is closed, then  $F$  is outer semicontinuous at  $x$ .

We chose to adopt the outer/inner semicontinuity formalism because of its appealing symmetry and its clear relation with the limits (inferior and superior) in the Painlevé-Kuratowski set convergence, see Definition 2.1. Since in our applications we will always consider set-valued maps which are bounded and with closed values, these subtleties will not play any role. On the other hand we find it important to clarify this point, in order to avoid possible misunderstandings, since in what follows we will possibly refer to results which use the upper/lower semicontinuity formalism. For the historical reasons behind these notions, we refer to the discussion in [105, Commentary of Chapter 5].

Any single-valued map  $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$  can be seen as a set-valued map  $F^f : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$  with  $F^f(x) := \{f(x)\}$ , that is, a set-valued map with singleton values. It is thus interesting to note the following equivalences:

- $f$  is continuous at  $x$ ;
- $F^f$  is continuous at  $x$ ;
- $F^f$  is outer semicontinuous and locally bounded at  $x$ ;
- $F^f$  is inner semicontinuous at  $x$ .

### 2.1.1 Existence and Completeness for Differential Inclusions

We are now in a position to give an existence result for differential inclusions.

#### Proposition 2.2: Existence of Solutions for Differential Inclusions

Consider  $\mathcal{C} \subset \mathbb{R}^n$  and a set-valued map  $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$  with  $\mathcal{C} \subset \text{dom}(F)$ , locally bounded in  $\mathcal{C}$ , outer semicontinuous on  $\mathcal{C}$  with closed and convex values. Then, for every  $x_0 \in \text{int}(\mathcal{C})$  there exists a solution of (1.1) with  $\phi(0) = x_0$ . The same holds if outer semicontinuity on  $\mathcal{C}$  is replaced by inner semicontinuity on  $\mathcal{C}$ .

The proof of this result can be found, for example, in [5, Theorems 2.1, 2.3], in [45] or, for the outer semicontinuous case only, [114, Theorem 4.7]. Note that in Proposition 2.2, we are considering initial conditions  $x_0$  in the *interior* of  $\mathcal{C}$ . Initial conditions on the boundary of  $\mathcal{C}$  could be problematic in view of existence, since the differential inclusion can possibly force the solution to “escape”  $\mathcal{C}$ . This problem is known in literature under the name of *viability*, and we refer to [5, Chapter 4] for a formal treatment. For simplicity, from now on in this subsection we will consider differential inclusions defined on the whole space  $\mathbb{R}^n$ , i.e. the right-hand side has  $\text{dom}(F) = \mathbb{R}^n$ . Proposition 2.2 can be seen as an extension of the Cauchy-Peano Existence Theorem, in the sense that we recover the classical hypothesis when considering continuous single-valued maps  $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ .

Since, as we said, we are not interested in uniqueness results, we can now ask if we can give conditions under which the solutions not only exist but are (forward)-*complete*, in the sense that their domains can be extended to the whole line  $\mathbb{R}_+$ . In the single-valued setting, completeness is often ensured by imposing an anti-“*blow up in finite times*” condition, which can be obtained by asking a linear growth behavior of the vector field. This idea can be obviously generalized to the set-valued context, see for example [53, Theorem 2.2] and references therein. In the following, we formalize this idea.

#### Lemma 2.3: Linear Growth and Completeness

Consider a set valued map  $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$  with  $\text{dom}(F) = \mathbb{R}^n$ ; if

- $F$  satisfies the hypotheses of Proposition 2.2 on  $\mathbb{R}^n$ ;
- There exist constants  $k$  and  $a \in \mathbb{R}$  such that  $|f| \leq k|x| + a$ , for all  $f \in F(x)$ , for all  $x \in \mathbb{R}^n$ ,

then for any maximal solution  $\phi$  of (1.1) it holds that  $\text{dom}(\phi) = [0, +\infty)$ .



We underline that a linear growth property of the right hand side is only a sufficient condition for forward completeness. For example, the vector field  $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ ,  $f(x) = -|x|x$  cannot be bounded in norm by a linear function, but the solutions of  $\dot{x} = f(x)$  are forward complete, as an explicit computation shows. Interestingly, this system can be seen as a particular case of differential inclusion

$$\dot{x} \in -F(x),$$

where  $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$  is a *maximal monotone operator*, as defined in [105, Definition 12.5]. For this kind of differential inclusions, uniqueness and forward completeness of solutions is proved in [23, Théorème 3.1], see also [26, Theorem 5.8] for a time-varying counterpart.

For a thorough discussion about existence, uniqueness and completeness of solutions for differential inclusions, the interested reader can see, for example, the books [45], [114], [54].

## 2.2 Regularization of State-Dependent Switched Systems

In this section, we consider again switching systems as defined in Section 1.1.2. In particular, we want to investigate and deeply discuss the state-dependent switched systems case, by properly defining the notion of *state-dependent switching signal*. More specifically, we first clarify what we mean by partition of the state-space.

### Definition 2.4: Proper State-Space Partition

Given a discrete index set  $\mathcal{I} \subset \mathbb{N}$ , let us consider a collection of closed sets  $\{X_i\}_{i \in \mathcal{I}} \subset \mathcal{P}(\mathbb{R}^n)$  and a collection of open sets  $\{\mathcal{O}_i\}_{i \in \mathcal{I}} \subset \mathcal{P}(\mathbb{R}^n)$  such that

- a)  $\bigcup_{i \in \mathcal{I}} X_i = \mathbb{R}^n$ ,
- b)  $X_i \subset \mathcal{O}_i$ , for all  $i \in \mathcal{I}$ ,
- c)  $\overline{\text{int}(X_i)} = X_i$ , for all  $i \in \mathcal{I}$ ,
- d) For every  $i \in \mathcal{I}$ ,  $\text{bd}(X_i)$  has zero Lebesgue measure,
- e)  $X_i \cap X_j = \text{bd}(X_i) \cap \text{bd}(X_j)$ , for all  $i, j \in \mathcal{I}$ ,  $i \neq j$ .

In this situation, we say that  $\mathcal{X} := \{X_i, \mathcal{O}_i\}_{i \in \mathcal{I}}$  is a *proper partition* of  $\mathbb{R}^n$ . We define  $\partial\mathcal{X} := \bigcup_{i \in \mathcal{I}} \text{bd}(X_i)$ , and we underline that  $\partial\mathcal{X}$  has zero Lebesgue measure.

The collection of open sets  $\{\mathcal{O}_i\}_{i \in \mathcal{I}}$  is somewhat arbitrary, since, given any collection of closed set  $\{X_i\}_{i \in \mathcal{I}}$ , we can always define, for every  $i \in \mathcal{I}$ ,  $\mathcal{O}_i := X_i + \mathbb{B}(0, \varepsilon)$ , for an arbitrary  $\varepsilon > 0$ , satisfying property b) of Definition 2.4. On the other hand, it will be useful to have the collection  $\{\mathcal{O}_i\}_{i \in \mathcal{I}}$  directly in the definition of proper partition, since all the continuity/differentiability properties that we need will be easily defined based on the open sets  $\mathcal{O}_i$ .

**Definition 2.5: State-Dependent Switched System**

Given  $\mathcal{X} = \{X_i, \mathcal{O}_i\}_{i \in \mathcal{I}}$  a proper partition of  $\mathbb{R}^n$ , consider  $f_i \in \mathcal{C}^1(\mathcal{O}_i, \mathbb{R}^n)$ ,  $i \in \mathcal{I}$ . A *state-dependent switching signal* associated to  $\mathcal{X}$  is a function  $\sigma : \mathbb{R}^n \rightarrow \mathcal{I}$  such that

$$\sigma(x) = i, \quad \text{if } x \in \text{int}(X_i), \quad (2.1)$$

and the *state-dependent switched system* associated to  $\{X_i, \mathcal{O}_i, f_i\}_{i \in \mathcal{I}}$  is the differential equation

$$\dot{x} = f_{\sigma(x)}(x). \quad (2.2)$$

We note here that, given a proper partition  $\mathcal{X} = \{X_i, \mathcal{O}_i\}$ , a state dependent switching signal associated to it is not uniquely defined: the value of  $\sigma$  remains unspecified on the null-measure set  $\partial\mathcal{X}$ . In what follows we clarify why this ambiguity does not affect the solution set of the corresponding regularized state-dependent switched system.

Given a proper partition  $\mathcal{X}$  of  $\mathbb{R}^n$ , we can introduce an “index indicator map”, that is a set valued map  $\mathcal{I}_{\mathcal{X}} : \mathbb{R}^n \rightrightarrows \mathcal{I}$  defined as

$$\mathcal{I}_{\mathcal{X}}(x) := \{i \in \mathcal{I} \mid x \in X_i\}. \quad (2.3)$$

Intuitively, the map  $\mathcal{I}_{\mathcal{X}} : \mathbb{R}^n \rightrightarrows \mathcal{I}$  can be seen as a set-valued counterpart of the signal  $\sigma : \mathbb{R}^n \rightarrow \mathcal{I}$  associated to  $\mathcal{X}$ . From Definition 2.5,  $\mathcal{I}_{\mathcal{X}}$  is almost everywhere single valued and, by item e) of Definition 2.5, if  $x \in \text{int}(X_\ell)$  for some  $\ell \in \mathcal{I}$  then  $\mathcal{I}_{\mathcal{X}}(x) = \{\ell\} = \{\sigma(x)\}$ . On the other hand, given a point  $x \in \partial\mathcal{X}$ ,  $\mathcal{I}_{\mathcal{X}}(x)$  is multivalued, and contains all the indexes of sets to which  $x$  belongs.

System (2.2) has a discontinuous right-hand side in the first argument and thus it may not have any Carathéodory solution due to the discontinuity points of  $f_{\sigma(\cdot)}(\cdot)$ , see [39]. Many possible definitions of “generalized solutions” for discontinuous dynamical system are possible (see for example [30] or [39]); we consider the concept of *Filippov solutions*, introduced firstly in [54]. More formally, we define  $F^{\text{sw}} : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ , the Filippov regularization of the discontinuous map  $f_\sigma$ , as

$$F^{\text{sw}}(x) := \text{Fil}(f_{\sigma(x)})(x) := \bigcap_{\delta > 0} \bigcap_{\mu_L(S)=0} \overline{\text{co}}\{f_{\sigma(y)}(y) \mid y \in \mathbb{B}(x, \delta) \setminus S, y \neq x\}$$

for any  $x \in \mathbb{R}^n$ . Under the hypotheses in Definitions 2.4 and 2.5, it can be proven that

$$F^{\text{sw}}(x) = \text{co}\{f_i(x) \mid i \in \mathcal{I}_{\mathcal{X}}(x)\},$$

see for example [54], [39] and [63]. Summarizing, we have “regularized” the discontinuous differential equation in (2.2), by transforming it into the differential inclusion

$$\dot{x} \in F^{\text{sw}}(x) = \text{co}\{f_i(x) \mid i \in \mathcal{I}_{\mathcal{X}}(x)\}. \quad (2.4)$$

A *Filippov solution* of system (2.2) is, by definition, a solution of the differential inclusion (2.4), in the sense presented in Section 1.1.1. As proved in [54, Chapter 2], or in [39, Proposition 3], if  $f_i$  are continuous then the set-valued map  $F^{\text{sw}} : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$  is outer semicontinuous and locally bounded in  $\mathbb{R}^n$  with convex and compact values,

and thus, recalling Proposition 2.2, we have that for any  $x_0 \in \mathbb{R}^n$  there exists a Filippov solution of (2.2) starting at  $x_0$ . Moreover, if  $\mathcal{I}$  is finite and all the subsystems satisfy a linear growth condition, that is there exist  $k_i, a_i \in \mathbb{R}$ , for any  $i \in \mathcal{I}$  such that

$$|f_i(x)| \leq k_i|x| + a_i, \quad \text{for all } x \in X_i,$$

then system (2.4) satisfies the conditions in Lemma 2.3, by choosing  $k = \max_{i \in \mathcal{I}} k_i$ ,  $a = \max_{i \in \mathcal{I}} a_i$ , proving completeness of solutions. Let us note that this is true in particular for state-dependent switched systems among a finite set of linear subsystems, i.e. the case  $f_i(x) = A_i x$ , for some  $A_i \in \mathbb{R}^{n \times n}$ ,  $i \in \mathcal{I}$ .

## 2.3 Solutions and Stability for Hybrid Systems

In this section we consider again the framework introduced in Section 1.1.3, and we introduce the notion of solutions, stability and finally we give a hybrid counterpart of the Lyapunov Theorem 1.3.

Due to the composite structure of hybrid system  $\mathcal{H} = (\mathcal{C}, \mathcal{D}, F, G)$  as defined in (1.8), its solutions takes values on *hybrid-time domains*, which we define in the following statement.

### Definition 2.6: Hybrid Time Domains

Consider a set  $E \subset \mathbb{R}_+ \times \mathbb{N}$ . We say that  $E$  is a *compact hybrid time domain* if  $E = \bigcup_{j=0}^J ([t_j, t_{j+1}], j)$ , for some  $J \in \mathbb{N}$  and for some times  $0 = t_0 \leq t_1 \leq \dots \leq t_{J+1}$ .

We say that  $E$  is a *hybrid time domain* if, for each  $(T, J) \in E$ , the set  $E \cap ([0, T] \times \{0, 1, \dots, J\})$  is a compact hybrid time domain.

Intuitively a hybrid time domain is a (finite or infinite) union of closed (except the last one, if exists) and consecutive intervals (possibly reduced to a singleton), which are labeled by a discrete counter.

The candidate solutions to (1.8) will be defined on hybrid time domains, and must be *sufficiently regular* on non-trivial intervals defining their domains. In the following statement we formalize this intuition.

### Definition 2.7: Hybrid Arcs

A function  $\phi : \text{dom}(\phi) \rightarrow \mathbb{R}^n$  is an *hybrid arc* if

- The set  $\text{dom}(\phi) \subset \mathbb{R}_+ \times \mathbb{N}$  is a hybrid time domain,
- For every  $j \in \mathbb{N}$ , the function  $t \mapsto \phi(t, j)$  is absolutely continuous on the interval  $I^j := \{t \in \mathbb{R}_+ \mid (t, j) \in \text{dom}(\phi)\}$ .

Now, given a hybrid system  $\mathcal{H}$  as in (1.8), we have all that we need to properly define the notion of solution.

**Definition 2.8: Solutions to Hybrid Systems**

A hybrid arc  $\phi : \text{dom}(\phi) \rightarrow \mathbb{R}^n$  is a solution to the hybrid systems  $\mathcal{H} = (\mathcal{C}, \mathcal{D}, F, G)$  if

- The initial condition satisfies  $\phi(0, 0) \in \bar{\mathcal{C}} \cup \mathcal{D}$ ;
- For all  $j \in \mathbb{N}$  such that  $I^j := \{t \in \mathbb{R}_+ \mid (t, j) \in \text{dom}(\phi)\}$  has non empty interior, we have
  - $\phi(t, j) \in \mathcal{C}$ , for all  $t \in \text{int}(I^j)$ ,
  - $\dot{\phi}(t, j) \in F(\phi(t, j))$  for almost all  $t \in I^j$ .
- For all  $(t, j) \in \text{dom}(\phi)$  such that  $(t, j + 1) \in \text{dom}(\phi)$ , we have
  - $\phi(t, j) \in \mathcal{D}$ ,
  - $\phi(t, j + 1) \in G(\phi(t, j))$ .

For the problem of existence of solutions, we refer to [58, Chapter 6] for a complete discussion. Since the flow and jump dynamics in (1.8) are defined by set-valued maps, in general the solutions from a given starting point, if any, are not unique.

We now formulate the stability concepts and results in the context of hybrid systems. This case requires some particular care since, as we noted, solutions of (1.8) are defined on hybrid-time domains.

**Definition 2.9: Stability for Hybrid Systems**

Consider an hybrid system  $\mathcal{H} = (\mathcal{C}, \mathcal{D}, F, G)$ . Given a closed set  $\mathcal{A} \subset \mathbb{R}^n$ , we say that

1. The set  $\mathcal{A}$  is *uniformly globally stable* for (1.8) if, there exists a function  $\alpha \in \mathcal{K}_\infty$  such that, for any solution  $\phi$  of (1.8), we have that  $|\phi(t, j)|_{\mathcal{A}} \leq \alpha(|\phi(0, 0)|_{\mathcal{A}})$ , for all  $(t, j) \in \text{dom}(\phi)$ .
2. The set  $\mathcal{A}$  is *uniformly globally attractive* for (1.8) if, for any  $\varepsilon > 0$  and any  $r > 0$  there exists a  $T > 0$  such that, for any solution  $\phi$  of (1.8) with  $|\phi(0, 0)|_{\mathcal{A}} \leq r$ , we have that  $|\phi(t, j)|_{\mathcal{A}} \leq \varepsilon$ , for any  $(t, j) \in \text{dom}(\phi)$  such that  $t + j \geq T$ .
3. The set  $\mathcal{A}$  is *uniformly globally asymptotically stable* (UGAS) for (1.8) if it is uniformly globally Lyapunov stable and uniformly globally attractive.

These definitions are very similar to the ones in Definition 1.2. We want to underline one important fact: in the continuous-time setting, given a  $T > 0$ , the *length* of an interval  $(0, T) \subset \mathbb{R}_+$  is given by  $T$ . In the hybrid setting, considering a solution  $\phi$ , and its domain  $\text{dom}(\phi) \subset \mathbb{R}_+ \times \mathbb{N}$  we define the *length* or distance between the initial hybrid time  $(0, 0)$  and an arbitrary point  $(t, j) \in \text{dom}(\phi)$  as the positive number  $d_{\mathcal{H}}((0, 0), (t, j)) := t + j$ . This is consistent with the intuitive idea

of “length”, since, given any hybrid time domain  $E$  as in Definition 2.6, the relation defined by  $(t_1, j_1) \leq_E (t_2, j_2) \Leftrightarrow t_1 + j_1 \leq t_2 + j_2$  is a partial ordering on  $E$ .

It is now easy to adapt the Lyapunov Theorem 1.3 in this context.

### Theorem 2.10: Smooth Hybrid Lyapunov Theorem

Consider a hybrid system  $\mathcal{H} = (\mathcal{C}, \mathcal{D}, F, G)$  and a closed set  $\mathcal{A} \subset \mathbb{R}^n$ . We say that a function  $V : \text{dom } V \rightarrow \mathbb{R}$  is a *smooth Lyapunov function* for  $\mathcal{H}$  with respect to  $\mathcal{A}$  if  $\bar{\mathcal{C}} \cup \mathcal{D} \cup G(\mathcal{D}) \subset \text{dom } V \subset \mathbb{R}^n$ ,  $V$  is continuously differentiable in an open set containing  $\bar{\mathcal{C}}$ , there exist  $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$  such that

$$\alpha_1(|x|_{\mathcal{A}}) \leq V(x) \leq \alpha_2(|x|_{\mathcal{A}}), \quad \forall x \in \mathcal{C} \cup \mathcal{D} \cup G(\mathcal{D}), \quad (2.5)$$

and there exists  $\gamma \in \mathcal{PD}$  such that

$$\langle \nabla V(x), f \rangle \leq -\gamma(|x|_{\mathcal{A}}), \quad \forall x \in \mathcal{C}, \forall f \in F(x), \quad (2.6)$$

$$V(g) - V(x) \leq -\gamma(|x|_{\mathcal{A}}), \quad \forall x \in \mathcal{D}, \forall g \in G(x). \quad (2.7)$$

If there exists a smooth Lyapunov function for  $\mathcal{H}$  with respect to  $\mathcal{A}$ , then the set  $\mathcal{A}$  is UGAS for  $\mathcal{H}$ .

The proof of this theorem basically follows the idea presented in the proof of Theorem 1.3, and can be found in [58, Theorem 3.18].

## 2.4 Locally Lipschitz Lyapunov Functions

In many situations, the existence of a *smooth* Lyapunov function is not only sufficient but also necessary (and thus equivalent) to uniform global asymptotic stability for differential inclusions (1.1). In the case  $\mathcal{C} = \mathbb{R}^n$ , see the results in [36], [122], [86], and for a hybrid counterpart the interested reader can be referred to [58, Theorem 7.31]. Nevertheless, it often happens that a *nonsmooth* function  $V$  may be easier to describe and construct.

To illustrate this idea in a particular case, let us consider a *linear* differential inclusion (LDI), that is,

$$\dot{x} \in \text{co}\{A_i x \mid i \in \{1, \dots, M\}\}, \quad (2.8)$$

for some  $A_i \in \mathbb{R}^{n \times n}$ ,  $i = 1, \dots, M$ . As discussed in Section 1.1.2, let us recall that (2.8) can represent the dynamics of a switching system under an arbitrary switching rule, given the family of subsystems  $\dot{x} = A_i x$ ,  $i \in \{1, \dots, M\}$ . In this case, global asymptotic stability of the origin is *equivalent* to the existence of a *smooth* Lyapunov function that is convex and homogeneous of degree 2, as shown in [42], [92]. Despite this powerful theoretical result, given a set  $\{A_1, \dots, A_M\}$  and supposing that the resulting LDI (2.8) is UGAS, the problem of finding such a *smooth* convex Lyapunov function is a challenging task from both the theoretical and computational points of view. Various classes of *non-smooth* functions have been proposed to approximate them, for example maxima of quadratic functions and their convex conjugates [57], [59], or functions with convex polyhedral level sets

[92]. More generally, stability results for differential inclusions relying on nonsmooth Lyapunov functions appear in [30], [121].

When considering locally Lipschitz functions (and thus not necessarily differentiable) as candidate Lyapunov functions for a given dynamical system, it is crucial to define a convenient notion of “*derivative of the Lyapunov function along solutions*”. More precisely, given a locally Lipschitz function  $V : \mathbb{R}^n \rightarrow \mathbb{R}$  and an absolutely continuous function  $\phi : \mathbb{R}_+ \rightarrow \mathbb{R}^n$  the function  $V \circ \phi : \mathbb{R}_+ \rightarrow \mathbb{R}$  is absolutely continuous, and thus its derivative exists almost everywhere. On the other hand, the main strength of the Lyapunov direct method is that the derivative of  $V$  along solutions is computed without needing the explicit expression of solutions, but via a condition (as in (1.11)) that only depends on point  $x$  in the state space. In the smooth case, this is done exploiting the chain rule, for which the differentiability of  $V$  is essential. If  $V$  is not differentiable, in general  $\nabla V$  is not everywhere defined, and so we need an “enlarged/generalized” concept of gradient for locally Lipschitz functions. Many generalizations are possible, depending on the context and/or on the “degree of smoothness” of the considered family of functions (continuous, locally Lipschitz, convex, etc). For locally Lipschitz functions, the main extension of the theory of differential calculus is presented in [34], and summarized in [105].

### 2.4.1 Generalized Gradients/Derivatives for Locally Lipschitz Functions

While in the books [34] and [105], non-smooth analysis theory is developed in a fairly general context, considering functions defined on Banach spaces and not necessarily locally Lipschitz, we specialize here the same concepts in a more restrictive setting, considering functions  $V : \text{dom}(V) \rightarrow \mathbb{R}$ , where  $\text{dom}(V) \subset \mathbb{R}^n$  is a given *open set*. This choice is motivated by the fact that this assumption simplifies the presentation and is exactly the hypothesis that we need when defining a candidate Lyapunov function for finite-dimensional dynamical systems.

#### Definition 2.11: Local Lipschitzness

Consider  $V : \text{dom}(V) \rightarrow \mathbb{R}$ , with  $\text{dom}(V) \subset \mathbb{R}^n$  open. The function  $V$  is said to be *locally Lipschitz* if, for every  $x \in \text{dom}(V)$  there exist a neighborhood  $\mathcal{U}(x) \subset \text{dom}(V)$  of  $x$  and a constant  $L > 0$  such that

$$|V(y) - V(z)| \leq L|y - z|, \quad \text{for any } y, z \in \mathcal{U}(x). \quad (2.9)$$

Given an open set  $\mathcal{O} \subset \mathbb{R}^n$  we will denote by  $Lip_{\text{loc}}(\mathcal{O}, \mathbb{R})$  the set of locally Lipschitz functions  $V : \mathcal{O} \rightarrow \mathbb{R}$ .

Given a not necessarily open set  $\mathcal{C} \subset \text{dom}(V)$  we say that  $V$  is *Lipschitz at  $x$  relative to  $\mathcal{C}$*  if (2.9) holds for any  $y, z \in \mathcal{U}(x) \cap \mathcal{C}$ . The function  $V$  is said to be *locally Lipschitz relative to  $\mathcal{C}$*  if it is Lipschitz relative to  $\mathcal{C}$  at each  $x \in \mathcal{C}$ .

It is important to recall the following inclusions

$$\mathcal{C}^1(\mathcal{O}, \mathbb{R}) \subsetneq Lip_{\text{loc}}(\mathcal{O}, \mathbb{R}) \subsetneq \mathcal{C}^0(\mathcal{O}, \mathbb{R}),$$

and thus, in general, locally Lipschitz functions are not everywhere differentiable, but we have the following remarkable result.

**Theorem 2.12: Rademacher Theorem**

A function  $V \in Lip_{loc}(\mathcal{O}, \mathbb{R})$  is almost everywhere differentiable in  $\mathcal{O}$ , i.e. there exists a set  $\mathcal{N}_V \subset \mathcal{O}$ , with  $\mu_L(\mathcal{N}_V) = 0$  such that  $\nabla V(x)$  exists  $\forall x \in \mathcal{O} \setminus \mathcal{N}_V$ .

We now introduce the concept of generalized gradient for locally Lipschitz functions. The following definition is in fact an equivalent expression [34, Theorem 2.5.1]. The original definition in a more general context is given in [34, Section 2.1].

**Definition 2.13: Clarke Generalized Gradient**

Consider a locally Lipschitz function  $V : \text{dom } V \rightarrow \mathbb{R}$ . The *Clarke generalized gradient* at  $x \in \text{dom } V$  is the set

$$\partial V(x) := \text{co} \left\{ v \in \mathbb{R}^n \mid \exists x_k \rightarrow x, x_k \notin \mathcal{N}_V, \text{ s.t. } v = \lim_{k \rightarrow \infty} \nabla V(x_k) \right\}, \quad (2.10)$$

where  $\mathcal{N}_V := \{x \in \text{dom } V \mid \nabla V(x) \text{ does not exist}\}$  has zero Lebesgue measure, by Rademacher theorem.

Intuitively, the Clarke generalized gradient at  $x$  is obtained by considering all the convergent sequences of “classical gradients” in a neighborhood of  $x$ . In [34, Theorem 2.5.1] the existence of at least one sequence  $x_k$  as considered in (2.10) is proved, implying  $\partial V(x) \neq \emptyset$ , for all  $x \in \text{dom}(V)$ . We can thus see the Clarke generalized gradient as a set-valued map  $\partial V : \text{dom } V \rightrightarrows \mathbb{R}^n$ , with non-empty, compact and convex values. Moreover  $\partial V : \text{dom } V \rightrightarrows \mathbb{R}^n$  is outer semicontinuous, as proved in [37, Proposition 1.5]. Considering a point  $x \in \text{dom}(V)$  and supposing that  $V$  is continuously differentiable in a open neighborhood of  $x$ , it is easy to prove that  $\partial V(x) = \{\nabla V(x)\}$ , formalizing the fact that the Clarke gradient is a proper generalization of the “classical” gradient. Moreover, considering any *convex* function  $V : \mathbb{R}^n \rightarrow \mathbb{R}$ ,  $V$  is clearly locally Lipschitz, and the Clarke gradient coincides with the “usual” subgradient defined in convex analysis theory, i.e.

$$\partial V(x) = \partial_{\text{conv}} V(x) := \{v \in \mathbb{R}^n \mid V(y) \geq V(x) + \langle v, y - x \rangle, \forall y \in \mathcal{U}(x)\},$$

for any  $x \in \mathbb{R}^n$ , where  $\mathcal{U}(x)$  is any open neighborhood of  $x$ .

It is important to note that Clarke gradient somehow inherits many properties from classical differential calculus (linearity, chain rules, etc.) as proved in [34, Chapter 2.3]. More precisely, given  $V_1, V_2 : \mathbb{R}^n \rightarrow \mathbb{R}$  locally Lipschitz,  $g \in \mathcal{C}^1(\mathbb{R}^n, \mathbb{R}^n)$ ,  $\lambda \in \mathbb{R}$  and for any point  $x \in \mathbb{R}^n$ , we have

$$\begin{aligned} \partial[\lambda V_1](x) &= \lambda \partial V_1(x), \\ \partial[V_1 + V_2](x) &\subset \partial V_1(x) + \partial V_2(x) \\ \partial[V_1 \cdot V_2](x) &\subset V_1(x) \partial V_2(x) + V_2(x) \partial V_1(x) \\ \partial[V_1 \circ g](x) &\subset (J_g(x)) \partial V_1(g(x)) \end{aligned} \quad (2.11)$$

where  $J_g(x)$  denotes the Jacobian matrix of  $g$  at  $x$ . It turns out that these properties are exact, in the sense that the inclusions in (2.11) become equalities (see [34,



Chapter 2]), if the considered functions are not only locally Lipschitz but also *regular*, as we define in what follows.

**Definition 2.14: (Clarke)-Regular Functions**

Given an open set  $\mathcal{O} \subset \mathbb{R}^n$ , a locally Lipschitz function  $V : \mathcal{O} \rightarrow \mathbb{R}$  is *regular* at  $x \in \mathcal{O}$  if, for every  $v \in \mathbb{R}^n$ , the directional derivative  $V'(x; v) := \lim_{h \rightarrow 0^+} \frac{V(x+hv) - V(x)}{h}$  exists and the equality

$$V'(x; v) = \max \{ w^\top v \mid w \in \partial V(x) \}, \quad \forall v \in \mathbb{R}^n, \quad (2.12)$$

holds.  $V$  is called *regular* if it is regular at each  $x \in \mathcal{O}$ .

Definition 2.14 is in fact a characterization of regularity for locally Lipschitz functions, which follows from [34, Proposition 2.1.2]. For an alternative definition we refer to [34, Definition 2.3.4].

The right-hand side of (2.12) is also called the *Clarke generalized directional derivative* of  $V$  at  $x$  along  $v$  (denoted by  $V^0(x, v)$  and defined in [34, Section 2.1]). The results and definitions of this section could be equivalently stated by referring to Clarke generalized directional derivatives instead of the Clarke generalized gradient, but we chose to base our exposition on the Clarke generalized gradient concept. Even if, from a historical point of view, the generalized derivative  $V^0(x, v)$  “came first” in generalizing the sub-differential calculus from convex to locally Lipschitz functions, definition (or characterization) (2.10) simply involves limits of classical gradients (which is a familiar and “comfortable” notion) and we think that our choice may make the presentation more pleasant for the reader.

## 2.4.2 Clarke Derivative

In our context, the Clarke generalized gradient is particularly useful, since it allows us to introduce a notion of generalized derivative of locally Lipschitz functions along absolutely continuous functions, as defined in the following statement.

**Lemma 2.15**

Consider a locally Lipschitz function  $V : \text{dom}(V) \rightarrow \mathbb{R}$  and an absolutely continuous function  $\phi : \text{dom}(\phi) \subset \mathbb{R} \rightarrow \text{dom}(V)$ . Then, for almost every  $t \in \text{dom}(\phi)$ , there exists a  $v \in \partial V(\phi(t))$  such that

$$\frac{d}{dt} V \circ \phi(t) = \langle v, \dot{\phi}(t) \rangle.$$

A similar statement and the proof can be found in [30, Proposition 4]. Intuitively speaking, this lemma provides the required generalization of the chain rule and it motivates the following definition, where we introduce a possible generalization of the concept of “derivative of locally Lipschitz function with respect to a differential inclusions”.



**Definition 2.16: Set-Valued Clarke Derivative**

Consider a set  $\mathcal{C} \subset \mathbb{R}^n$ ,  $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$  with  $\text{dom}(F) \supset \mathcal{C}$  and a locally Lipschitz function  $V : \text{dom}(V) \rightarrow \mathbb{R}$  such that  $\text{dom}(V) \supset \mathcal{C}$ . Given  $x \in \mathcal{C}$  we define the *Clarke derivative of  $V$  with respect to  $F$*  as

$$\dot{V}_F(x) := \{ \langle v, f \rangle \mid v \in \partial V(x), f \in F(x) \}. \quad (2.13)$$

Since the scalar product  $\langle \cdot, \cdot \rangle : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$  is bilinear and  $\partial V(x)$  is a compact and convex subset of  $\mathbb{R}^n$  for any  $x \in \text{dom}(V)$ , if also the map  $F$  has compact and convex values,  $\dot{V}_F(x)$  is a non-empty, compact and convex interval of  $\mathbb{R}$  for any  $x \in \mathcal{C}$  and thus it has a well-defined maximum  $\max \dot{V}_F(x)$ . The importance of this set-valued notion of derivative is mainly given by the following result.

**Lemma 2.17**

Consider a set  $\mathcal{C} \subset \mathbb{R}^n$ ,  $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$  with  $\text{dom}(F) \supset \mathcal{C}$  with compact and convex values. Given  $V : \text{dom}(V) \rightarrow \mathbb{R}$  locally Lipschitz and such that  $\text{dom}(V) \supset \mathcal{C}$ , for any solution  $\phi : \text{dom}(\phi) \rightarrow \mathbb{R}^n$  of (1.1), we have

$$\frac{d}{dt} V \circ \phi(t) \in \dot{V}_F(\phi(t)), \quad \text{for almost every } t \in \text{dom}(\phi).$$

*Proof :* The proof trivially follows from Lemma 2.15, and recalling that, by definition, any solution  $\phi$  of (1.1) satisfies  $\dot{\phi}(t) \in F(\phi(t))$  almost everywhere in  $\text{dom}(\phi)$ .  $\diamond$

Lemma 2.17 allows us to give a straightforward generalization of Theorem 1.3 relaxing the smoothness assumption on the candidate Lyapunov function.

**Theorem 2.18: Clarke Lyapunov Theorem**

Consider a set  $\mathcal{C} \subset \mathbb{R}^n$ ,  $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$  with  $\text{dom}(F) \supset \mathcal{C}$  and with convex and compact values, and a closed set  $\mathcal{A} \subset \mathbb{R}^n$ . A function  $V : \text{dom} V \rightarrow \mathbb{R}$  is a (locally Lipschitz) *Clarke Lyapunov function* on  $\mathcal{C}$  for system (1.1) with respect to  $\mathcal{A}$  if  $\mathcal{C} \subset \text{dom} V \subset \mathbb{R}^n$ ,  $V$  is locally Lipschitz, there exist  $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$  such that

$$\alpha_1(|x|_{\mathcal{A}}) \leq V(x) \leq \alpha_2(|x|_{\mathcal{A}}), \quad \forall x \in \mathcal{C},$$

and there exists  $\gamma \in \mathcal{PD}$  such that

$$\max \dot{V}_F(x) \leq -\gamma(|x|_{\mathcal{A}}), \quad \forall x \in \mathcal{C}. \quad (2.14)$$

If there exists a Clarke Lyapunov function for system (1.1) with respect to  $\mathcal{A}$ , then the set  $\mathcal{A}$  is UGAS for (1.1). If  $V$  satisfies (2.14) with  $\gamma \equiv 0$ , then the set  $\mathcal{A}$  is UGS for (1.1).

Thanks to Lemma 2.17, the proof can follow exactly the same steps as the proof of Theorem 1.3.

The idea behind this theorem was suggested in [37, Chapter 4], using a relaxed chain rule as in Lemma 2.17. In the past years several stability results have been

proposed for differential inclusions relying on locally Lipschitz Lyapunov functions, making use of the Clarke generalized gradient as in Theorem 2.18. Among other examples, the Clarke generalized gradient is used in [11] considering piecewise affine Lyapunov functions for state-dependent switching systems, in [63] in the context of interconnected piecewise  $\mathcal{C}^1$  vector fields and in [85] for interconnected hybrid systems.

### 2.4.3 Non-Pathological Functions and Lie Derivative

Before proceeding, let us carefully analyze the conditions of Theorem 2.18 and in particular inequality (2.14): it requires, at each point of the state space, to find an upper bound on the maximum of the set  $\dot{V}_F(x)$ , that is, the maximum over *all the possible scalar products* between vectors  $v \in \partial V(x)$  and directions  $f \in F(x)$ , at any point  $x \in \mathcal{C}$ . Without any further assumption on  $V$  and  $F$  condition (2.14) cannot in general be relaxed, as we will discuss in Chapter 3. On the other hand, in many specific cases, it can be seen that this condition is rather conservative, when the systems and/or the candidate Lyapunov functions satisfy some further assumptions.

For example, in [112], it is shown how stability can be proven by studying a proper subset of  $\dot{V}_F(x)$ , in the context of Filippov regularization of discontinuous differential equations, considering regular (Definition 2.14) candidate Lyapunov functions. This idea has been further generalized some years later in [8] and [7], and in this thesis we will follow the formalism introduced there.

To propose tighter conditions compared to Theorem 2.18, as we said, we will consider functions that are not only locally Lipschitz, but fortunately not much more: we will consider the set of *non-pathological functions*, formally introduced in the following definition.

#### Definition 2.19: Non-Pathological Functions

A locally Lipschitz function  $V : \text{dom}(V) \rightarrow \mathbb{R}$  is *non-pathological* if, given any absolutely continuous function  $\phi : \mathbb{R}_+ \rightarrow \text{dom}(V)$ , we have that for almost every  $t \in \mathbb{R}_+$  there exists  $a_t \in \mathbb{R}$  such that

$$\langle v, \dot{\phi}(t) \rangle = a_t, \quad \text{for all } v \in \partial V(\phi(t)).$$

In other words, for almost every  $t \in \mathbb{R}_+$ ,  $\partial V(\phi(t))$  is a subset of an affine subspace orthogonal to  $\dot{\phi}(t)$ .

This definition was originally given in [124]; non-pathological functions form a large class of functions which clearly includes  $\mathcal{C}^1(\mathbb{R}^n, \mathbb{R})$ , we recall here some important properties of this family of functions, for the proofs we refer to [124] and [8].

**Lemma 2.20: Properties of Non-Pathological Functions**

The set of non-pathological functions is closed under addition, multiplication by scalars and pointwise maximum. More precisely, if  $V_1, V_2 : \mathbb{R}^n \rightarrow \mathbb{R}$  are non-pathological then  $V_1 + V_2$ ,  $\max\{V_1, V_2\}$  and  $\lambda V_1$  ( $\lambda \in \mathbb{R}$ ) are non-pathological. Moreover, if  $V : \mathbb{R}^n \rightarrow \mathbb{R}$  is locally Lipschitz and has *at least* one of the following properties:

- continuously differentiable,
- convex/concave,
- Clarke-regular (Definition 2.14),
- semiconvex/semiconcave,

then  $V$  is non-pathological.

On the other hand “pathological” locally Lipschitz functions do exist:

**Example 2.1 (Pathological function)** *Given any compact interval  $[a, b]$ , (with  $a < b$ ) consider a locally Lipschitz function  $V : \mathbb{R} \rightarrow \mathbb{R}$  such that  $\partial V(x) = [a, b]$ , for any  $x \in \mathbb{R}$ . The existence of such a function is proved in a theoretical setting in [18], but for a constructive definition the interested reader can see [104, pag. 129] on which is also based subsequent Example 3.1. Considering now the absolutely continuous function  $\phi : \mathbb{R}_+ \rightarrow \mathbb{R}$  defined by  $\phi(t) = t$  (the identity) we have that  $\dot{\phi}(t) = 1$  for any  $t \in \mathbb{R}_+$ . Consider now any  $t \in \mathbb{R}_+$  and given any  $v \in \partial V(\phi(t)) = [a, b]$  we have  $\langle v, \dot{\phi}(t) \rangle = \langle v, 1 \rangle = v$  and thus  $V$  does not satisfy Definition 2.19.*

*Interestingly, from a theoretical point of view, this property of having “fat” Clarke gradients (and, in particular, of being “pathological”) is the generic case in the set of locally Lipschitz functions, as proved in [19]. On the other hand, Lemma 2.20 ensures that at least the classes of functions commonly used in control theory are, fortunately, non-pathological.  $\triangle$*

The usefulness of non-pathological functions is mainly given by the following result.

**Lemma 2.21**

If  $V : \mathbb{R}^n \rightarrow \mathbb{R}$  is non-pathological and  $\phi : \mathbb{R}_+ \rightarrow \mathbb{R}^n$  is an absolutely continuous function, then the set

$$\{\langle v, \dot{\phi}(t) \rangle \mid v \in \partial V(\phi(t))\},$$

is equal to the singleton  $\{\frac{d}{dt}V(\phi(t))\}$  for almost every  $t \in \mathbb{R}_+$ .

The proof of this lemma can be found in [124], and motivates the following definition.

**Definition 2.22: Set-Valued Lie Derivative**

Consider a  $\mathcal{C} \subset \mathbb{R}^n$ ,  $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$  with  $\text{dom}(F) \supset \mathcal{C}$  and locally Lipschitz function  $V : \text{dom}(V) \rightarrow \mathbb{R}$  such that  $\text{dom}(V) \supset \mathcal{C}$ . Given  $x \in \mathcal{C}$  we define the *Lie derivative of  $V$  with respect to  $F$*  as

$$\dot{\bar{V}}_F(x) := \{a \in \mathbb{R} \mid \exists f \in F(x) : \langle v, f \rangle = a, \forall v \in \partial V(x)\} \quad (2.15)$$

If, in addition, the map  $F$  has compact and convex values,  $\dot{\bar{V}}_F(x)$  is a compact interval of  $\mathbb{R}$ , for any  $x \in \mathcal{C}$ , but possibly empty, as we will clarify in Chapter 4. Moreover, it is clear that we have

$$\dot{\bar{V}}_F(x) \subset \dot{V}_F(x). \quad (2.16)$$

In fact, given  $a \in \dot{\bar{V}}_F(x)$ , there exists  $f \in F(x)$  such that  $a = \langle p, f \rangle$ , for all  $p \in \partial V(x)$  and thus in particular  $a \in \dot{V}_F(x)$ . Intuitively, it means that when defining  $\dot{\bar{V}}_F(x)$  we do not consider *every* possible scalar product between vectors of  $\partial V(x)$  and  $F(x)$  (as in the definition of  $\dot{V}_F(x)$ ). Rather, we only consider directions  $f \in F(x)$  that are “meaningful” in the sense of possible flowing directions of solutions.

**Lemma 2.23**

Consider a set  $\mathcal{C} \subset \mathbb{R}^n$ ,  $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$  with  $\text{dom}(F) \supset \mathcal{C}$  with compact and convex values. Given a non-pathological function  $V : \text{dom}(V) \rightarrow \mathbb{R}$  such that  $\text{dom}(V) \supset \mathcal{C}$ , for any solution  $\phi : \text{dom}(\phi) \rightarrow \mathbb{R}^n$  of (1.1), we have that

$$\frac{d}{dt} V(\phi(t)) \in \dot{\bar{V}}_F(\phi(t)) \quad (2.17)$$

for almost every  $t \in \text{dom}(\phi)$ .

*Proof :* By Lemma 2.21 and by definition of solutions we have that

$$\begin{aligned} \left\{ \frac{d}{dt} V(\phi(t)) \right\} &= \{ \langle p, \dot{\phi}(t) \rangle \mid p \in \partial V(\phi(t)) \} \\ &\subset \{ a \in \mathbb{R} \mid \exists f \in F(\phi(t)) : \langle p, f \rangle = a, \forall p \in \partial V(\phi(t)) \} = \dot{\bar{V}}_F(\phi(t)), \end{aligned}$$

for almost every  $t \in \text{dom}(\phi)$ . ◇

It is clear that this lemma is the “non-pathological” counterpart of Lemma 2.17, and thus it allows stating another “relaxed” Lyapunov Theorem.

**Theorem 2.24: Lie Lyapunov Theorem**

Consider a set  $\mathcal{C} \subset \mathbb{R}^n$ ,  $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$  with  $\text{dom}(F) \supset \mathcal{C}$  and  $F$  taking convex and compact values, and a closed set  $\mathcal{A} \subset \mathbb{R}^n$ . We say that a function  $V : \text{dom } V \rightarrow \mathbb{R}$  is a (locally Lipschitz) *Lie Lyapunov function* for system (1.1) with respect to  $\mathcal{A}$  if  $\mathcal{C} \subset \text{dom } V \subset \mathbb{R}^n$ ,  $V$  is non-pathological, there exist  $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$  such that

$$\alpha_1(|x|_{\mathcal{A}}) \leq V(x) \leq \alpha_2(|x|_{\mathcal{A}}), \quad \forall x \in \mathcal{C},$$

and there exists  $\gamma \in \mathcal{PD}$  such that

$$\max \dot{\bar{V}}_F(x) \leq -\gamma(|x|_{\mathcal{A}}), \quad \forall x \in \mathcal{C}. \quad (2.18)$$

If there exists a Lie Lyapunov function for system (1.1) with respect to  $\mathcal{A}$ , then the set  $\mathcal{A}$  is UGAS for (1.1). If  $V$  satisfies (2.18) with  $\gamma \equiv 0$ , then the set  $\mathcal{A}$  is UGS for (1.1).

The proof again follows the same ideas of proof of Theorem 1.3, thanks to Lemma 2.23. When considering condition (2.18), we are assuming by definition that  $\max \emptyset = -\infty$ ; in fact we recall that  $\dot{\bar{V}}_F(x)$  can be possibly the empty interval, for some  $x \in \mathcal{C}$ . In other words, we are avoiding to check condition (2.18) at points  $x \in \mathcal{C}$  such that  $\dot{\bar{V}}_F(x) = \emptyset$ . We will clarify why this is justified with some specific examples in Chapter 4.

Over the past years, stability results involving conditions relying on the Lie derivative concept have attracted attention in the control community. Among other examples, the study of the Lie derivative of regular functions has been recently used in [78] to identify and remove infeasible directions of a differential inclusion of the form (1.1), while in [79] the authors proposed a Lie derivative-based invariance principle for state-dependent switched systems, based on the ideas already introduced in [109].

Concluding this chapter, we propose a numerical example that illustrates in practice the computation and the behavior of the Clarke generalized gradient, Clarke and Lie derivatives.

**Example 2.2** Consider the locally Lipschitz function  $V : \mathbb{R} \rightarrow \mathbb{R}$  defined as  $V(x) = |x|$ . Since the set where  $V$  is not differentiable is  $\mathcal{N}_V = \{0\}$ , and  $\nabla V(x) = \text{sgn}(x)$ , for any  $x \in \mathbb{R} \setminus \mathcal{N}_V$ , from definition (2.10), the Clarke generalized gradient at 0 is

$$\partial V(0) = [-1, 1].$$

Now, suppose that a set-valued map  $F : \mathbb{R} \rightrightarrows \mathbb{R}$  is given such that  $F(0) := [f_1, f_2] \subset \mathbb{R}$ . Using (2.13), we compute

$$\begin{aligned} \dot{V}_F(0) &= \{pf \mid p \in [-1, 1], f \in [f_1, f_2]\} \\ &= [-\max\{|f_1|, |f_2|\}, \max\{|f_1|, |f_2|\}]. \end{aligned}$$

On the other hand, using (2.15) and noting that  $p_1 f = p_2 f$  for each  $p_1, p_2 \in [-1, 1]$

if and only if  $f = 0$ , we get

$$\dot{\bar{V}}_F(0) = \begin{cases} \{0\} & \text{if } 0 \in [f_1, f_2], \\ \emptyset & \text{if } 0 \notin [f_1, f_2]. \end{cases}$$

It is easily verified that  $\dot{\bar{V}}_F(0)$  is a subset of  $\dot{V}_F(0)$  and it is possibly empty.  $\triangle$



# 3

## Piecewise Constructed Lyapunov Functions and Densely Checked Conditions

In this chapter, given a set  $\mathcal{C} \subset \mathbb{R}^n$  and a set-valued map  $F : \text{dom}(F) \rightrightarrows \mathbb{R}^n$ , with  $\mathcal{C} \subset \text{dom}(F)$  we study the differential inclusion

$$\dot{x} \in F(x), \quad x \in \mathcal{C}. \quad (3.1)$$

In particular, we propose a class of locally Lipschitz functions with piecewise structure for use as candidate Lyapunov functions. Subject to the hypothesis of inner semicontinuity of the set valued map  $F$ , we show that Lyapunov inequalities can be checked only on a dense set and thus we avoid checking them at points of non differentiability of the Lyapunov function. Connections to other classes of locally Lipschitz or piecewise regular functions are also discussed and applications to hybrid dynamical systems are included. We provide several examples as an illustration of the usefulness and the limitations of our approach. This chapter is mainly based on the publications [46] and [47].

### 3.1 Overview

Lyapunov tools for stability analysis of equilibria, or compact, or just closed sets for differential inclusions have been the subject of extensive research. As showed in Chapter 1, even if in many situations the existence of a smooth Lyapunov function is sufficient and necessary for global asymptotic stability of (3.1), it often happens that a *nonsmooth* function  $V$  may be easier to describe and construct. Stability results for differential inclusions relying on nonsmooth Lyapunov functions appear in [30], [121], [37]. In most of these works, the authors use the Clarke generalized gradient to formulate Lyapunov conditions at points where the Lyapunov function  $V$  is not differentiable. Among other examples, the Clarke generalized gradient is used in [11] for piecewise affine Lyapunov functions for state-dependent switching systems and in [85] for interconnected hybrid systems. This strategy, effective for classical continuous-time nonlinear systems, was suggested in [37, Chapter 4] and then well summarized in [121, p.99]. The main drawback of the Clarke generalized gradient condition is that it considers *all* the scalar products between generalized gradients



and *all* values of the differential inclusion. A different approach is presented in [62], where the nonsmooth function is studied by exploiting regularization-via-convolution techniques. One possible way to overcome the limitations of generalized gradients is to propose stability conditions that only need to be checked away from the points where the candidate Lyapunov function is not differentiable.

With this motivation in mind, we introduce in this chapter a class of locally Lipschitz functions for which the Lyapunov inequalities need to be checked only on a dense subset of  $\mathcal{C}$ , under further hypotheses on the map  $F : \mathcal{C} \rightrightarrows \mathbb{R}^n$ , but without further assumptions on  $\mathcal{C}$ . The class includes functions that can be built using the pointwise maximum and the pointwise minimum of continuously differentiable functions (see [49] and references therein) but is more general. When  $\mathcal{C}$  is a closed set (possibly with an empty interior) we show that our conditions are less restrictive than the Clarke gradient-based conditions of Theorem 2.18. We also relate the proposed class to *piecewise continuously differentiable functions*, introduced in [31] and [110] for optimization purposes. As main application, we use the proposed class of Lyapunov functions for sufficient conditions for asymptotic stability of hybrid dynamical systems.

Collecting our relaxed Lyapunov conditions on the *flow* set  $\mathcal{C}$  with proper Lyapunov *jump* conditions on the set  $\mathcal{D}$ , we propose a stability result that generalizes the standard sufficient Lyapunov conditions for hybrid systems in [58, Theorem 3.18]. This result allows revisiting a classical example from the reset control literature: the Clegg integrator in feedback with an integrating plant. This example has been shown to overcome intrinsic limitations of linear feedback systems in [14]. For this example, both [131] and [96] provided numerical and analytic nonconvex (and nonsmooth) Lyapunov functions. We give here a pair of new (arguably simpler) nonconvex functions, and also a new convex Lyapunov function.

### 3.1.1 Chapter Organization

The chapter is structured as follows. In Section 3.2, we provide some example to illustrate the possible extensions and the limitations of the approach of Theorem 2.18. In Section 3.3, we present our main stability statements, while in Section 3.4 we deeply investigate the relations between our results and the existing literature on locally Lipschitz Lyapunov functions. In Section 3.5, we introduce the concept of *global* piecewise functions which simplifies our analysis. Finally, in Section 3.6 we apply all our previous results in the context of hybrid dynamical systems, presenting, as an example, the Clegg integrator.

## 3.2 Everywhere and Almost Everywhere Conditions

We recall how the conditions of Theorem 2.18 require, in general, to find a positive definite function  $V : \text{dom}(V) \rightarrow \mathbb{R}$  with  $\mathcal{C} \subset \text{dom}(V)$ , to compute the Clarke generalized gradient of  $V$  at *each point* in  $\mathcal{C}$  and finally check the condition

$$\langle v, f \rangle \leq -\gamma(|x|), \quad \forall x \in \mathcal{C}, \forall v \in \partial V(x), \forall f \in F(x), \quad (3.2)$$

with  $\gamma \equiv 0$  if one is interested in (UGS) or  $\gamma \in \mathcal{PD}$  in the (UGAS) case.

We see here how condition (3.2) can be straightforwardly relaxed in some cases, as already noted in [121] for continuous vector fields and differential equations.

### Proposition 3.1: Almost Everywhere Conditions

Consider an open set  $\mathcal{O} \subset \mathbb{R}^n$ , a locally Lipschitz function  $V : \mathcal{O} \rightarrow \mathbb{R}$ , an inner semicontinuous and locally bounded set valued map  $F : \mathcal{O} \rightrightarrows \mathbb{R}^n$ , a closed set  $\mathcal{A} \subset \mathbb{R}^n$  and a continuous function  $\gamma : \mathbb{R} \rightarrow \mathbb{R}$ . If

$$\langle \nabla V(y), f \rangle \leq -\gamma(|y|_{\mathcal{A}}), \quad \forall f \in F(y), \forall y \in \mathcal{O} \setminus \mathcal{N}_V, \quad (3.3)$$

it holds that

$$\langle v, f \rangle \leq -\gamma(|x|_{\mathcal{A}}) \quad \forall x \in \mathcal{O}, \forall v \in \partial V(x), \forall f \in F(x).$$

We recall that  $\mathcal{N}_V \subset \mathbb{R}^n$  was defined as the set of point at which  $V$  is not differentiable, that by Rademacher Theorem has measure zero, i.e.  $\mu_L(\mathcal{N}_V) = 0$ . For that reason (3.3) can be regarded as an “almost everywhere” Lyapunov condition.

In Proposition 3.1 we don't specify any particular property of the continuous function  $\gamma : \mathbb{R}_+ \rightarrow \mathbb{R}$  to cover both  $\gamma \equiv 0$  (certifying UGS) and  $\gamma \in \mathcal{PD}$  (certifying UGAS) as in Theorem 2.18, as well as a variety of more general decrease/increase conditions. The following proof is simply an adaptation, in the context of inner semicontinuous set-valued maps, of the reasoning presented in [122] for continuous functions  $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ .

*Proof :* Consider  $x \in \mathcal{O}$ , take any sequence  $x_k \rightarrow x$ ,  $x_k \in \mathcal{O} \setminus \mathcal{N}_V$ , such that  $\lim_{k \rightarrow \infty} \nabla V(x_k)$  exists. Denote  $w := \lim_{k \rightarrow \infty} \nabla V(x_k)$  and pick any  $f \in F(x)$ . By inner semicontinuity and local boundedness of  $F$ , there exists a sequence  $f_k \in F(x_k)$  such that  $f_k \rightarrow f$ . By equation (3.3), and by continuity of  $\gamma$  and of the scalar product, we have

$$\begin{aligned} \langle \nabla V(x_k), f_k \rangle &\leq -\gamma(|x_k|_{\mathcal{A}}), \\ \downarrow \quad \downarrow & \\ \langle w, f \rangle &\leq -\gamma(|x|_{\mathcal{A}}). \end{aligned}$$

Recalling Definition 2.13, any  $v \in \partial V(x)$  is obtained as convex combination of vectors  $v = \lim_{k \rightarrow \infty} \nabla V(x_k)$ . Thus, by arbitrariness of  $x_k \rightarrow x$  and  $f \in F(x)$ , we can conclude.  $\diamond$

Summarizing, when  $F$  is inner semicontinuous and locally bounded, it suffices to check the Lyapunov inequality *almost everywhere* in the open set  $\mathcal{O}$ , that is at points where  $\nabla V$  is defined and then the Clarke decrease condition (2.14) holds everywhere in  $\mathcal{O}$ . Combining Theorem 2.18 and Proposition 3.1, it is possible to guarantee UGAS (or UGS) using locally Lipschitz functions  $V$  by only certifying the decrease at the points where  $V$  is differentiable. In the following section we explore the limits and the drawbacks of this “relaxed” approach.

### 3.2.1 Counterexamples: dense sets and non-inner semicontinuous maps

Given an open set  $\mathcal{O} \subset \mathbb{R}^n$ , note that a full measure subset of  $\mathcal{O}$ , that is a set  $\mathcal{S} \subset \mathcal{O}$  such that  $\mu_L(\mathcal{O} \setminus \mathcal{S}) = 0$ , is always a *dense* subset of  $\mathcal{O}$ , in the sense that  $\overline{\mathcal{S}}$  contains  $\mathcal{O}$ . The converse is not true in general: for example  $\mathcal{S} = \mathbb{Q} \subset \mathbb{R}$  is such that  $\overline{\mathbb{Q}} = \mathbb{R}$  but  $\mu_L(\mathbb{Q}) = 0$ . One can ask: is it sufficient, for a general locally Lipschitz function, to check Lyapunov decrease inequalities only on a *dense subset* of  $\mathcal{C}$ ? The answer is “no”, even if the dynamic is continuous, as illustrated by the following example.

**Example 3.1 (Checking on a dense set)** *The main idea of this example is taken from [104]: Consider  $\lambda \in (\frac{1}{2}, 1)$ , and a measurable set  $M \subset \mathbb{R}_{\geq 0}$  such that, for every non-empty interval  $I \subset \mathbb{R}_{\geq 0}$ , it holds that*

$$0 < \mu_L(M \cap I) < \mu_L(I), \quad (3.4)$$

(such sets are called interval-splitting sets) and, additionally,

$$\mu_L(M \cap [0, t]) \geq \lambda t, \quad \forall t > 0. \quad (3.5)$$

The construction of such a set is provided in [41, Lemma 2], see also [107] for the original construction of interval-splitting sets. In [41], interval-splitting sets with the additional property (3.5) are used to construct locally Lipschitz functions for which the steepest descent / subdifferential flow generated by the Clarke subdifferential has quite pathological behavior: it fails to decrease and in fact increases the functions. Given a set  $N \subset \mathbb{R}$ , define the characteristic function of  $N$  as

$$\chi_N(s) := \begin{cases} 1, & \text{if } s \in N, \\ 0, & \text{if } s \notin N. \end{cases}$$

Consider the function  $W : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}$  defined as

$$W(s) := \int_0^s \chi_M(r) - \chi_{M^c}(r) dr,$$

where  $M^c := \mathbb{R}_{\geq 0} \setminus M$ . Using the same reasoning as in [104], it can be proven that  $W$  is locally Lipschitz, and (3.4) implies that  $\partial W(x) = [-1, 1]$  for all  $x \in \mathbb{R}_{\geq 0}$  and the sets

$$\begin{aligned} \mathcal{X}_1 &:= \{x \in \mathbb{R}_{\geq 0} \mid \nabla W(x) \text{ exists} \wedge \nabla W(x) = 1\}, \\ \mathcal{X}_{-1} &:= \{x \in \mathbb{R}_{\geq 0} \mid \nabla W(x) \text{ exists} \wedge \nabla W(x) = -1\}, \end{aligned}$$

are both dense subsets of  $\mathbb{R}_{\geq 0}$ . In fact  $\nabla W(\cdot) = \chi_M(\cdot) - \chi_{M^c}(\cdot)$  almost everywhere, and by (3.4) and Rademacher Theorem, for any  $x \in \mathbb{R}_{\geq 0}$  and any  $\varepsilon > 0$ , there exist  $x_1^\varepsilon \in \mathcal{X}_1$  and  $x_{-1}^\varepsilon \in \mathcal{X}_{-1}$  such that  $|x - x_j^\varepsilon| < \varepsilon$ , for any  $j \in \{-1, 1\}$ . We prove next that (3.5) ensures that the function  $W$  satisfies the bounds

$$(2\lambda - 1)s \leq W(s) \leq s, \quad \forall s \in \mathbb{R}_{\geq 0}. \quad (3.6)$$

The upper bound is straightforward as

$$W(s) = \mu_L(M \cap [0, s]) - \mu_L(M^c \cap [0, s]) \leq \mu_L([0, s]) = s.$$

The lower bound is obtained as follows from (3.5):

$$\begin{aligned} W(s) &= \mu_L(M \cap [0, s]) - \mu_L(M^c \cap [0, s]) \\ &\geq \lambda s - \mu_L([0, s] \setminus (M \cap [0, s])) \geq \lambda s - (1 - \lambda)s = (2\lambda - 1)s. \end{aligned}$$

Consider now the differential equation

$$\dot{x} = f(x) = x,$$

and the candidate Lyapunov function  $V : \mathbb{R} \rightarrow \mathbb{R}$  defined by  $V(x) := W(|x|)$ . The function  $V : \mathbb{R} \rightarrow \mathbb{R}$  is locally Lipschitz and, by (3.6), it is also positive definite and radially unbounded. Moreover, it holds that

$$\nabla V(x) = \begin{cases} -1, & \forall x > 0, x \in \mathcal{X}_{-1}, \\ +1, & \forall x < 0, x \in -\mathcal{X}_{-1}. \end{cases}$$

Consequently

$$\langle \nabla V(x), f(x) \rangle = -|x|, \quad \forall x \in \mathcal{S},$$

where  $\mathcal{S} := \mathcal{X}_{-1} \cup -\mathcal{X}_{-1} \setminus \{0\}$  is a dense subset of  $\mathbb{R}$  by construction. In other words,  $V$  is a positive definite and radially unbounded locally Lipschitz function for which the Lyapunov decrease inequality

$$\langle \nabla V(x), f(x) \rangle \leq -|x|,$$

is satisfied on a dense subset  $\mathcal{S}$  of  $\mathbb{R}$ . On the other hand, the equilibrium point 0 is clearly unstable.  $\triangle$

In the next section, we present a subclass of locally Lipschitz functions for which it is enough to check the Lyapunov decrease inequality on a dense subset of  $\mathcal{C}$ .

Another question of interest in generalizing Proposition 3.1 is whether the inner semicontinuity hypothesis is in general necessary, or it can be replaced by outer semicontinuity, without losing this ‘‘almost everywhere’’ stability certificate. The following example actually illustrates how inner semicontinuity is indeed necessary.

**Example 3.2 (Violating Inner Semicontinuity)** Consider the differential inclusion (3.1), with  $\mathcal{C} = \mathbb{R}^2$ , and the considered set-valued map is defined as the Filippov regularization  $F_f^{Fil} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  of the state dependent switched system

$$\dot{x} = \begin{cases} A_1 x, & \text{if } x^\top Q x \geq 0, \\ A_2 x, & \text{if } x^\top Q x < 0 \end{cases} \quad (3.7)$$

where

$$A_1 := \begin{bmatrix} -0.3 & -1 \\ 5 & -0.3 \end{bmatrix}, \quad A_2 := \begin{bmatrix} -0.3 & 5 \\ -1 & -0.3 \end{bmatrix}, \quad Q := \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Defining  $X_i := \{x \in \mathbb{R}^2 \mid (-1)^i x^\top Q x \leq 0\}$ , it is easy to see that  $\mathcal{X} = \{X_i, \mathbb{R}^2\}_{i \in \{1,2\}}$  is a proper state space partition, as introduced in Definition 2.4. For the details of

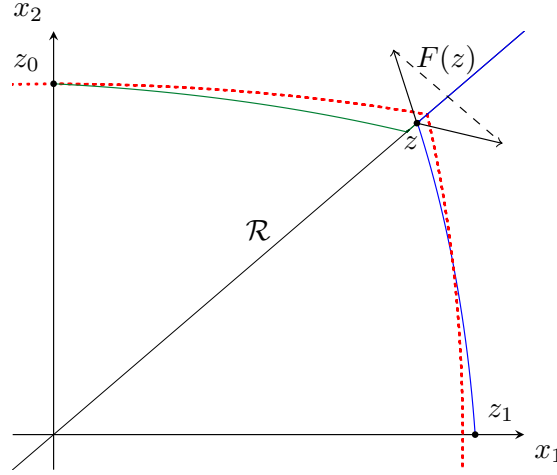


Figure 3.1: Example 3.2: In red dashed line a level set of  $V$  in (3.10), in green and blue the trajectories of two solutions. The solution starting at  $z_1$  enters the plotted sublevel set, reaches the point  $z \in \mathcal{R}$  and then starts “sliding” towards infinity.

*Filippov regularization of state dependent switched system we refer to Section 2.2. By definition of Filippov regularization, we have*

$$F_f^{Fil} = \begin{cases} \{A_1x\}, & \text{if } x^\top Qx > 0, \text{ equivalently } x \in \text{int}(X_1), \\ \{A_2x\}, & \text{if } x^\top Qx < 0, \text{ equivalently } x \in \text{int}(X_2), \\ \text{co}\{A_1x, A_2x\} & \text{if } x^\top Qx = 0. \end{cases} \quad (3.8)$$

*As showed in Chapter 1,  $F$  is outer semicontinuous on  $\mathbb{R}^2$ , locally bounded, has compact and convex values, but it is not inner semicontinuous at points  $x \in \mathbb{R}^2 \setminus \{0\}$  satisfying  $x^\top Qx = 0$ . Indeed, considering any  $y \neq 0$  such that  $y^\top Qy = 0$ , we have that  $\tilde{f} := \frac{1}{2}A_1y + \frac{1}{2}A_2y \in F_f^{Fil}(y)$ , but considering a sequence  $y_k \rightarrow y$  such that  $y_k^\top Qy_k > 0$  (i.e.  $y_k \in \text{int}(X_1)$ ), for any  $k \in \mathbb{N}$ , we have that  $F_f^{Fil}(y_k) = \{A_1y_k\}$  and thus we can not “obtain”  $\tilde{f}$  as limits of vectors  $f_k \in F_f^{Fil}(y_k)$ , violating the definition inner semicontinuous (recall Definition 2.1).*

*We want to study the stability of  $\mathcal{A} = \{0\}$  for the differential inclusion*

$$\dot{x} \in F_f^{Fil}(x), \quad x \in \mathbb{R}^2, \quad (3.9)$$

*trying to construct a locally Lipschitz Lyapunov function.*

*Consider the function*

$$V(x) := \max \{x^\top P_1x, x^\top P_2x\}, \quad (3.10)$$

*where*

$$P_1 := \begin{bmatrix} 5 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad P_2 := \begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix}.$$

*It is easy to check that  $P_iA_i + A_i^\top P_i < 0$ , for all  $i \in \{1, 2\}$  and if  $x^\top Qx > 0$  then  $x^\top P_1x > x^\top P_2x$ . As a consequence we have*

$$\langle \nabla V(x), f \rangle < 0; \quad \forall x \in \mathbb{R}^2 \setminus (\text{bd}(X_1) \cup \text{bd}(X_2)), \quad \forall f \in F(x),$$

which implies (3.3). On the other hand, condition (3.2) does not hold on  $\mathcal{R} := \{x \in \mathbb{R}^2 \mid x_1 = x_2\}$ , and one can see that every solution of (3.9) starting at some  $x_0 \in \mathcal{R}$ ,  $x_0 \neq 0$  goes to infinity sliding along  $\mathcal{R}$ . In particular, the origin is unstable. See Figure 3.1 for a graphical representation.  $\triangle$

In Chapters 4 and 5 we will consider not-inner semicontinuous differential inclusion arising from state-dependent switched systems as (3.7), but the analysis will be carried on with different tools with respect to the ones proposed in the following section.

### 3.3 Main Stability result

Given a set  $\mathcal{C} \subset \mathbb{R}^n$ , we present here a class of locally Lipschitz Lyapunov functions associated to system (3.1), for which it suffices to check the Lyapunov inequality on a dense subset of  $\mathcal{C}$ . We want to underline how, at this stage, we are not imposing any topological assumption on the set  $\mathcal{C} \subset \mathbb{R}^n$ , which can be open, closed (or neither open nor closed), and bounded or unbounded. Most of the efforts of this chapters exactly lie in proposing “flexible” conditions, permitting the analysis of differential inclusions constrained to non-necessarily “well-behaved” sets.

#### Definition 3.2: The Class $\mathcal{L}(\mathcal{C})$

Consider  $\mathcal{C} \subset \mathbb{R}^n$ . A function  $V : \text{dom } V \rightarrow \mathbb{R}$  (with  $\text{dom } V$  open), is a *locally Lipschitz and locally finitely generated function on  $\mathcal{C}$*  (and we write  $V \in \mathcal{L}(\mathcal{C})$ ) if  $\mathcal{C} \subset \text{dom } V$  and

- (a) For each  $x \in \mathcal{C}$ , there exist an open neighborhood  $\mathcal{U}(x) \subset \mathbb{R}^n$  and an  $L > 0$  such that  $|V(x') - V(x'')| \leq L|x' - x''|$ ,  $\forall x', x'' \in \mathcal{U}(x) \cap \mathcal{C}$  (i.e.,  $V$  is locally Lipschitz relative to  $\mathcal{C}$ , recall Definition 2.11).
- (b) There exists a set  $\mathcal{S}(x) \subset \mathcal{C} \cap \mathcal{U}(x)$ , such that  $\nabla V(y)$  exists for all  $y \in \mathcal{S}(x)$ , satisfying

$$\overline{\mathcal{S}(x)} \supset \mathcal{C} \cap \mathcal{U}(x) \quad (\text{i.e. } \mathcal{S}(x) \text{ is dense in } \mathcal{C} \cap \mathcal{U}(x)). \quad (3.11)$$

- (c) There exists a finite index set  $\mathcal{I}(x)$ , and for each  $i \in \mathcal{I}(x)$ , there are open sets  $\mathcal{U}_i \subset \mathbb{R}^n$  and  $\mathcal{C}^1$  functions  $V_i : \mathcal{U}_i \rightarrow \mathbb{R}$  such that each  $y \in \mathcal{S}(x)$  satisfies, for some  $i \in \mathcal{I}(x)$ ,

$$y \in \mathcal{U}_i, \quad V(y) = V_i(y), \quad \text{and} \quad \nabla V(y) = \nabla V_i(y). \quad (3.12)$$

This definition intuitively presents a subclass of locally Lipschitz functions having a particular local structure relying on a finite family of continuously differentiable functions. Given a set valued map  $F : \mathcal{C} \rightrightarrows \mathbb{R}^n$ , this structure is crucial in defining relaxed conditions ensuring decreasing properties of a  $V \in \mathcal{L}(\mathcal{C})$  along solution of (3.1), as presented in the following statement.

**Definition 3.3: The Class  $\mathcal{L}_F(\mathcal{A}, \mathcal{C})$** 

Considering a map  $F : \mathcal{C} \rightrightarrows \mathbb{R}^n$  and a closed set  $\mathcal{A} \subset \mathbb{R}^n$ , we say that  $V$  is a *locally Lipschitz and locally finitely generated strong Lyapunov function for  $\mathcal{A}$  on  $\mathcal{C}$*  ( $V \in \mathcal{L}_F(\mathcal{A}, \mathcal{C})$ ) if  $V \in \mathcal{L}(\mathcal{C})$  and

(L1) There exist  $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$  such that

$$\alpha_1(|x|_{\mathcal{A}}) \leq V(x) \leq \alpha_2(|x|_{\mathcal{A}}), \quad \forall x \in \mathcal{C}. \quad (3.13)$$

(L2) There exists  $\gamma \in \mathcal{PD}$  such that, for each  $x \in \mathcal{C}$ , each  $y \in \mathcal{S}(x)$  satisfies (3.12) for some  $i \in \mathcal{I}(x)$  and moreover,

$$\langle \nabla V_i(y), f \rangle \leq -\gamma(|y|_{\mathcal{A}}), \quad \forall f \in F(y). \quad (3.14)$$

Finally, we say that  $V$  is a locally Lipschitz and locally finitely generated *weak* Lyapunov function for  $\mathcal{A}$  on  $\mathcal{C}$  ( $V \in \mathcal{L}_F^\circ(\mathcal{A}, \mathcal{C})$ ) if all the previous conditions hold with  $\gamma \equiv 0$  in (L2).  $\triangle$

We now prove that if  $V$  is in  $\mathcal{L}_F(\mathcal{A}, \mathcal{C})$  ( $V$  is in  $\mathcal{L}_F^\circ(\mathcal{A}, \mathcal{C})$ , resp.) then the value of  $V$  decreases (does not increase, resp.) along the solutions of system (3.1).

**Theorem 3.4: Main Stability Result**

If  $V \in \mathcal{L}_F(\mathcal{A}, \mathcal{C})$  and  $F : \mathcal{C} \rightrightarrows \mathbb{R}^n$  is locally bounded and inner semicontinuous, then, for every  $\phi \in \mathcal{S}_{F, \mathcal{C}}$  and almost every  $t \in \text{dom}(\phi) := [0, T_\phi)$ ,

$$\frac{d}{dt} V(\phi(\cdot))(t) \leq -\gamma(|\phi(t)|_{\mathcal{A}}). \quad (3.15)$$

Thus,  $\mathcal{A}$  is UGAS for (3.1). Similarly, if  $V \in \mathcal{L}_F^\circ(\mathcal{A}, \mathcal{C})$ , then (3.15) holds with  $\gamma \equiv 0$  and thus  $\mathcal{A}$  is UGS for (3.1).

*Proof :* Consider any solution  $\phi \in \mathcal{S}_{F, \mathcal{C}}$ . Since  $\phi$  is absolutely continuous by definition so is  $t \mapsto V(\phi(t))$ . Then,  $\phi$  and  $V \circ \phi$  are differentiable almost everywhere, i.e., for almost every  $t \in \text{dom} \phi = [0, T_\phi)$ ,  $\dot{\phi}(t)$  and  $\frac{d}{dt} V(\phi(\cdot))(t)$  exist, and for almost every such  $t$ ,  $\dot{\phi}(t) \in F(\phi(t))$ . Thus consider any such time  $t \in [0, T_\phi)$ , and denote  $x := \phi(t) \in \mathcal{C}$  and  $f := \dot{\phi}(t) \in F(x)$ . Consider the neighborhood  $\mathcal{U}(x)$  of  $x$ , the associated index set  $\mathcal{I}(x)$ , the sets  $\mathcal{U}_i$  and functions  $\{V_i\}_{i \in \mathcal{I}(x)}$  given by Definition 3.2. Let us consider a sequence of times  $t_k \searrow t$ , and denote by  $x_k := \phi(t_k) \in \mathcal{C}$ ; by continuity of  $\phi$ ,  $x_k \rightarrow x$ . Without loss of generality we can suppose  $x_k \in \mathcal{U}(x) \cap \mathcal{C}$ ,  $\forall k \in \mathbb{N}$  (possibly disregarding some initial points and relabeling). Consider the set  $\mathcal{S}(x)$  given in Definition 3.2; since  $\mathcal{C} \cap \mathcal{U}(x) \subset \mathcal{S}(x)$ , for each  $k \in \mathbb{N}$  there exists a sequence  $x_{k,l} \in \mathcal{S}(x)$  converging to  $x_k$ . Moreover, by property (c) of Definition 3.2, for each  $l \in \mathbb{N}$ , there exists  $i_{k,l} \in \mathcal{I}(x)$  satisfying  $V(x_{k,l}) = V_{i_{k,l}}(x_{k,l})$  and  $\nabla V(x_{k,l}) = \nabla V_{i_{k,l}}(x_{k,l})$ . Since  $\mathcal{I}(x)$  is finite, we may consider, without relabeling, a subsequence of  $x_{k,l}$ , such that  $i_{k,l} = i_k$  for some  $i_k \in \mathcal{I}_x$ , for every  $l \in \mathbb{N}$ . Similarly we may consider, without relabeling, a subsequence of  $x_k$

such that  $i_k = i$ , for some  $i \in \mathcal{I}(x)$  and every  $k \in \mathbb{N}$ . Then  $V(x_{k,l}) = V_i(x_{k,l})$ ,  $\forall l \in \mathbb{N}$ , and, by continuity of  $V$  and  $V_i$ , the following holds

$$\begin{aligned} V(x_k) &= \lim_{l \rightarrow \infty} V(x_{k,l}) = \lim_{l \rightarrow \infty} V_i(x_{k,l}) = V_i(x_k), \\ V(x) &= \lim_{k \rightarrow \infty} V(x_k) = \lim_{k \rightarrow \infty} V_i(x_k) = V_i(x). \end{aligned} \quad (3.16)$$

Hence, recalling that  $V_i \in \mathcal{C}^1(U_i, \mathbb{R})$  and using identities (3.16), we have

$$\begin{aligned} \frac{d}{dt} V(\phi(\cdot))(t) &= \lim_{\tau \searrow t} \frac{V(\phi(\tau)) - V(\phi(t))}{\tau - t} = \lim_{k \rightarrow \infty} \frac{V(x_k) - V(x)}{t_k - t} \\ &= \lim_{k \rightarrow \infty} \frac{V_i(x_k) - V_i(x)}{t_k - t} = \lim_{\tau \searrow t} \frac{V_i(\phi(\tau)) - V_i(\phi(t))}{\tau - t} \\ &= \frac{d}{dt} V_i(\phi(\cdot))(t) = \langle \nabla V_i(x), \dot{\phi}(t) \rangle = \langle \nabla V_i(x), f \rangle. \end{aligned} \quad (3.17)$$

Now, for each  $k \in \mathbb{N}$ , we can choose a large enough  $l = l_k$  so that  $x_{k,l_k} \rightarrow x$  as  $k \rightarrow \infty$ . By inner semicontinuity of  $F$  there exists a sequence  $f_k \in F(x_{k,l_k})$  such that  $f_k \rightarrow f$  as  $k \rightarrow \infty$ . Finally by continuity of  $\nabla V_i$ ,  $\gamma$  and the scalar product we have, as  $k \rightarrow \infty$ ,

$$\begin{aligned} (3.14) \quad \Rightarrow \quad \langle \nabla V_i(x_{k,l_k}), f_k \rangle &\leq -\gamma(|x_{k,l_k}|_{\mathcal{A}}), \\ \downarrow \qquad \qquad \downarrow & \\ \langle \nabla V_i(x), f \rangle &\leq -\gamma(|x|_{\mathcal{A}}), \end{aligned}$$

and by (3.17), we can conclude that (3.15) holds. By a standard comparison argument, (3.15) implies that  $\mathcal{A}$  is (UGAS) for (3.1). The same argument could be used to infer that  $V \in \mathcal{L}_F^\circ(\mathcal{A}, \mathcal{C})$  implies that (3.15) holds with  $\gamma \equiv 0$  and thus  $\mathcal{A}$  is (UGS) for (3.1).  $\diamond$

## 3.4 Comparisons with other Classes of Functions

In this section, we prove that the class  $\mathcal{L}(\mathcal{C})$  introduced in Definition 3.2 has strong connections with the class of piecewise  $\mathcal{C}^1$  functions introduced in [110], in the case where  $\mathcal{C} \subset \mathbb{R}^n$  is an *open set*. This also allows us to investigate the relations between the class  $\mathcal{L}_F(\mathcal{A}, \mathcal{C})$  and the concept of Clarke locally Lipschitz Lyapunov functions defined in Theorem 2.18.

### 3.4.1 Piecewise $\mathcal{C}^1$ functions vs $\mathcal{L}(\mathcal{C})$ functions

In [110], the following class of functions is introduced.



**Definition 3.5: Piecewise  $\mathcal{C}^1$  function; [110]**

Given an *open* set  $\mathcal{O} \subset \mathbb{R}^n$ , a *continuous* function  $V : \mathcal{O} \rightarrow \mathbb{R}$  is called *piecewise  $\mathcal{C}^1$  function on  $\mathcal{O}$*  if for each  $x \in \mathcal{O}$  there exist an open neighborhood  $\mathcal{U}(x) \subset \mathcal{O}$  of  $x$ , an index set  $\mathcal{I}(x) = \{1, \dots, K\}$ , a family  $\mathcal{F} = \{V_1, \dots, V_K\} \subset \mathcal{C}^1(\mathcal{U}(x), \mathbb{R})$  such that

$$V(z) \in \{V_i(z) \mid i \in \mathcal{I}(x)\}, \quad \forall z \in \mathcal{U}(x). \quad (3.18)$$

Roughly speaking, a piecewise  $\mathcal{C}^1$  function at every point  $x \in \mathcal{O}$  is locally defined as a continuous selection (or patching) of a finite number of “pre-defined” continuously differentiable functions.

We now prove that the conditions of Definition 3.5 are equivalent to those conditions of Definition 3.2 for  $\mathcal{L}(\mathcal{C})$  when  $\mathcal{C}$  is an *open set*. We may then think of  $\mathcal{L}(\mathcal{C})$  functions as generalizations of piecewise  $\mathcal{C}^1$  functions over not necessarily open sets. This extension is relevant in Section 3.6, when  $\mathcal{C}$  is a closed set. More precisely, we have the following proposition.

**Proposition 3.6**

Consider an *open* set  $\mathcal{O} \subset \mathbb{R}^n$ . A function  $V : \text{dom}(V) \rightarrow \mathbb{R}$  is in  $\mathcal{L}(\mathcal{O})$  if and only if it is piecewise  $\mathcal{C}^1$  on  $\mathcal{O}$ .

*Proof:* To prove the equivalence, first suppose that  $V \in \mathcal{L}(\mathcal{O})$  and we show that it is piecewise  $\mathcal{C}^1$  on  $\mathcal{O}$ . The continuity of  $V$  on  $\mathcal{O}$  is trivial, since  $V$  is locally Lipschitz relative to  $\mathcal{O}$ , by item (a) of Definition 3.2. Take any  $x \in \mathcal{O}$  and consider the neighborhood  $\mathcal{U}(x)$ , the set  $\mathcal{S}(x)$ , the index set  $\mathcal{I}(x)$ , the sets  $\mathcal{U}_i$ , and the functions  $V_i : \mathcal{U}_i \rightarrow \mathbb{R}$  given in Definition 3.2 of  $\mathcal{L}(\mathcal{O})$ . Since  $\mathcal{O}$  is open, without loss of generality, we suppose that  $\mathcal{U}(x) \subset \mathcal{O}$ . We want to prove that

$$\mathcal{U}(x) \subset \bigcup_{i \in \mathcal{I}(x)} \mathcal{V}_i := \bigcup_{i \in \mathcal{I}(x)} \{z \in \mathcal{U}_i \mid V(z) = V_i(z)\}. \quad (3.19)$$

To prove (3.19), take any  $z \in \mathcal{U}(x)$ , if  $z \in \mathcal{S}(x)$ , then  $V(z) = V_i(z)$  for some  $i \in \mathcal{I}(x)$  and thus  $z \in \bigcup_{i \in \mathcal{I}(x)} \mathcal{V}_i$ . If  $z \notin \mathcal{S}(x)$ , then from conditions (b) and (c) in Definition 3.2 there exist  $z_k \in \mathcal{S}(x)$ ,  $z_k \rightarrow z$  such that  $V(z_k) = V_{i_k}(z_k)$  for some  $i_k \in \mathcal{I}(x)$ . By finiteness of  $\mathcal{I}(x)$ , without loss of generality we can assume  $i_k = i$  for some  $i \in \mathcal{I}(x)$  and for all  $k \in \mathbb{N}$ . Then by continuity of  $V$  and  $V_i$ , we have

$$V(z) = \lim_{k \rightarrow \infty} V(z_k) = \lim_{k \rightarrow \infty} V_i(z_k) = V_i(z),$$

which shows that  $z \in \mathcal{V}_i \subset \bigcup_{i \in \mathcal{I}(x)} \mathcal{V}_i$ . Concluding, define  $\mathcal{I}'(x) := \{i \in \mathcal{I}(x) \mid x \in \mathcal{U}_i\}$  and  $\mathcal{U}'(x) \subset \mathcal{U}(x)$  as  $\mathcal{U}'(x) := \bigcap_{i \in \mathcal{I}'(x)} \mathcal{U}_i \cap \mathcal{U}(x)$ . By (3.19),  $\mathcal{I}'(x)$  is not empty and thus  $\mathcal{U}'(x)$  is an (open) neighborhood of  $x$ . Again by (3.19) we have

$$V(z) \in \{V_i(z) \mid i \in \mathcal{I}'(x)\}, \quad \forall z \in \mathcal{U}'(x),$$

concluding the implication.

For the converse implication, let us suppose  $V : \mathcal{O} \rightarrow \mathbb{R}^n$  is a piecewise  $\mathcal{C}^1$  function on  $\mathcal{O}$ . We recall the following facts:

- If  $V$  is piecewise  $\mathcal{C}^1$  on  $\mathcal{O}$  then it is locally Lipschitz continuous, see [110, Prop. 4.1.2],
- There exists an open and dense subset  $\mathcal{S} \subset \mathcal{O}$  where  $V$  is continuously differentiable. Moreover, consider any  $x \in \mathcal{O}$ , the neighborhood  $\mathcal{U}(x)$  and the associated index set  $\mathcal{I}(x)$  in Definition 3.5. For every  $z \in \mathcal{U}(x) \cap \mathcal{S}$  it follows from [110, Prop. 4.1.5] that there exists an  $i \in \mathcal{I}(x)$  such that  $\nabla V(z) = \nabla V_i(z)$ .

The two facts above imply that a piecewise  $\mathcal{C}^1$  function satisfies properties (a), (b) and (c) of Definition 3.2, concluding the proof.  $\diamond$

We recall here some useful properties of piecewise  $\mathcal{C}^1$  functions. Let us consider a piecewise  $\mathcal{C}^1$  function  $V : \mathcal{O} \rightarrow \mathbb{R}$  on an open set  $\mathcal{O}$ , any  $x \in \mathcal{O}$  and the related neighborhood  $\mathcal{U}(x)$  and the index set  $\mathcal{I}(x)$  given in Definition 3.5. Define the following set of *essentially active indexes* at  $x \in \mathcal{O}$ :

$$\mathcal{I}^e(x) = \left\{ i \in \mathcal{I}(x) \mid x \in \overline{\text{int}(\{y \in \mathcal{U}(x) \mid V(y) = V_i(y)\})} \right\}. \quad (3.20)$$

It follows by continuity that  $V(x) = V_i(x)$  for all  $i \in \mathcal{I}^e(x)$ , and in particular there exists a neighborhood  $\mathcal{U}'(x)$  of  $x$ ,  $\mathcal{U}'(x) \subset \mathcal{U}(x)$ , such that

$$V(z) \in \{V_i(z) \mid i \in \mathcal{I}^e(x)\}, \quad \text{for all } z \in \mathcal{U}'(x),$$

see [110, Propostion 4.1.1] for the proof.

Moreover in [110, Proposition 4.3.1] the following characterization of the Clarke gradient for piecewise  $\mathcal{C}^1$  functions is proven; we recall it here, together with the instructive proof.

### Lemma 3.7

Given  $V : \mathcal{O} \rightarrow \mathbb{R}$  piecewise  $\mathcal{C}^1$  and any  $x \in \mathcal{O}$ , it holds that

$$\partial V(x) = \text{co}\{\nabla V_i(x) \mid i \in \mathcal{I}^e(x)\}. \quad (3.21)$$

*Proof :* Consider  $x \in \mathcal{O}$  and the related neighborhood  $\mathcal{U}(x)$ , define

$$S_1(x) := \{\nabla V_i(x) \mid i \in \mathcal{I}^e(x)\},$$

$$S_2(x) := \left\{ v \in \mathbb{R}^n \mid \begin{array}{l} \exists x_k \rightarrow x, x_k \notin \mathcal{N}_V, \text{ s.t.} \\ v = \lim_{k \rightarrow \infty} \nabla V(x_k) \end{array} \right\}.$$

Recalling Definition 2.13, if we prove that  $S_1(x) = S_2(x)$ ; then (3.21) holds.

$S_1(x) \subset S_2(x)$ : Let us call  $C_i = \{y \in \mathcal{U}(x) \mid V(y) = V_i(y)\}$ , and consider  $v \in S_1(x)$ , that is there exists an  $i \in \mathcal{I}^e(x)$  such that  $v = \nabla V_i(x)$ . By (3.20), there exists a sequence  $x_k \rightarrow x$  such that  $x_k \in \text{int}(C_i)$  for all  $k \in \mathbb{N}$ . By definition of  $C_i$  we have that  $V$  is differentiable at any  $x_k$ , and moreover  $\nabla V(x_k) = \nabla V_i(x_k)$ , for any  $k \in \mathbb{N}$ . Thus, we have

$$\nabla V_i(x) = \lim_{k \rightarrow \infty} \nabla V_i(x_k) = \lim_{k \rightarrow \infty} \nabla V(x_k)$$

proving that  $v = \nabla V_i(x) \in S_2(x)$ .

$S_2(x) \subset S_1(x)$ : Consider  $v \in S_2(x)$ , namely there exists a sequence  $x_k \rightarrow x$ , such that  $V$  is differentiable at any  $x_k$ , and such that  $\lim_{k \rightarrow \infty} \nabla V(x_k) = v$ . We can suppose, without loss of generality that  $x_k \in \mathcal{U}(x)$ , for any  $k \in \mathbb{N}$ . In [110, Proposition 4.1.3] it is proved that for each  $k \in \mathbb{N}$  there exists  $i_k \in \mathcal{I}^e(x)$  such that  $\nabla V(x_k) = \nabla V_{i_k}(x_k)$ . By finiteness of  $\mathcal{I}^e(x)$  we can extract (relabeling  $k$  into  $h$ ) the subsequence  $x_h$  with constant indices  $i_h \equiv i$  for some  $i \in \mathcal{I}^e(x)$ , obtaining

$$\lim_{k \rightarrow \infty} \nabla V(x_k) = \lim_{k \rightarrow \infty} \nabla V_i(x_k) = \nabla V_i(x),$$

and hence  $v \in S_1(x)$ . ◇

### 3.4.2 Clarke Lyapunov Functions vs $\mathcal{L}_F(\mathcal{A}, \mathcal{C})$ functions

We study here the relation between *Clarke locally Lipschitz Lyapunov functions* (in the sense of Theorem 2.18 and locally finitely generated Lyapunov functions  $\mathcal{L}_F(\mathcal{A}, \mathcal{C})$  (in the sense of Definition 3.3). The next lemma shows that a function  $V \in \mathcal{L}_F(\mathcal{A}, \mathcal{C})$  also satisfies the condition (2.14) in the interior of  $\mathcal{C}$ , and thus is a Clarke locally Lipschitz Lyapunov function if  $\mathcal{C}$  is open.

#### Lemma 3.8

Given a closed set  $\mathcal{A} \subset \mathbb{R}^n$ , a set  $\mathcal{C} \subset \mathbb{R}^n$  and  $F : \mathcal{C} \rightrightarrows \mathbb{R}^n$  a locally bounded and inner semicontinuous set-valued map. Consider a function  $V \in \mathcal{L}_F(\mathcal{A}, \mathcal{C})$ , then for every  $x \in \text{int}(\mathcal{C})$ , we have

$$\langle v, f \rangle \leq -\gamma(|x|_{\mathcal{A}}), \quad \forall v \in \partial V(x), \quad \forall f \in F(x). \quad (3.22)$$

If  $V \in \mathcal{L}_F^\circ(\mathcal{A}, \mathcal{C})$  then, for every  $x \in \text{int}(\mathcal{C})$ , inequality (3.22) holds with  $\gamma \equiv 0$ .

*Proof* : Consider any  $x \in \text{int}(\mathcal{C})$ , and the open neighborhood  $\mathcal{U}(x)$  given by Definition 3.2; we can suppose, without loss of generality, that  $\mathcal{U}(x) \subset \text{int}(\mathcal{C})$ . Thanks to Lemma 3.6,  $V$  is piecewise  $\mathcal{C}^1$  on  $\mathcal{U}(x)$ , and thus, recalling (3.21), we have

$$\partial V(x) = \text{co}\{\nabla V_i(x) \mid i \in \mathcal{I}^e(x)\}.$$

The proof is then carried out by showing

$$\langle \nabla V_i(x), f \rangle \leq -\gamma(|x|_{\mathcal{A}}), \quad \forall i \in \mathcal{I}^e(x), \quad \forall f \in F(x). \quad (3.23)$$

For proving (3.23), consider any  $i \in \mathcal{I}^e(x)$ . By definition (3.20), there exists a sequence  $x_k \rightarrow x$  such that  $x_k \in \text{int}(\mathcal{V}_i)$  for all  $k \in \mathbb{N}$ , where  $\mathcal{V}_i := \{z \in \mathcal{U}(x) \mid V(z) = V_i(z)\}$ . By density of  $\mathcal{S}(x)$ , given in (3.11) in Definition 3.2, for each  $k \in \mathbb{N}$  there exists a sequence  $x_{k,l} \rightarrow x_k$  as  $l \rightarrow \infty$ , such that  $x_{k,l} \in \mathcal{S}(x) \cap \text{int}(\mathcal{V}_i)$ , for all  $l \in \mathbb{N}$ . For each  $k \in \mathbb{N}$ , we choose a large enough  $l = l_k$  so that  $x_{k,l_k} \rightarrow x$  as  $k \rightarrow \infty$ . By construction  $x_{k,l_k} \in \mathcal{S}(x) \cap \text{int}(\mathcal{V}_i)$ , for all  $k \in \mathbb{N}$ . Consider any  $f \in F(x)$ , by inner semicontinuity of  $F$  we can find a sequence  $f_k \in F(x_{k,l_k})$  such that  $f_k \rightarrow f$  as  $k \rightarrow \infty$ . By definition of  $\mathcal{V}_i$  and equation (3.14), we have

$$\langle \nabla V_i(x_{k,l_k}), f_k \rangle \leq -\gamma(|x_{k,l_k}|_{\mathcal{A}}), \quad \forall k \in \mathbb{N},$$

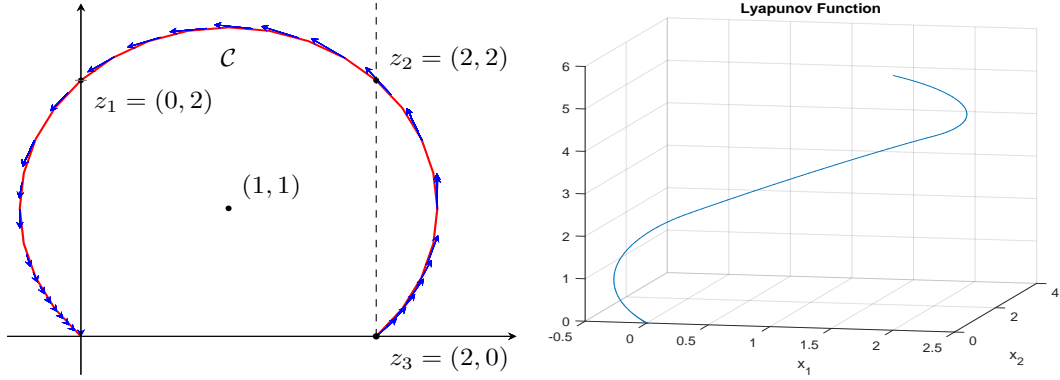


Figure 3.2: Example 3.3, the system (3.24) and the behavior of the function  $V$ .

and by continuity of  $\nabla V_i$  and  $\gamma$  this implies

$$\langle \nabla V_i(x), f \rangle \leq -\gamma(|x|_{\mathcal{A}}).$$

Since  $x \in \text{int}(\mathcal{C})$ ,  $i \in \mathcal{I}^e(x)$  and  $f \in F(x)$  are arbitrary, we can conclude the proof. The same argument can be used in the case  $V \in \mathcal{L}_F^\circ(\mathcal{A}, \mathcal{C})$  with  $\gamma \equiv 0$ .  $\diamond$

The lemma showed, roughly speaking, that a function  $V \in \mathcal{L}_F(\mathcal{A}, \mathcal{C})$  is a Clarke locally Lipschitz Lyapunov function if  $\mathcal{C}$  is open. It is not clear how, under the conditions of Definition 3.3, to consider the condition (3.22) at points on the boundary  $\text{bd}(\mathcal{C})$ . The next example shows how a function in  $\mathcal{L}_F(\mathcal{A}, \mathcal{C})$  can “do the job” when  $\mathcal{C}$  has no interior.

**Example 3.3** Consider the closed set

$$\mathcal{C} := \{x := (x_1, x_2) \in \mathbb{R}^2 \mid (x_1 - 1)^2 + (x_2 - 1)^2 = 2 \wedge x_2 \geq 0\},$$

represented by the red line in Figure 3.2, and the vector field  $f : \mathcal{C} \rightarrow \mathbb{R}^2$ , defining system

$$\dot{x} = f(x) = |x|(Ax + b), \quad x \in \mathcal{C}, \quad (3.24)$$

where  $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$  and  $b = (1, -1)$ .

Note that  $f(0) = 0$  and  $f(x) \neq 0$  for all  $x \in \mathcal{C} \setminus \{0\}$  (since  $A^{-1}b = (1, 1) \notin \mathcal{C}$ ). Denoting by  $T_{\mathcal{C}}(x)$  the tangent cone of  $\mathcal{C}$  at  $x$  (see e.g. [105, Definition 6.1]), we have  $f(x) \in T_{\mathcal{C}}(x)$ , for every  $x \in \mathcal{C}$ , see Figure 3.2 for a graphical representation.

We want to prove that  $\mathcal{A} = \{0\}$  is UGAS for system (3.24), constructing a function  $V \in \mathcal{L}_F(\{0\}, \mathcal{C})$ . To this end, consider three functions  $V_i \in \mathcal{C}^1(\mathbb{R}^2, \mathbb{R})$ ,  $i \in \{1, 2, 3\}$  given by

$$V_1(x) := x_2, \quad V_2(x) := x_1 + 2, \quad V_3(x) := -x_2 + 6. \quad (3.25)$$

We show below that the function  $V : \mathbb{R}^2 \rightarrow \mathbb{R}$ , given below, satisfies  $V \in \mathcal{L}_F(\{0\}, \mathcal{C})$ .

$$V(x) := \begin{cases} V_1(x) & \text{if } x_1 \leq 0, \\ V_2(x) & \text{if } 0 < x_1 < 2, \\ V_3(x) & \text{if } x_1 \geq 2. \end{cases}$$

**Step 1:** *Local Lipschitz continuity and positive definiteness in  $\mathcal{C}$ :*

By definition,  $V$  is continuously differentiable in  $\mathcal{C} \setminus \{z_1, z_2, z_3\}$ , where  $z_1 := (0, 2)$ ,  $z_2 := (2, 2)$  and  $z_3 := (2, 0)$ , as shown in Figure 3.2. At these points  $V$  is continuous because

$$\begin{aligned} V(z_1) &= V_1(z_1) = V_2(z_1) = 2, \\ V(z_2) &= V_2(z_2) = V_3(z_2) = 4, \\ V(z) &= V_3(z), \quad \forall z \in \mathbb{B}(z_3, 1) \cap \mathcal{C}. \end{aligned}$$

It is also easily checked that  $V$  is locally Lipschitz relative to  $\mathcal{C}$  with constant  $L(x) \equiv 1$ . Finally, from a case-by-case analysis, it is easily checked that  $\frac{|x|}{2} \leq V(x) \leq 3|x|$ ,  $\forall x \in \mathcal{C}$ .

**Step 2:** *Lyapunov conditions:*

By definition of  $V$  and  $f$  we have

$$\nabla V(x) = \begin{cases} (0, 1)^\top & \text{if } x_1 < 0 \\ (1, 0)^\top & \text{if } 0 < x_1 < 2 \\ (0, -1)^\top & \text{if } x_1 > 2 \end{cases}$$

and computing we obtain

$$\langle \nabla V(x), f(x) \rangle = \begin{cases} |x|(x_1 - 1) & \text{if } x_1 < 0, \\ |x|(-x_2 - 1) & \text{if } 0 < x_1 < 2, \\ |x|(1 - x_1) & \text{if } x_1 > 2. \end{cases}$$

Concluding, we may prove the almost everywhere condition

$$\langle \nabla V(x), f(x) \rangle \leq -|x|, \quad \forall x \in \mathcal{C} \setminus \{z_1, z_2, z_3\}. \quad (3.26)$$

It is thus clear that conditions (b), (c) and (L2) of Definitions 3.2 and 3.3 hold, by choosing  $\mathcal{U}(z_i) = \mathbb{B}(z_i, 1)$ ,  $\mathcal{I}(z_i) = \{i, i + 1\}$ ,  $\mathcal{S}(z_i) = \mathcal{U}(z_i) \setminus \{z_i\}$ , for  $i \in \{1, 2\}$ . For  $z_3$  it suffices to choose  $\mathcal{U}(z_3) = \mathbb{B}(z_3, 1)$ ,  $\mathcal{I}(z_3) = \{3\}$ ,  $\mathcal{S}(z_3) = \mathcal{U}(z_3) \setminus \{z_3\}$ . For the other points in  $\mathcal{C}$  the claim is trivial from (3.26). We have thus proved that  $V \in \mathcal{L}_F(\{0\}, \mathcal{C})$ , and Theorem 3.4 implies (UGAS) of the origin for system (3.24). See Figure 3.2 on the right for a graphical representation of the function  $V$ .

Comparison with Clarke Lyapunov functions:

It is easily seen that the function  $V$  fails to be continuous outside  $\mathcal{C}$ ; more precisely, for every open set  $\mathcal{O}$  such that  $\mathcal{C} \subset \mathcal{O}$ ,  $V$  is not continuous on  $\mathcal{O}$ , because discontinuity is inevitable in any neighborhood of  $z_1$  and  $z_2$ . It is thus clear that the Clarke gradient cannot be defined at  $z_1$  and  $z_2$ , and thus  $V$  is not a Clarke locally Lipschitz Lyapunov function, as defined in Theorem 2.18. We emphasize that alternative constructions are possible but what is appealing about our design is its intuitive nature.

△

### 3.5 Global Piecewise structure

The properties in Definitions 3.2 and 3.5 are, in a sense, local, because they require that each  $x \in \text{dom } V$  has a neighborhood on which  $V$  has a piecewise structure and is built from finitely many smooth functions, but this structure and the smooth functions may be different for different  $x$ . Below, we introduce a family of locally Lipschitz functions for which the piecewise structure is global, in the sense that  $V$  is globally obtained by “gluing together” a finite number of smooth functions.

#### Definition 3.9: Proper Piecewise $\mathcal{C}^1$ functions

Let  $\mathcal{O} \subset \mathbb{R}^n$  be an open set. A continuous function  $V : \mathcal{O} \rightarrow \mathbb{R}$  is called a *proper piecewise  $\mathcal{C}^1$  function* on  $\mathcal{O}$  if there exist  $\mathcal{I} = \{1, \dots, K\}$ , closed sets  $\{\mathcal{X}_i\}_{i \in \mathcal{I}}$ , open sets  $\{\mathcal{O}_i\}_{i \in \mathcal{I}}$  and continuously differentiable functions  $V_i : \mathcal{O}_i \rightarrow \mathbb{R}$ ,  $i \in \mathcal{I}$ , such that:

- (A)  $\mathcal{X}_i \cap \mathcal{O} \subset \mathcal{O}_i$ , for all  $i \in \mathcal{I}$ ,
- (B)  $\overline{\text{int}(\mathcal{X}_i)} = \mathcal{X}_i$ , (namely  $\mathcal{X}_i$  is *regular-closed*), for all  $i \in \mathcal{I}$ ,
- (C)  $\mathcal{O} \subset \bigcup_{i \in \mathcal{I}} \mathcal{X}_i$ ,
- (D)  $V(x) = V_i(x)$ , if  $x \in \mathcal{X}_i$ . △

If  $V : \mathcal{O} \rightarrow \mathbb{R}$  is a proper piecewise  $\mathcal{C}^1$  function, then it is piecewise  $\mathcal{C}^1$ , as in Definition 3.5. Indeed, it suffices to select  $\mathcal{U}(x) \equiv \mathcal{O}$ ,  $\mathcal{I}(x) \equiv \mathcal{I}$  and  $\mathcal{F} \equiv \{V_1, \dots, V_K\}$ .

For this class of proper piecewise  $\mathcal{C}^1$  functions, given a set valued map  $F : \mathcal{C} \rightrightarrows \mathbb{R}^n$ , in the case where  $\mathcal{C}$  is regular-closed (that is  $\mathcal{C} = \overline{\text{int}(\mathcal{C})}$ ), it may be computationally easier to check the conditions of Definition 3.3, as the next result suggests. The sufficient conditions below may hold when conditions based on the Clarke gradient don't; this is illustrated in Section 3.6.2.

#### Theorem 3.10

Consider a set  $\mathcal{C} \subset \mathbb{R}^n$  such that  $\overline{\text{int}(\mathcal{C})} = \mathcal{C}$  and let  $V : \mathcal{O} \rightarrow \mathbb{R}$  be a proper piecewise  $\mathcal{C}^1$  function with  $\mathcal{C} \subset \mathcal{O}$ . Consider a closed set  $\mathcal{A} \subset \mathbb{R}^n$  and a locally bounded and inner semicontinuous set-valued map  $F : \mathcal{C} \rightrightarrows \mathbb{R}^n$ . If there exist  $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ , and  $\gamma \in \mathcal{PD}$  such that

$$\alpha_1(|x|_{\mathcal{A}}) \leq V(x) \leq \alpha_2(|x|_{\mathcal{A}}), \quad \forall x \in \mathcal{C}, \quad (3.27)$$

$$\langle \nabla V_i(x), f \rangle \leq -\gamma(|x|_{\mathcal{A}}), \quad \begin{cases} \forall x \in \text{int}(\mathcal{X}_i) \cap \text{int}(\mathcal{C}), \\ \forall f \in F(x), \forall i \in \mathcal{I}, \end{cases} \quad (3.28)$$

then  $V \in \mathcal{L}_F(\mathcal{A}, \mathcal{C})$ . Thus  $\mathcal{A}$  is (UGAS) for system (3.1). Considering  $\gamma \equiv 0$  in (3.28), then  $V \in \mathcal{L}_F^\circ(\mathcal{A}, \mathcal{C})$  and  $\mathcal{A}$  is (UGS) for system (3.1).

*Proof :* We prove the theorem by showing that all the hypotheses of Definition 3.2 and Definition 3.3 hold. The facts that  $\mathcal{C} \subset \mathcal{O}$  and that  $V$  is locally Lipschitz

relative to  $\mathcal{C}$  are straightforward. Condition (3.27) is exactly (L1) of Definition 3.3. It only remains to prove (b),(c) of Definition 3.2 and (L2) of Definition 3.3. For any  $x \in \mathcal{C}$ , we consider  $\mathcal{U}(x) \equiv \mathcal{O}$  and define  $\mathcal{S}(x) \equiv \mathcal{S} := \bigcup_{i \in \mathcal{I}} \overline{\text{int}(\mathcal{X}_i)} \cap \text{int}(\mathcal{C})$ . We now prove that  $\overline{\mathcal{S}} \supset \mathcal{C}$ . Consider a point  $x \in \mathcal{C}$ , recalling that  $\text{int}(\mathcal{C}) = \mathcal{C}$  we consider a sequence  $x_k \in \text{int}(\mathcal{C})$  such that  $x_k \rightarrow x$ , as  $k \rightarrow \infty$ . By properties (B) and (C) of Definition 3.9, we have

$$\overline{\bigcup_{i \in \mathcal{I}} \text{int}(\mathcal{X}_i)} = \bigcup_{i \in \mathcal{I}} \overline{\text{int}(\mathcal{X}_i)} = \bigcup_{i \in \mathcal{I}} \mathcal{X}_i \supset \mathcal{O} \supset \mathcal{C} \supset \text{int}(\mathcal{C}).$$

Thus, for each  $k \in \mathbb{N}$ , there exists a sequence  $x_{k,l} \in \bigcup_{i \in \mathcal{I}} \text{int}(\mathcal{X}_i) \cap \text{int}(\mathcal{C})$  such that  $x_{k,l} \rightarrow x_k$  as  $l \rightarrow \infty$ . For each  $k \in \mathbb{N}$ , we choose a large enough  $l = l_k$  so that  $x_{k,l_k} \rightarrow x$  as  $k \rightarrow \infty$ . By construction,  $x_{k,l_k} \in \bigcup_{i \in \mathcal{I}} \text{int}(\mathcal{X}_i) \cap \text{int}(\mathcal{C})$ , proving the density of  $\mathcal{S}$  in  $\mathcal{C}$  (and thus (b) of Definition 3.2). Now, for each  $y \in \mathcal{S}$  there exists  $i \in \mathcal{I}$  such that  $y \in \text{int}(\mathcal{X}_i)$  and since by condition (D),  $V$  coincides with  $V_i$  in the open set  $\text{int}(\mathcal{X}_i)$ , we have  $\nabla V(y) = \nabla V_i(y)$  proving condition (c) of Definition 3.2. Finally, by condition (3.28) we obtain (L2) of Definition 3.3, concluding the proof that  $V \in \mathcal{L}_F(\mathcal{A}, \mathcal{C})$ . Then UGAS follows from Theorem 3.4. The case with  $\gamma \equiv 0$  is completely analogous.  $\diamond$

We prove that the family of proper piecewise  $\mathcal{C}^1$  functions is closed under the pointwise maximum and pointwise minimum operators.

### Proposition 3.11

Consider an open set  $\mathcal{O} \subset \mathbb{R}^n$ . The class of proper piecewise  $\mathcal{C}^1$  functions on  $\mathcal{O}$  is closed under the operations of pointwise maximum and pointwise minimum of finitely many functions. More precisely, given  $V_1, \dots, V_K : \mathcal{O} \rightarrow \mathbb{R}$  proper piecewise  $\mathcal{C}^1$  functions on  $\mathcal{O}$ , the functions  $V_M, V_m : \mathcal{O} \rightarrow \mathbb{R}$  defined by

$$\begin{aligned} V_M(x) &:= \max_{i=1, \dots, K} \{V_i(x)\}, \\ V_m(x) &:= \min_{i=1, \dots, K} \{V_i(x)\}, \end{aligned} \quad \forall x \in \mathcal{O},$$

are proper piecewise  $\mathcal{C}^1$  on  $\mathcal{O}$ .

*Proof* : [Sketch of the Proof] We note that if  $V$  is proper piecewise  $\mathcal{C}^1$  so is  $W := -V$ . Moreover, by definition of pointwise maximum and pointwise minimum operators,  $\max\{a, b, c\} = \max\{a, \max\{b, c\}\}$  and  $\min\{a, b\} = -\max\{-a, -b\}$  for all  $a, b, c \in \mathbb{R}$ . Therefore, it suffices to prove that, given  $V_1, V_2 : \mathcal{O} \rightarrow \mathbb{R}$  proper piecewise  $\mathcal{C}^1$  functions, the max function  $V_M : \mathcal{O} \rightarrow \mathbb{R}$  defined by

$$V_M(x) := \max\{V_1(x), V_2(x)\}, \quad \forall x \in \mathcal{O},$$

is proper piecewise  $\mathcal{C}^1$  on  $\mathcal{O}$ , since the general statement follows iterating this property. Toward this end, let us consider  $V_1, V_2 : \mathcal{O} \rightarrow \mathbb{R}$  proper piecewise  $\mathcal{C}^1$  functions. For both  $j = 1, 2$ , we can consider  $\mathcal{I}^j = \{1, \dots, K_j\}$ , closed sets  $\{\mathcal{X}_i^j\}_{i \in \mathcal{I}^j}$ , open sets  $\{\mathcal{O}_i^j\}_{i \in \mathcal{I}^j}$  and continuously differentiable functions  $V_i^j : \mathcal{O}_i^j \rightarrow \mathbb{R}$ ,  $i \in \mathcal{I}^j$ , such that

Definition 3.9 is satisfied for  $V_j$ ,  $j = 1, 2$ . Define the open sets

$$\begin{aligned}\mathcal{Y}_0 &:= \text{int}(\{x \in \mathcal{O} \mid V_1(x) = V_2(x)\}), \\ \mathcal{Y}_1 &:= \{x \in \mathcal{O} \mid V_1(x) > V_2(x)\}, \\ \mathcal{Y}_2 &:= \{x \in \mathcal{O} \mid V_1(x) < V_2(x)\}.\end{aligned}$$

Let us take  $\mathcal{I}^M := \{1, \dots, K_1, K_1 + 1, \dots, 2K_1, 2K_1 + 1, \dots, 2K_1 + K_2\}$  and define

$$\mathcal{X}_i^M := \begin{cases} \overline{\mathcal{X}_i^1 \cap \mathcal{Y}_0}, & \text{if } 1 \leq i \leq K_1, \\ \overline{\mathcal{X}_{i-K_1}^1 \cap \mathcal{Y}_1}, & \text{if } K_1 + 1 \leq i \leq 2K_1, \\ \overline{\mathcal{X}_{i-2K_1}^2 \cap \mathcal{Y}_2}, & \text{if } 2K_1 + 1 \leq i \leq 2K_1 + K_2, \end{cases}$$

and

$$(\mathcal{O}_i^M, V_i^M) := \begin{cases} (\mathcal{O}_i^1, V_i^1) & \text{if } 1 \leq i \leq K_1, \\ (\mathcal{O}_{i-K_1}^1, V_{i-K_1}^1) & \text{if } K_1 + 1 \leq i \leq 2K_1, \\ (\mathcal{O}_{i-2K_1}^2, V_{i-2K_1}^2) & \text{if } 2K_1 + 1 \leq i \leq 2K_1 + K_2. \end{cases}$$

It can be shown that  $V_M$ ,  $\mathcal{I}^M$ ,  $\{\mathcal{X}_i^M\}_{i \in \mathcal{I}^M}$ ,  $\{\mathcal{O}_i^M\}_{i \in \mathcal{I}^M}$  and  $\{V_i^M\}_{i \in \mathcal{I}^M}$  satisfy Definition 3.9. Conditions (A) and (D) are straightforward; Condition (B) follows from the fact that given  $\mathcal{X} \subset \mathbb{R}^n$  such that  $\mathcal{X} = \overline{\text{int}(\mathcal{X})}$  and an open set  $\mathcal{U} \subset \mathbb{R}^n$ , one has

$$\overline{\overline{\text{int}(\mathcal{X} \cap \mathcal{U})}} = \overline{\mathcal{X} \cap \mathcal{U}}.$$

Condition (C) holds recalling that  $\cup_{i \in \mathcal{I}^1} \text{int}(\mathcal{X}_i^1)$  and  $\cup_{i \in \mathcal{I}^2} \text{int}(\mathcal{X}_i^2)$  are dense in  $\mathcal{O}$  and  $\mathcal{O} \subset \mathcal{Y}_0 \cup \overline{\mathcal{Y}_1} \cup \overline{\mathcal{Y}_2}$ .  $\diamond$

Let  $\mathcal{O} \subset \mathbb{R}^n$  be an open set, and consider  $V_1, \dots, V_K : \mathcal{O} \rightarrow \mathbb{R}$  proper piecewise  $\mathcal{C}^1$  functions. We consider a *max-min* function  $V_{\text{Mm}} : \mathcal{O} \rightarrow \mathbb{R}$  defined by

$$V_{\text{Mm}}(x) := \max_{j \in \{1, \dots, J\}} \left\{ \min_{k \in S_j} \{V_k(x)\} \right\}, \quad \forall x \in \mathcal{O} \quad (3.29)$$

where  $J \geq 1$  and  $S_1, \dots, S_J \subset \{1, \dots, K\}$  are non-empty subsets, see also [49] and references therein for a thorough discussion about this family of functions. Iterating the result in Proposition 3.11, it trivially holds that  $V_{\text{Mm}}$  is proper piecewise  $\mathcal{C}^1$  on  $\mathcal{O}$ . Remarkably, any piecewise *affine* function (PWA) can be represented in the form (3.29) with  $V_1, \dots, V_K : \mathbb{R}^n \rightarrow \mathbb{R}$  affine functions, that is the so-called lattice representation, see [129] and references therein. We will intensively study this particular family of functions in the subsequent Chapter 4 in the context of state-dependent switched systems.

## 3.6 Application to Hybrid Dynamical Systems

In this section we show how Theorem 3.4 can be used in the hybrid systems setting, proposing a result which generalizes the “classical” hybrid Lyapunov Theorem 2.10. We recall here the main definition: Given  $\mathcal{C}, \mathcal{D} \subset \mathbb{R}^n$ ,  $F : \text{dom } F \rightrightarrows \mathbb{R}^n$ ,  $G :$



$\text{dom } G \rightrightarrows \mathbb{R}^n$ , such that  $\mathcal{C} \subset \text{dom } F$  and  $\mathcal{D} \subset \text{dom } G$ , a hybrid dynamical system  $\mathcal{H} = (\mathcal{C}, \mathcal{D}, G, F)$  is

$$\mathcal{H} : \begin{cases} \dot{x} \in F(x), & x \in \mathcal{C}, \\ x^+ \in G(x), & x \in \mathcal{D}. \end{cases} \quad (3.30)$$

For all relevant discussion we refer to Sections 1.1.3 and 2.3. A possible extension of Theorem 2.10 now follows.

### Theorem 3.12: “Relaxed” Hybrid Lyapunov Theorem

Given hybrid system (3.30), suppose that  $F : \text{dom } F \rightrightarrows \mathbb{R}^n$  is locally bounded and inner semicontinuous with  $\mathcal{C} \subset \text{dom } F$ . Given a closed set  $\mathcal{A}$ , suppose that  $V : \text{dom } V \rightarrow \mathbb{R}$  is such that

- (a)  $V \in \mathcal{L}_F(\mathcal{A}, \mathcal{C})$ ;
- (b)  $V$  is a discrete-time Lyapunov function for  $\mathcal{A}$  in  $\mathcal{D}$ , in the sense that  $\text{dom } V \supset \mathcal{D} \cup G(\mathcal{D})$  and there exist  $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ ,  $\gamma \in \mathcal{PD}$  satisfying

$$\alpha_1(|x|_{\mathcal{A}}) \leq V(x) \leq \alpha_2(|x|_{\mathcal{A}}), \quad \forall x \in \mathcal{D} \cup G(\mathcal{D}) \quad (3.31a)$$

$$V(g) - V(x) \leq -\gamma(|x|_{\mathcal{A}}) \quad \forall x \in \mathcal{D}, g \in G(x). \quad (3.31b)$$

Then  $\mathcal{A}$  is UGAS for hybrid system (1.8). Moreover, if  $V \in \mathcal{L}_F^\circ(\mathcal{A}; \mathcal{C})$  and condition (b) is satisfied with  $\gamma \equiv 0$ , then  $\mathcal{A}$  is UGS for hybrid system (1.8).

*Proof :* Consider any solution of (3.30)  $\psi : \text{dom } \psi \rightarrow \mathbb{R}^n$ . Consider any  $j \in \mathbb{N}$  such that the  $j$ -th interval of flow,  $I^j$ , has nonempty interior. In this case, considering the restriction  $\psi(\cdot, j) : I^j \rightarrow \mathcal{C}$ , we have  $\psi(\cdot, j) \in \mathcal{S}_{F, \mathcal{C}}$ . Applying Theorem 3.4 we have that  $V \circ \psi(\cdot, j) : I^j \rightarrow \mathbb{R}$  is strictly decreasing, with the rate determined by  $\gamma$  in (3.14). Conditions (3.31a) guarantee the decrease of  $V \circ \psi$  during jumps of  $\psi$ , with the rate determined by  $\gamma$  in (3.31a). Thus  $\mathcal{A}$  is UGAS, following the same steps as in the proof of [58, Theorem 3.18]. The case with  $V \in \mathcal{L}_F^\circ(\mathcal{A}; \mathcal{C})$  and  $\gamma \equiv 0$  ensuring (UGS) of  $\mathcal{A}$  for system (3.30) is straightforward.  $\diamond$

There is no loss of generality in using the same class  $\mathcal{K}_\infty$  functions  $\alpha_1, \alpha_2$  for positive definiteness in  $\mathcal{C}$  and in  $\mathcal{D} \cup G(\mathcal{D})$ : if they were different, one considers the point-wise minimum for the lower bound and the point-wise maximum for the upper bound. The same reasoning applies to  $\gamma$ . Thus merging (L1) of Definition 3.3 and (3.31a) of Theorem 3.12 yields  $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$  such that

$$\alpha_1(|x|_{\mathcal{A}}) \leq V(x) \leq \alpha_2(|x|_{\mathcal{A}}) \quad \forall x \in \mathcal{C} \cup \mathcal{D} \cup G(\mathcal{D}), \quad (3.32)$$

consistently with the hypothesis of Theorem 2.10. For the case when  $\mathcal{A}$  is compact, it is customary to relax this condition to the equivalent property that  $V$  be positive definite w.r.t  $\mathcal{A}$ , and radially unbounded relative to  $\mathcal{C} \cup \mathcal{D}$ , see [82, Lemma 4.3]. Similarly, one can show that (3.31a) is equivalent to requiring that  $G(\mathcal{D} \cap \mathcal{A}) \subset \mathcal{A}$  and

$$V(g) - V(x) < 0, \quad \forall x \in \mathcal{D} \setminus \mathcal{A}, \forall g \in G(x), \quad (3.33)$$

so that there is no need to compute explicitly the function  $\gamma$ , see [111].

### 3.6.1 Homogeneous Hybrid Dynamics

In this section we study a class of hybrid dynamical systems given by

$$\begin{cases} \dot{x} = A_F x, & x \in \mathcal{C} = \{x \in \mathbb{R}^n \mid x^\top Q_F x \geq 0\}, \\ x^+ = A_J x, & x \in \mathcal{D} = \{x \in \mathbb{R}^n \mid x^\top Q_J x \geq 0\}, \end{cases} \quad (3.34)$$

where  $A_F, A_J \in \mathbb{R}^{n \times n}$  and  $Q_F, Q_J \in \text{Sym}(\mathbb{R}^n) := \{S \in \mathbb{R}^{n \times n} \mid S = S^\top\}$ . Such (3.34) satisfies the *hybrid basic conditions* defined in [58, Assumption 6.5]. Moreover,  $\mathcal{C}$  and  $\mathcal{D}$  are symmetric cones, that is, if  $x \in \mathcal{C}$  ( $x \in \mathcal{D}$  resp.) then  $\lambda x \in \mathcal{C}$  ( $\lambda x \in \mathcal{D}$  resp.), for all  $\lambda \in \mathbb{R}$ . Noting that the flow and jump maps are linear, system (3.34) is *homogeneous with respect to the standard dilation*, as defined in [60] (see also [58, Chapter 9]). Consistently with the converse result in [123], we consider (proper piecewise  $\mathcal{C}^1$ ) candidate Lyapunov functions *homogeneous of degree 2*.

#### Definition 3.13: Proper Piecewise Quadratic Functions

A proper piecewise  $\mathcal{C}^1$  function  $V : \mathbb{R}^n \rightarrow \mathbb{R}$  (recall Definition 3.9) is a *proper piecewise quadratic function* ( $V \in \mathcal{PQ}(\mathbb{R}^n)$ ) if

- For each  $i \in \mathcal{I}$ , there exists  $R_i \in \text{Sym}(\mathbb{R}^n)$  such that  $\mathcal{X}_i = \{x \in \mathbb{R}^n \mid x^\top R_i x \geq 0\}$ ;
- For each  $i \in \mathcal{I}$  there exists  $P_i \in \text{Sym}(\mathbb{R}^n)$  such that  $V_i(x) := x^\top P_i x$ , for all  $x \in \mathbb{R}^n$ .

By definition of  $\{\mathcal{X}_i\}_{i \in \mathcal{I}}$  and  $\{V_i\}_{i \in \mathcal{I}}$ , a proper piecewise quadratic function is in particular *even* and *homogeneous of degree 2*, that is,  $V(\lambda x) = \lambda^2 V(x)$ ,  $\forall \lambda \in \mathbb{R}$ ,  $\forall x \in \mathbb{R}^n$ .

We now state a useful corollary of Theorem 3.12 in the context of homogeneous hybrid dynamical systems (3.34).

#### Corollary 3.14

Consider system (3.34), with  $Q_F$  not negative semi-definite. Suppose that there exist  $V \in \mathcal{PQ}$  and  $\lambda_1, \lambda_2 > 0$  such that

$$\lambda_1 |x|^2 \leq V(x) \leq \lambda_2 |x|^2, \quad \forall x \in \mathcal{C} \cup \mathcal{D} \cup A_J(\mathcal{D}), \quad (3.35)$$

and, for all  $x \in \mathbb{R}^n$ , for all  $i \in \mathcal{I}$ , it holds that

$$x^\top Q_F x > 0 \wedge x^\top R_i x > 0 \Rightarrow x^\top P_i A_F x < 0. \quad (3.36)$$

Moreover, suppose that for all  $x \in \mathbb{R}^n$ , for all  $(j, i) \in \mathcal{I} \times \mathcal{I}$ , it holds that

$$\begin{aligned} x^\top Q_J x \geq 0 \wedge x^\top R_j x \geq 0 \wedge x^\top A_J^\top R_i A_J x \geq 0 \\ \Downarrow \\ x^\top A_J P_i A_J x - x^\top P_j x < 0. \end{aligned} \quad (3.37)$$

Then the origin is UGAS for hybrid system (3.34).

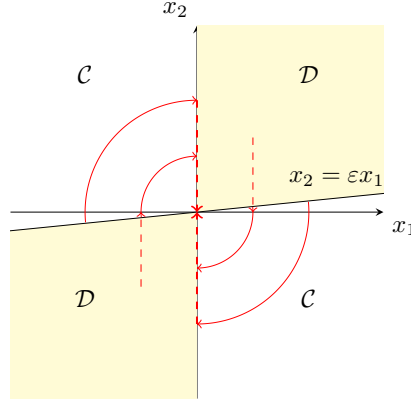


Figure 3.3: Trajectories of system (3.38)

*Proof* : [Sketch of the proof] The hypothesis of  $Q_F$  being not negative semi-definite ensures that  $\text{int}(\mathcal{C}) = \mathcal{C} = \{x \in \mathbb{R}^n \mid x^\top Q_F x \geq 0\}$ , as required in Theorem 3.10. It is clear that (3.35) implies (3.32). Implication (3.36) ensures that  $V \in \mathcal{L}_F(\{0\}; \mathcal{C})$ , since it implies condition (3.28) of Theorem 3.10. Implication (3.37) ensures (3.31b). Applying Theorem 3.12 we conclude that  $\{0\}$  is (UGAS) for system (3.34).  $\diamond$

We note that condition (3.35) does *not* necessary imply  $P_i > 0$  for all  $i \in \mathcal{I}$ . In fact it is sufficient to ensure that the functions  $V_i(x) = x^\top P_i x$  are positive definite in their region of activation  $\mathcal{X}_i = \{x \in \mathbb{R}^n \mid x^\top R_i x \geq 0\}$ , for each  $i \in \mathcal{I}$ . The overall function  $V$  then satisfies bounds as in (3.32); the quadratic bounds in (3.35) can be obtained by homogeneity of  $V$ . Moreover, Corollary 3.14 is particularly appealing because conditions (3.35), (3.36) and (3.37) could be reduced, via S-Procedure, as a system of LMIs, but paying a price in term of conservatism.

### 3.6.2 Clegg Integrator

The Clegg integrator connected to an integrating plant has been shown to overcome intrinsic limitations of linear feedback [14], (see also [102]). More specifically, using the  $\varepsilon$ -regularization suggested in [96], we focus on the hybrid closed-loop

$$\begin{cases} \dot{x} = A_F x, & x \in \mathcal{C} = \{x \in \mathbb{R}^2 \mid x^\top Q x \geq 0\}, \\ x^+ = A_J x, & x \in \mathcal{D} = \{x \in \mathbb{R}^2 \mid x^\top Q x \leq 0\}, \end{cases} \quad (3.38)$$

with

$$A_F = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad A_J = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad Q = \begin{bmatrix} 1 & -\frac{1}{2\varepsilon} \\ -\frac{1}{2\varepsilon} & 0 \end{bmatrix},$$

and  $\varepsilon > 0$  being a small regularization parameter, see Figure 3.3 for a graphical representation.

Following [131], there does not exist a *quadratic* Lyapunov function. In fact, given any symmetric and positive definite matrix  $P = \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix}$ , and considering points

$z_1 = (-1, 0)$  and  $z_2 = (0, 1)$ , we have that  $z_1, z_2 \in \mathcal{C}$  and the Lyapunov inequalities  $z_1^\top P A_F z_1 < 0$  and  $z_2^\top P A_F z_2 < 0$  would imply  $p_{12} < 0$  and  $p_{12} > 0$  respectively and hence, a contradiction. UGAS of (3.38) was established with nonconvex numerical piecewise quadratic constructions in [131], and then via a nonconvex analytic construction in [96]. We illustrate the use of Corollary 3.14 by building three proper piecewise quadratic Lyapunov functions, one of them convex. In what follows, we fix  $\varepsilon = 0.1$ , but the functions that we construct work for any  $\varepsilon$  such that  $0 < \varepsilon \leq 0.1$ .

**Max of Quadratics:**

We first use a max function of two quadratics

$$V_M(x) := \max\{x^\top P_1 x, x^\top P_2 x\}, \quad (3.39)$$

$$\text{with } P_1 = \begin{bmatrix} 1 & -0.1 \\ -0.1 & 0.5 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 2.5 & 1.4 \\ 1.4 & 0.5 \end{bmatrix},$$

where we illustrate the use of quadratics not necessarily positive definite (see also [83]). We prove next that (3.39) satisfies the hypotheses of Corollary 3.14. First of all, the function  $V_M$  satisfies (3.35) (positive definiteness): this is straightforward by noting that  $P_1 > 0$ . Let us define  $X_1, X_2 \subset \mathbb{R}^2$ , as

$$\begin{aligned} \mathcal{X}_1 &:= \{x \mid V_1(x) \geq V_2(x)\} = \{x \mid x^\top (P_1 - P_2)x \geq 0\}, \\ \mathcal{X}_2 &:= \{x \mid V_2(x) \geq V_1(x)\} = \{x \mid x^\top (P_2 - P_1)x \geq 0\}, \end{aligned}$$

and denoting  $R := (P_1 - P_2) = \begin{bmatrix} -1.5 & -1.5 \\ -1.5 & 0 \end{bmatrix}$ , we can conclude

$$\mathcal{X}_1 = \{x \mid x^\top R x \geq 0\}, \quad \mathcal{X}_2 = \{x \mid x^\top R x \leq 0\}.$$

For the jump condition (3.37), we first note that  $\mathcal{D} \subset \{x \mid x_1 x_2 \geq 0\} \subset \mathcal{X}_2$ : given  $x = (x_1, x_2)$  with  $x_1 x_2 \geq 0$  we have  $x^\top R x \leq 0$ . Moreover, since  $G(\mathcal{D}) = \{x \mid x_2 = 0\}$ , we conclude that  $\mathcal{D} \cup G(\mathcal{D}) \subset \mathcal{X}_2$ . In other words, the jump set and its image are contained in the region where  $V \equiv V_2$ . We can thus simply obtain the inequality

$$\begin{aligned} V_2(x^+) - V_2(x) &= x^\top A_J^\top P_2 A_J x - x^\top P_2 x \\ &= 2.5x_1^2 - 2.5x_1^2 - 3x_1 x_2 - 0.5x_2^2 < 0, \end{aligned}$$

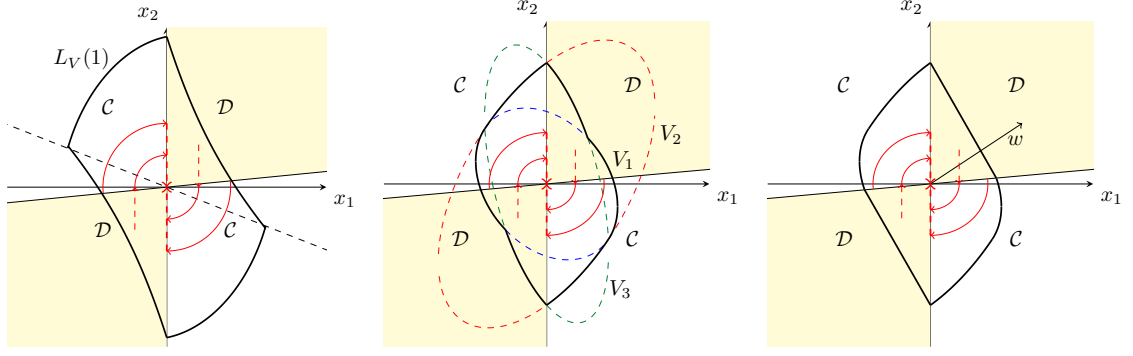
$\forall x \in \mathcal{D} \setminus \{0\}$ , proving (3.37).

For the flow condition, for simplicity, we split the flow set  $\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2 := (\mathcal{X}_1 \cap \mathcal{C}) \cup (\mathcal{X}_2 \cap \mathcal{C})$ . From the fact that  $\mathcal{D} \subset \mathcal{X}_2$ , we note that  $(\mathcal{X}_1 \cap \mathcal{C}) = \mathcal{X}_1$ . Thus for  $\mathcal{C}_1$ , it suffices to find a  $\mu_1 \geq 0$  such that the LMI

$$P_1 A_F + A_F^\top P_1 + \mu_1 S + 0.1 \text{Id} < 0$$

is satisfied, and this is the case choosing, for example,  $\mu_1 = 0.4$ . Secondly, we note that the region  $\mathcal{C}_2 := \mathcal{X}_2 \cap \mathcal{C}$  can be rewritten as  $\mathcal{C}_2 = \{x \mid x^\top Q_2 x \geq 0\}$ , where

$$Q_2 := \begin{bmatrix} 0.2 & -0.8 \\ -0.8 & -4 \end{bmatrix}.$$



(a) In black we plotted the 1-level set of the max of quadratics Lyapunov function  $V_M$  defined in (3.39). (b) The 1-level set of the mid of quadratics Lyapunov function  $V_{\text{mid}}$  defined in (3.40). (c) The 1-level set of the convex Lyapunov function  $V_{\text{conv}}$  defined in (3.41).

Figure 3.4: Level sets of the constructed Lyapunov functions for system (3.38); in red some particular solutions.

Thus, for proving

$$x \in \mathcal{X}_2 \Rightarrow x^\top P_2 A_F x < 0$$

it suffices to find a  $\mu_2 \geq 0$  such that the LMI

$$P_2 A_F + A_F^\top P_2 + \mu_2 Q_2 + 0.1 \text{Id} < 0,$$

is satisfied. This is the case, for example, with  $\mu_2 = 2$ .

To summarize, all the conditions of Corollary 3.14 are satisfied by  $V_M$  and we can conclude that  $\mathcal{A} = \{0\}$  is (UGAS) for system (3.38). See Figure 3.4(a) for a graphical representation of our construction, where nonconvexity of  $V_M$  emerges from the fact that  $P_2$  is not sign-definite.

#### Mid of quadratics:

Consider the symmetric matrices

$$P_1 = \begin{bmatrix} 1 & 0.25 \\ 0.25 & 0.7 \end{bmatrix}, P_2 = \begin{bmatrix} 0.55 & -0.2 \\ -0.2 & 0.25 \end{bmatrix}, P_3 = \begin{bmatrix} \frac{25}{16} & \frac{49}{160} \\ \star & 0.25 \end{bmatrix}$$

and consider the function

$$\begin{aligned} V_{\text{mid}}(x) &:= \text{mid}\{V_1(x), V_2(x), V_3(x)\} \\ &:= \max\{\min\{V_1, V_2\}, \min\{V_2, V_3\}, \min\{V_1, V_3\}\}, \end{aligned} \quad (3.40)$$

where  $V_i(x) := x^\top P_i x$ . Intuitively, the ‘‘mid’’ operator selects the function whose value lies between the two others. Taking  $I = \{1, 2, 3\}$ , we introduce as in Definition 3.13, the sets as

$$\begin{aligned} \mathcal{X}_i &:= \text{cl}\left(\{x \in \mathbb{R}^2 \mid x^\top P_j x < x^\top P_i x < x^\top P_k x\} \right. \\ &\quad \left. \cup \{x \in \mathbb{R}^2 \mid x^\top P_k x < x^\top P_i x < x^\top P_j x\}\right) \end{aligned}$$

for every  $i \in I$ ,  $j \neq k \in I \setminus \{i\}$ . The sets  $X_i$  could of course be rewritten in the form  $\mathcal{X}_i = \{x \in \mathbb{R}^n \mid x^\top R_i x \geq 0\}$  for some  $R_i \in \text{Sym}(\mathbb{R}^n)$ ,  $i \in \{1, 2, 3\}$ , but this is not necessary for our analysis, since we rely on the general Theorem 3.10, instead of Corollary 3.14.

Positive definiteness of  $V_{\text{mid}}$  is immediate, noting that  $P_1, P_2$  and  $P_3$  are all positive definite.

The jump condition on  $\mathcal{D}$  is trivial, and can be obtained following the same steps as with function  $V_M$ .

For the flow condition we will prove (3.28). It can be seen that

$$\begin{aligned}\mathcal{X}_1 \cap \text{int}(\mathcal{C}) &= \{x \mid x^\top Q_1 x > 0\}, \\ \mathcal{X}_2 \cap \text{int}(\mathcal{C}) &= \{x \mid x^\top Q_2 x > 0\}, \\ \mathcal{X}_3 \cap \text{int}(\mathcal{C}) &= \emptyset,\end{aligned}$$

where

$$\begin{aligned}Q_1 &= \theta_1 \theta_2^\top + \theta_2 \theta_1^\top = \begin{bmatrix} 0.2 & -0.9 \\ -0.9 & -2 \end{bmatrix}, \\ Q_2 &= \theta_2 \theta_3^\top + \theta_3 \theta_2^\top = \begin{bmatrix} -2 & -1 \\ -1 & 0 \end{bmatrix}.\end{aligned}$$

where  $\theta_1 = (-0.1, 1)$ ,  $\theta_2 = (-1, -1)$  and  $\theta_3 = (1, 0)$  are properly chosen vectors perpendicular to the lines that represent the boundaries of  $\mathcal{X}_1 \cap \text{int}(\mathcal{C})$  and  $\mathcal{X}_2 \cap \text{int}(\mathcal{C})$ , respectively. We prove the flow condition (3.28) looking for scalars  $\mu_1, \mu_2 \geq 0$  such that the LMIs

$$\begin{aligned}P_1 A_F + A_F^\top P_1 + \mu_1 Q_1 + 0.1 \text{Id} &< 0, \\ P_2 A_F + A_F^\top P_2 + \mu_2 Q_2 + 0.1 \text{Id} &< 0,\end{aligned}$$

are satisfied; this turns out to be true choosing, for example,  $\mu_1 = \mu_2 = 0.5$ . We have proved that

$$\begin{aligned}\langle \nabla V_1(x), A_F x \rangle &\leq -0.1|x|^2, \quad \forall x \in \mathcal{C}_1, \quad \text{and} \\ \langle \nabla V_2(x), A_F x \rangle &\leq -0.1|x|^2, \quad \forall x \in \mathcal{C}_2\end{aligned}$$

which imply the flow conditions (3.28). Theorem 3.12 holds and we conclude that  $\mathcal{A} = \{0\}$  is (UGAS) for the system (3.38). See Fig. 3.4(b) for a graphical representation of our construction, which shows again nonconvex level sets of  $V_{\text{mid}}$ .

### Convex Lyapunov Function

The two Lyapunov functions above are both nonconvex. We construct here a convex one, starting from  $V_{\text{mid}}$ . Looking at the level set  $L_{V_{\text{mid}}}(1)$ , the idea is to connect the points of intersection of  $L_{V_{\text{mid}}}(1)$  with the two lines that form the boundary of  $\mathcal{D}$  using a straight line. We thus define

$$V_{\text{conv}}(x) = \begin{cases} V_{\text{mid}}(x), & \text{if } x \in \mathcal{C}, \\ \langle w, x \rangle^2, & \text{if } x \in \mathcal{D}, \end{cases} \quad (3.41)$$

where  $w = (0.9574, 0.7071)$  is a vector tangent to the line of interest, suitably scaled to ensure continuity. This function satisfies the conditions of Theorem 3.12 from the properties of  $V_{\text{mid}}$ . It is represented in Fig. 3.4(c).

### 3.7 Summary

We studied the problem of stability for a class of differential inclusions constrained in a set  $\mathcal{C}$ , with a particular interest in applications to hybrid dynamical systems. We provided sufficient Lyapunov conditions for a particular class of piecewise-defined locally Lipschitz functions, requiring to check the Lyapunov decrease inequality only on a dense subset of a given domain of interest  $\mathcal{C}$ . We then studied the relations between our proposed class of locally Lipschitz functions and the piecewise  $\mathcal{C}^1$  functions recently proposed in the literature. Our approach has also been compared with the existing literature on locally Lipschitz Lyapunov functions, showing that our conditions are in general less restrictive than the Clarke gradient-based conditions presented in Section 2.4. We finally applied our approach in the context of hybrid dynamical systems, with particular care to the homogeneous case. We apply our result to a classical example from the reset systems literature: the Clegg integrator model. Several examples are provided to show the novelty and the usefulness of our results.

# 4

## Max-Min Locally Lipschitz Lyapunov Functions for Switched Systems

In this chapter we consider (unconstrained) differential inclusions of the form

$$\dot{x} \in F(x), \quad (4.1)$$

proposing stability conditions using a particular class of locally Lipschitz functions, obtained by iterating pointwise maximum and minimum operators over a finite family of continuously differentiable functions. We focus our attention on outer semicontinuous maps  $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$  in the right-hand side of (4.1), since we are interested in applications to switched systems. In particular, we consider:

- Switching systems under arbitrary switching rule, that is differential inclusions where the set-valued right-hand-side comprises the convex hull of a finite number of vector fields.
- Autonomous switched systems with a state-dependent switching signal.

Since regularization of state-dependent switched systems are not inner semicontinuous in general (as showed by Example 3.2) the tools presented in Chapter 3 cannot be directly applied in this setting, forcing us to follow other paths for the stability analysis.

More precisely, we analyze the notions of directional derivatives (introduced in Section 2.4) for max-min functions, and use them in deriving stability conditions with various degrees of conservatism, where more conservative conditions are numerically more tractable. The proposed constructions also provide nonconvex Lyapunov functions, which are shown to be useful for systems with state-dependent switching that do not admit a convex Lyapunov function. Several examples are included to illustrate the results. The content of this chapter is based on the articles [49] and [51].

### 4.1 Overview

Starting with a finite family of vector fields  $\{f_1, \dots, f_M\} \subset \mathcal{C}^1(\mathbb{R}^n, \mathbb{R}^n)$  we are interested in studying stability properties in the switched systems context. As introduced



in Section 1.1.2, when the evolution of state trajectories results from *arbitrary switching* among the individual subsystems, the stability analysis problem is equivalently addressed by considering the differential inclusion (DI), described by

$$\dot{x} \in \text{co}\{f_i(x) \mid i \in \{1, \dots, M\}\}. \quad (4.2)$$

For the *linear* differential inclusion (LDI) case (that is  $f_i(x) = A_i x$  for some  $A_i \in \mathbb{R}^{n \times n}$ ), it is shown in [42], [92] that asymptotic stability is equivalent to the existence of a common Lyapunov function that is convex, homogeneous of degree 2, and  $\mathcal{C}^1(\mathbb{R}^n, \mathbb{R})$ . By addressing a similar question, the paper [90] establishes the existence of a common homogeneous polynomial Lyapunov function for asymptotically stable LDIs. Various parameterizations can approximate such homogeneous convex functions, such as maximum of quadratic functions and its convex conjugates [57], [59], which are shown to be universal in [68]. Constructions involving functions with convex polyhedral level sets are proposed in [92] and in [16]. Without any structural conditions, the Lyapunov functions for system (4.2) are, in general, not finitely constructible.

In contrast to studying stability uniformly over all possible switching signals as in (4.1), it is also of interest to study dynamical systems driven by a *given switching function*  $\sigma : \mathbb{R}^n \rightarrow \{1, \dots, M\}$ , resulting in

$$\dot{x} = f_{\sigma(x)}(x), \quad (4.3)$$

so that the solution set for system (4.3) is a strict subset of the solution set of system (4.2). As already mentioned, existence of a convex Lyapunov function is necessary for asymptotic stability of LDIs. However, it is possible that system (4.3) is asymptotically stable with  $\sigma$  fixed, but does not admit a convex Lyapunov function [17]. In the linear setting (i.e.  $f_i(x) \equiv A_i x$ , for some  $A_1, \dots, A_m \in \mathbb{R}^{n \times n}$ ) it is possible to provide sufficient conditions for a minimum of quadratics (clearly non-convex) to be a Lyapunov function in this context, see [69] and [128]. Several other techniques involving construction of common Lyapunov functions also exist: with piecewise quadratic functions [77], using parameter-dependent polytopic selection of quadratic functions [40, 91], LMI based formulation for multiple Lyapunov functions [44], [100], and polynomial Lyapunov functions using sum-of-squares techniques. Beyond piecewise quadratics, sum-of-squares techniques have been used for polynomial Lyapunov functions [99].

In this article, the problem of interest is to construct a Lyapunov function for systems (4.1) and (4.3) which guarantees asymptotic stability of the origin  $\{0\} \subset \mathbb{R}^n$ . We consider maps obtained by taking the maximum, minimum, or their combination over a finite family of continuously differentiable positive definite functions, see Definition 4.2 for details. Such *max-min* type of Lyapunov functions were recently proposed in the context of discrete-time switching systems [3], [101]. For the continuous-time case treated in this paper, studying this class of functions naturally requires certain additional tools from nonsmooth and set-valued analysis, and one such fundamental tool is the generalized directional derivative. An overview of these concepts for locally Lipschitz function was given in Section 2.4. The construction of non-smooth Lyapunov functions for system (4.3) using the Clarke's generalized

gradient concept is also presented in [11]. However, this notion turns out to be rather conservative as is seen in several examples (including the one given in Section 4.2). To overcome this conservatism due to Clarke's generalized derivative, we work with the *set-valued Lie derivative*. Focusing on this latter notion of generalized directional derivative for the class of max-min Lyapunov functions, the major contributions of this chapter are listed as follows:

- Describe max-min functions and study generalized notions of set-valued derivatives for such functions.
- Provide stability results for systems (4.2) and (4.3) using set-valued derivative notions.
- Obtain stability conditions using matrix inequalities for the case of linear vector fields in (4.1) and (4.3), and Lyapunov functions obtained by max-min of quadratics.

For the *max-min* candidate Lyapunov functions studied in this paper, which are not Clarke-regular in general, we compute set-valued Lie derivatives and use them to derive stability conditions for systems (4.1) and (4.3). The resulting conditions turn out to be less conservative than the ones obtained by using Clarke's derivative, which are here recovered as a corollary. When restricting the attention to the linear case  $f_i(x) = A_i x$ , and max-min functions obtained from quadratic forms, the Lie-derivative conditions require solving nonlinear matrix inequalities.

It should be noted that, since we allow for the minimum operation in the construction, certain elements in our proposed class of Lyapunov functions are nonconvex. In our approach, when we construct a homogeneous of degree 2 nonconvex Lyapunov function for the LDI problem, a convexification of such functions also provides a Lyapunov function [59, Proposition 2.2]. In fact, the sublevel sets of max-min functions approximate the convex sublevel sets of a homogeneous of degree 2 convex Lyapunov function (which is known to exist) with nonconvex sets obtained via intersections and unions of ellipsoids.

When addressing system (4.3), our approach provides a more general class of nonconvex and nondifferentiable Lyapunov functions obtained via max-min operations. To describe the solutions of switched systems, we adopt Filippov regularizations [54], and establish stability conditions for the resulting system. Considering such regularized differential inclusions for the switched systems also allows considering *sliding motions* along the switching surfaces. In this setting, our adopted notion of set-valued Lie derivative turns out to be crucial and has an interesting geometrical interpretation in terms of the tangent subspace to the switching surface.

### 4.1.1 Chapter Organization

The chapter is organized as follows: In Section 4.2 we provide an example of a two-dimensional switched system that does not admit a convex Lyapunov function, but a max-min Lyapunov function can be found. In Section 4.3 the class of *max-min* functions is presented and we show our main stability results in the setting of differential inclusions. In Section 4.4 we apply our results to switched systems, written

as a differential inclusion using Filippov regularizations, and we study asymptotic stability along with an instructive example. In Section 4.5, we analyze deeply the case of *linear* switched systems and propose an algorithmic procedure to construct max-min Lyapunov functions, followed by some concluding remarks in Section 4.6.

## 4.2 A Motivating Example

We consider a switched system for which there does not exist any convex Lyapunov function. However, this system is asymptotically stable and our results will allow constructing a Lyapunov function  $V$  defined as

$$V(x) := \max \left\{ \min \{x^\top P_1 x, x^\top P_2 x\}, x^\top P_3 x \right\}, \quad (4.4)$$

for some positive definite matrices  $P_i \in \mathbb{R}^{2 \times 2}$ ,  $i = 1, 2, 3$ .

**Example 4.1** Consider a state-dependent switched system as with three subsystems and a state-dependent switching rule  $x \mapsto \sigma(x) \in \{1, 2, 3\}$ , namely

$$\dot{x} = A_{\sigma(x)} x \quad (4.5)$$

where

$$(A_1, A_2, A_3) = \left( \left[ \begin{array}{cc} -0.1 & 1 \\ -5 & -0.1 \end{array} \right], \left[ \begin{array}{cc} -0.1 & 5 \\ -1 & -0.1 \end{array} \right], \left[ \begin{array}{cc} 1.9 & 3 \\ -3 & -2.1 \end{array} \right] \right).$$

To define the switching signal  $\sigma$ , introduce matrices

$$(Q_1, Q_2, Q_3) := \left( \left[ \begin{array}{cc} -(1 + \sqrt{2}) & -\frac{2 + \sqrt{2}}{2} \\ -\frac{2 + \sqrt{2}}{2} & -1 \end{array} \right], \left[ \begin{array}{cc} \frac{-1}{(1 + \sqrt{2})} & -\frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & -1 \end{array} \right], \left[ \begin{array}{cc} 1 & \sqrt{2} \\ \sqrt{2} & 1 \end{array} \right] \right)$$

and the switching signal

$$\sigma(x) := \begin{cases} 1, & \text{if } x \in \mathcal{S}_1 := \{x^\top Q_1 x > 0\} \cup \mathcal{S}_{13}, \\ 2, & \text{if } x \in \mathcal{S}_2 := \{x^\top Q_2 x > 0\} \cup \mathcal{S}_{21}, \\ 3, & \text{if } x \in \mathcal{S}_3 := \{x^\top Q_3 x > 0\} \cup \mathcal{S}_{32}, \end{cases} \quad (4.6)$$

where the subspaces  $\mathcal{S}_{ij}$ ,  $i \neq j$  are defined as  $\mathcal{S}_{ij} := \{x \in \mathbb{R}^2 \mid x^\top Q_i x = x^\top Q_j x\}$ , namely

$$\begin{aligned} \mathcal{S}_{13} &:= \left\{ x \in \mathbb{R}^2 \mid x_2 = -(1 + \sqrt{2})x_1 \right\}, \\ \mathcal{S}_{21} &:= \left\{ x \in \mathbb{R}^2 \mid x_2 = -x_1 \right\}, \\ \mathcal{S}_{32} &:= \left\{ x \in \mathbb{R}^2 \mid x_2 = -\frac{1}{1 + \sqrt{2}}x_1 \right\}. \end{aligned}$$

We note that in (4.6), we have  $\mathcal{S}_1 \cup \mathcal{S}_2 \cup \mathcal{S}_3 = \mathbb{R}^2$  and that the only point of intersection among the three sets is the origin.

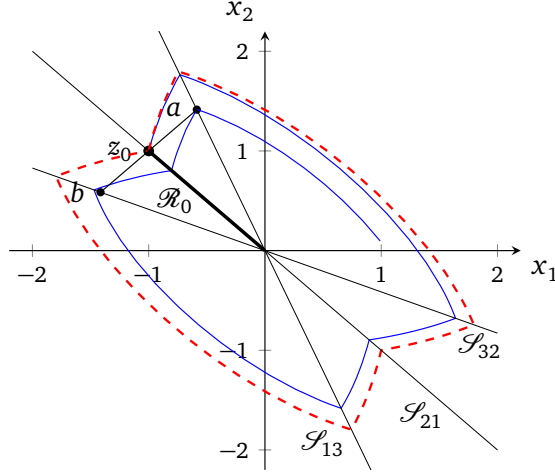


Figure 4.1: The solid blue line shows a trajectory of system (4.5) starting at  $z_0$  and moving in the clockwise direction. The red dashed line indicates a level set of the max-min Lyapunov function (4.4). The solid black line indicates the set  $\mathcal{R}_0$  used in the analysis.

In the following we note how the class of convex candidate Lyapunov functions is “not big enough” to handle this example, motivating the study of a class of locally Lipschitz functions not necessarily convex.

#### Proposition 4.1

There does not exist a convex Lyapunov function for system (4.5).

*Proof :* Given a set  $\mathcal{R}_0 \subset \mathbb{R}^2$  and a time  $T > 0$ , let  $\mathcal{C}(T; \mathcal{R}_0)$  be the set of reachable points of solutions of system (4.5) after time  $T$ , starting in  $\mathcal{R}_0$ , that is,

$$\mathcal{C}(T; \mathcal{R}_0) := \{x(t) \in \mathbb{R}^n \mid x \text{ solves (4.5), } x(0) \in \mathcal{R}_0, t \geq T\}.$$

Following [17, Lemma 2.1], if we show that there exists a compact set  $\mathcal{R}_0 \neq \{0\}$  and a  $T > 0$  such that  $\mathcal{R}_0 \subset \text{co}\{\mathcal{C}(T; \mathcal{R}_0)\}$ , then the system does not admit a convex Lyapunov function. Toward this end, we choose  $z_0 := [-1 \ 1]^\top \in \mathcal{S}_{21}$ , and the compact set  $\mathcal{R}_0 := \{\alpha z_0 : \alpha \in [0, 1]\} \subset \mathcal{S}_{21}$ , i.e. the line segment connecting 0 and  $z_0$ . We compute

$$\begin{aligned} e^{A_1 t} &= e^{-\frac{t}{10}} \begin{bmatrix} \cos(\sqrt{5}t) & \frac{\sqrt{5}}{5} \sin(\sqrt{5}t) \\ -\sqrt{5} \sin(\sqrt{5}t) & \cos(\sqrt{5}t) \end{bmatrix}, \\ e^{A_2 t} &= e^{-\frac{t}{10}} \begin{bmatrix} \cos(\sqrt{5}t) & \sqrt{5} \sin(\sqrt{5}t) \\ -\frac{\sqrt{5}}{5} \sin(\sqrt{5}t) & \cos(\sqrt{5}t) \end{bmatrix}, \\ e^{A_3 t} &= e^{-\frac{t}{10}} \begin{bmatrix} \frac{2}{\sqrt{5}} \sin(\sqrt{5}t) + \cos(\sqrt{5}t) & \frac{3}{\sqrt{5}} \sin(\sqrt{5}t) \\ -\frac{3}{\sqrt{5}} \sin(\sqrt{5}t) & \cos(\sqrt{5}t) - \frac{2}{\sqrt{5}} \sin(\sqrt{5}t) \end{bmatrix}, \end{aligned}$$

which allows us to write analytically the solution of the system starting from any given initial condition. We let  $t_1 > 0$  be the smallest time such that  $z_1 := e^{A_1 t_1} z_0 \in \mathcal{S}_{13}$ , and  $t_2 > t_1$  be the smallest time such that  $z_2 := e^{A_3(t_2-t_1)} z_1 \in \mathcal{S}_{32}$ . We finally choose  $t_3 > t_2$  as the smallest time such that  $z_3 := e^{A_2(t_3-t_2)} z_2 \in \mathcal{S}_{21}$ . It turns out that  $|z_3| = 1.2671$ . Thus, the half turn, starting with  $z_0 \in \mathcal{S}_{21}$  and reaching  $z_3 \in \mathcal{S}_{21}$ , decreases the norm of the state by a factor of  $\beta := \frac{|z_3|}{|z_0|} = 0.8961$ . Due to the central symmetry of the dynamics (that is, if  $x$  is a solution, then  $-x$  is also a solution) the solution will reach the set  $\mathcal{R}_0$  at the point  $\beta^2[-1 \ 1]^\top$  at time  $\tilde{t}_3 = 2t_3$ . Hence, the set  $\mathcal{R}_1 := \{\alpha z_0 : \alpha \in [0, \beta^2]\}$  is (strictly) contained in the set  $\mathcal{C}(t_1; \mathcal{R}_0)$ . To show that  $\mathcal{R}_0 \subset \text{co}\{\mathcal{C}(t_1; \mathcal{R}_0)\}$ , it thus remains to check that

$$\{\alpha z_0 : \alpha \in [\beta^2, 1]\} \subset \text{co}\{\mathcal{C}(t_1; \mathcal{R}_0)\}. \quad (4.7)$$

Property (4.7) is graphically illustrated in Figure 4.1 and is proven by the fact that points  $a = [\sqrt{2} - 2 \ \sqrt{2}]^\top \in \mathcal{S}_{13}$  and  $b = [-\sqrt{2} \ 2 - \sqrt{2}]^\top \in \mathcal{S}_{32}$  satisfy  $|a| < |z_1|$  and  $|b| < |e^{A_3(t_2-t_1)} e^{A_1 t_1} z_3|$ , and thus  $a, b \in \mathcal{C}(t_1; \mathcal{R}_0)$ , and  $z_0 = \frac{1}{2}a + \frac{1}{2}b \in \text{co}\{\mathcal{C}(t_1; \mathcal{R}_0)\}$ . Having already shown that  $0 \in \text{co}\{\mathcal{C}(t_1; \mathcal{R}_0)\}$ , property (4.7) indeed holds.  $\diamond$

In Section 4.4, we will study conditions that lead to the construction of a Lyapunov function for state-dependent switched systems. In particular, for the aforementioned example, we will find matrices  $P_i > 0$ ,  $i = 1, 2, 3$  to show that the function  $V$  in (4.4) is a Lyapunov function.

## 4.3 Stability Using Max-Min Functions

In this section, considering a particular class of locally Lipschitz functions, we use the generalized derivatives, establishing sufficient stability conditions for system (4.1).

### 4.3.1 Max-Min Functions

The following definition was introduced by [101] in the context of path-complete Lyapunov functions for discrete time switching systems.

#### Definition 4.2: Max-Min Functions

Consider an open and connected set  $\mathcal{O} \subset \mathbb{R}^n$ . Given  $K$  base functions  $V_1, \dots, V_K \in \mathcal{C}^1(\mathcal{O}, \mathbb{R})$ , a *max-min function*  $V_{\text{Mm}} : \mathcal{O} \rightarrow \mathbb{R}$  is either defined as

$$V_{\text{Mm}}(x) := \max_{j \in \{1, \dots, J\}} \left\{ \min_{k \in S_j} \{V_k(x)\} \right\}, \quad (4.8a)$$

for some  $J \geq 1$  and nonempty sets  $S_1, \dots, S_J \subset \{1, \dots, K\}$ , and, similarly, a *min-max function* is defined as

$$V_{\text{mM}}(x) = \min_{j \in \{1, \dots, J^*\}} \left\{ \max_{k \in S_j^*} \{V_k(x)\} \right\}, \quad (4.8b)$$

for some  $J^* \geq 1$  and nonempty sets  $S_1^*, \dots, S_{J^*}^* \subset \{1, \dots, K\}$ .

The following proposition states the equivalence between (4.8a) and (4.8b), which is obtained by applying the distributive property of the max min operators. For a formal proof we refer to [97] and references therein. In the sequel, all our derivations apply to both equivalent expressions (4.8a) and (4.8b) but for definiteness, we use the notation adopted in (4.8a).

### Proposition 4.3

Given  $J \geq 1$  (resp.  $J^* \geq 1$ ), and  $S_1, \dots, S_J$  (resp.  $S_1^*, \dots, S_{J^*}^*$ ) nonempty subsets of  $\{1, \dots, K\}$ , there exists  $J^* \geq 1$  (resp.  $J$ ) and nonempty subsets  $S_1^*, \dots, S_{J^*}^*$  (resp.  $S_1, \dots, S_J$ ) of  $\{1, \dots, K\}$  such that expressions (4.8a) and (4.8b) coincide, for all  $x \in \mathcal{O}$ , and for any  $V_1, \dots, V_K \in \mathcal{C}^1(\mathcal{O}, \mathbb{R}^n)$ .

We denote by  $\mathbf{Mm}(V_1, \dots, V_K)$  the set of all the possible max-min functions obtained from  $K$  base functions  $V_1, \dots, V_K$ . Given  $V \in \mathbf{Mm}(V_1, \dots, V_K)$ , it is noted that at each point  $x \in \mathcal{O}$  where a strict ordering holds between the values of the base functions, that is,  $V_{\ell_1}(x) < V_{\ell_2}(x) < \dots < V_{\ell_K}(x)$ , the function value  $V(x)$  coincides with  $V_{\tilde{\ell}}(x)$ , for some  $\tilde{\ell} \in \{1, \dots, K\}$ . At points where two or more base functions are equal, the function  $V$  may switch between different base functions. For every  $\ell \in \{1, \dots, K\}$ , we may define the set where the function  $V_\ell$  is active, more precisely

$$C_\ell := \{x \in \mathcal{O} \mid V(x) = V_\ell(x)\}, \quad (4.9)$$

which are closed by continuity of  $V, V_1, \dots, V_K$ . We can thus associate a mapping to every  $V \in \mathbf{Mm}(V_1, \dots, V_K)$ .

### Definition 4.4: Essentially-active index map

Given a function  $V \in \mathbf{Mm}(V_1, \dots, V_K)$ , the corresponding *essentially-active index map*  $\alpha_V : \mathcal{O} \rightrightarrows \{1, \dots, K\}$  is defined as

$$\alpha_V(x) := \{\ell \in \{1, \dots, K\} \mid x \in \overline{\text{int}(C_\ell)}\}, \quad (4.10)$$

Indexes  $\ell \in \alpha_V(x)$  are called *essentially-active indexes* of  $V$  at  $x$ .

We highlight that

$$\alpha_V(x) \subset \{\ell \in \{1, \dots, K\} \mid V(x) = V_\ell(x)\}, \quad \forall x \in \mathcal{O}. \quad (4.11)$$

To obtain the inclusion (4.11), consider any  $\ell \in \alpha_V(x)$ , then from Definition 4.4 and  $\mathcal{O}$  being open, there is a sequence  $x_k \rightarrow x$  such that  $x_k \in \text{int}(C_\ell)$ ,  $\forall k \in \mathbb{N}$ . By continuity of  $V$  and  $V_\ell$ , we have  $V(x) = \lim_{k \rightarrow \infty} V(x_k) = \lim_{k \rightarrow \infty} V_\ell(x_k) = V_\ell(x)$ .

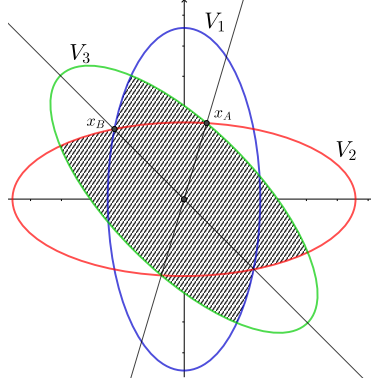


Figure 4.2: A particular sublevel set of  $V(x) = \max\{V_3(x), \min\{V_1(x), V_2(x)\}\}$ .

#### Remark 4.5: Max-Min functions as Piecewise $\mathcal{C}^1$ functions

Given  $V_1, \dots, V_K \in \mathcal{C}^1(\mathcal{O}, \mathbb{R}^n)$ , it is easy to prove that the class of functions  $\mathbf{Mm}(V_1, \dots, V_K)$  is in particular a subset of the more general class of *piecewise  $\mathcal{C}^1$  functions* as introduced in Definition 3.5. In this sense, the map  $\alpha_V : \mathcal{O} \rightrightarrows \{1, \dots, K\}$ , corresponds to the map  $\mathcal{I}^e$  introduced in (3.20); the  $\alpha_V$ -notation was chosen only for readability reasons. Moreover the set appearing in the right-hand side of inclusion (4.11) is called *active index set* in the context of piecewise  $\mathcal{C}^1$  functions, for example in [98] and [110, Chapter 4]. In the following we will use results from the piecewise  $\mathcal{C}^1$  theory, but we would like to underline how functions in  $\mathbf{Mm}(V_1, \dots, V_K)$  have even stronger properties, due to its structure. This structure will be useful in what follows to develop algorithms based on max-min candidate Lyapunov functions.

We emphasize that, in general, the inclusion in (4.11) is strict and equality does not necessarily hold. To see this, and better understand the definition of  $\alpha_V$ , let us consider an example with three quadratic and positive definite base functions  $V_1, V_2, V_3 : \mathbb{R}^2 \rightarrow \mathbb{R}$  and the max-min function  $V(x) = \max\{V_3(x), \min\{V_2(x), V_1(x)\}\}$  of the form (4.4). In Figure 4.2, we show a particular sublevel set of  $V$  and highlight two specific points  $x_A, x_B$  for checking the definition of  $\alpha_V$ . Locally around  $x_A$ , function  $V$  coincides with  $V_3$  and thus for (4.10) we have  $\alpha_V(x_A) = \{3\}$ , even though the active index set is  $\{2, 3\}$ , because  $x_A \notin (\text{int}(C_2))$ . On the other hand, if we consider  $x_B$ , where  $V_1(x_B) = V_2(x_B)$ , we have  $\alpha_V(x_B) = \{1, 2\}$  because  $V$  is equal to both  $V_1$  and  $V_2$  in some open sets around  $x_B$ .

Given  $x \in \mathcal{O}$ , the essentially active index set  $\alpha_V(x)$  encodes all the necessary informations to locally describe the function  $V$ , as formalized in what follows.

#### Lemma 4.6

Consider  $V \in \mathbf{Mm}(V_1, \dots, V_K)$ . For each  $x \in \mathcal{O}$  the set  $\alpha_V(x)$  is non empty and there exists a neighborhood  $\mathcal{U}$  of  $x$  such that

$$(z \in \mathcal{U}) \Rightarrow (\exists \ell_z \in \alpha_V(x) \text{ such that } V(z) = V_{\ell_z}(z)). \quad (4.12)$$

See [110, Propostion 4.1.1] for the proof in the general case of piecewise  $\mathcal{C}^1$  functions.

### 4.3.2 Stability Conditions

The following statement draws connections between Clarke's generalized gradient  $\partial V$  and the set-valued Clarke and Lie derivatives  $\dot{V}_F$  and  $\bar{\dot{V}}_F$  (we refer to Section 2.4 for the formal definitions) for a generic  $V \in \mathbf{Mm}(V_1, \dots, V_K)$ , using the map  $\alpha_V$ .

#### Proposition 4.7: Generalized Derivatives for Max-Min Functions

Given  $V \in \mathbf{Mm}(V_1, \dots, V_K)$  and  $x \in \mathcal{O}$ , the following equality holds

$$\partial V(x) = \text{co}\{\nabla V_\ell(x) \mid \ell \in \alpha_V(x)\}. \quad (4.13)$$

In particular, given  $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ , the Clarke and Lie derivatives read

$$\dot{V}_F(x) = \text{co}\{\langle \nabla V_\ell(x), f \rangle \mid \ell \in \alpha_V(x), f \in F(x)\} \quad (4.14)$$

$$\bar{\dot{V}}_F(x) = \{a \in \mathbb{R} \mid \exists f \in F(x) : a = \langle \nabla V_\ell(x), f \rangle, \forall \ell \in \alpha_V(x)\}. \quad (4.15)$$

*Proof:* The proof of (4.13), together with the general statement in the piecewise  $\mathcal{C}^1$  setting, can be found in Lemma 3.7. Equivalences (4.14) and (4.15) then follow recalling Definitions 2.16 and 2.22.  $\diamond$

We propose sufficient conditions for asymptotic stability of system (4.1) in terms of  $\bar{\dot{V}}_F$  given in (4.15), while adopting the convention  $\max \emptyset = -\infty$ . The following statement can be seen as a specification of the general result presented in Theorem 2.24, while restricting the class of candidate Lyapunov function to the set  $\mathbf{Mm}(V_1, \dots, V_K)$ .

#### Theorem 4.8: Max-Min Lyapunov Functions

Given system (4.1), an open and connected set  $\mathcal{O} \subset \mathbb{R}^n$  such that  $0 \in \mathcal{O}$ , and  $K$  positive-definite functions  $V_1, \dots, V_K \in \mathcal{C}^1(\mathcal{O}, \mathbb{R})$ , consider a max-min function  $V \in \mathbf{Mm}(V_1, \dots, V_K)$  with  $\bar{\dot{V}}_F$  given in (4.15). If there exists a function  $\gamma \in \mathcal{PD}$  such that, for every  $x \in \mathcal{O}$ ,

$$\max \bar{\dot{V}}_F(x) \leq -\gamma(|x|), \quad (4.16)$$

then  $V$  is a Lyapunov function and system (4.1) is locally AS. If  $\mathcal{O} = \mathbb{R}^n$  and in addition, each  $V_j$ ,  $j \in \{1, \dots, K\}$ , is radially unbounded, then the origin of (4.1) is GAS.

This result can be seen as a corollary of the more general statement presented in Theorem 2.24, since, by Lemma 2.20, functions in  $\mathbf{Mm}(V_1, \dots, V_K)$  are in particular non-pathological, since the class of non-pathological is in particular closed under pointwise maximum or minimum. On the other hand, we present in what follows an alternative and direct proof, which rely only on Definition 4.2. This reasoning



underlines how the properties of pointwise maximum and minimum operators somehow are reflected in the differential properties of functions  $V \in \mathbf{Mm}(V_1, \dots, V_K)$ , and for that reason it is presented here.

A fundamental result for proving Theorem 4.8 appears in Lemma 4.9 given below. The proof of Lemma 4.9 with some related discussions is deferred to Section 4.3.4.

#### Lemma 4.9

Consider a function  $V \in \mathbf{Mm}(V_1, \dots, V_K)$  and a solution  $\phi : [0, T) \rightarrow \mathcal{O}$  of the differential inclusion (4.1). For  $t \in [0, T)$ ,

$$\frac{d}{dt}V(\phi(t)) \text{ exists almost everywhere and} \quad (4.17a)$$

$$\frac{d}{dt}V(\phi(t)) \in \bar{V}_F(\phi(t)) \text{ almost everywhere.} \quad (4.17b)$$

*Proof* : [Proof of Theorem 4.8] Since, by Definition 4.2,  $V$  involves min/max operations over positive definite functions  $V_1, \dots, V_K$ , it follows that  $V$  is positive definite and, as shown in [82, Lemma 4.3], there exist class  $\mathcal{K}$  functions  $\underline{\chi}$  and  $\bar{\chi}$  such that

$$\underline{\chi}(|x|) \leq V(x) \leq \bar{\chi}(|x|), \quad \forall x \in \mathcal{O}. \quad (4.18)$$

For the local version, there exists  $\delta > 0$ , and a constant  $c > 0$  such that  $\bar{\mathbb{B}}(0, \delta) \subset \mathcal{E}_V(c) \subset \mathcal{O}$ , where  $\mathcal{E}_V(c) := \{x \in \mathbb{R}^n \mid V(x) \leq c\}$  denotes a sublevel set of  $V$ . Given a solution  $\phi(\cdot)$  of system (4.1) with  $\phi(0) \in \bar{\mathbb{B}}(0, \delta)$ , the function  $V \circ \phi$  is absolutely continuous. By Lemma 4.9,  $\frac{d}{dt}(V \circ \phi)(t)$  exists almost everywhere and by (4.16), we have

$$\frac{d}{dt}V(\phi(t)) \leq -\gamma(|\phi(t)|) \leq 0, \quad (4.19)$$

almost everywhere. Due to the right inequality in (4.19), any solution  $\phi(\cdot)$  starting in  $\bar{\mathbb{B}}(0, \delta)$  remains in  $\mathcal{E}_V(c)$  for all  $t$ , therefore it is defined on  $\mathbb{R}_{\geq 0}$  (i.e. it is complete). Thus, we have  $\frac{d}{dt}V(\phi(t)) \leq -\gamma \circ \bar{\chi}^{-1}(V(\phi(t)))$ , for almost all  $t \in \mathbb{R}_{\geq 0}$ , which ensures asymptotic stability of the origin by standard comparison results.

For the global version, since  $V_1, \dots, V_K$  are radially unbounded, then so is  $V$ , and we can choose  $\underline{\chi}, \bar{\chi}$  in (4.18) of class  $\mathcal{K}_\infty$ . In this case, if additionally (4.16) holds for all  $x \in \mathbb{R}^n$ , then the origin is GAS.  $\diamond$

Recalling that, given any locally Lipschitz function  $V : \mathbb{R}^n \rightarrow \mathbb{R}$  and any set valued map  $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ , it holds that  $\bar{V}_F(x) \subset \dot{V}_F(x)$  for any  $x \in \mathbb{R}^n$ , we can state the following result specifically for (4.2), using the notion of Clarke's generalized derivative, which is generally more conservative than Theorem 4.8.

**Corollary 4.10**

Consider the DI (4.2). Given an open and connected set  $\mathcal{O} \subset \mathbb{R}^n$  such that  $0 \in \mathcal{O}$  and  $K$  positive-definite functions  $V_1, \dots, V_K \in \mathcal{C}^1(\mathcal{O}, \mathbb{R})$ , consider a max-min function  $V \in \mathbf{Mm}(V_1, \dots, V_K)$ . Suppose that there exists a function  $\gamma \in \mathcal{PD}$ , such that for all  $x \in \mathcal{O}$ ,

$$\langle \nabla V_\ell(x), f_i(x) \rangle \leq -\gamma(|x|), \quad \forall \ell \in \alpha_V(x), \quad (4.20)$$

for all  $i \in \{1, \dots, M\}$ . Then the origin of (4.2) is AS and  $V$  is a Lyapunov function for system (4.2). If  $\mathcal{O} = \mathbb{R}^n$ , and in addition, each  $V_j$ ,  $j \in \{1, \dots, K\}$ , is radially unbounded, then the origin of (4.2) is GAS.

*Proof:* Consider a point  $x \in \mathcal{O}$ , and suppose that  $\alpha_V(x) = \{\ell_1, \dots, \ell_p\}$ . Recalling (4.13), for each  $v \in \partial V(x)$ , there exist  $\lambda_1, \dots, \lambda_p \geq 0$ ,  $\sum_{j=1}^p \lambda_j = 1$ , such that  $v = \sum_{j=1}^p \lambda_j \nabla V_{\ell_j}(x)$ . Consequently, for each  $i \in \{1, \dots, M\}$ , (4.20) yields

$$\langle v, f_i(x) \rangle = \sum_{j=1}^p \lambda_j \langle \nabla V_{\ell_j}(x), f_i(x) \rangle \leq -\sum_{j=1}^p \lambda_j \gamma(|x|) = -\gamma(|x|),$$

which implies that  $\langle v, f \rangle \leq -\gamma(|x|)$ , for each  $v \in \partial V(x)$ , and every  $f \in \text{co}\{f_i(x) \mid i \in \{1, \dots, M\}\}$ , i.e.  $\max \dot{V}_F(x) \leq -\gamma(|x|)$ , for all  $x \in \mathcal{O}$ . Since  $\dot{\bar{V}}_F(x) \subset \dot{V}_F(x)$ , the result follows from Theorem 4.8.  $\diamond$

This result, as the precedent, can be seen as the specification, for max min functions, of the general statement presented in Theorem 2.18. While Theorem 4.8 is stated for a general differential inclusion (4.1), we chose to present Corollary 4.10 for system (4.2) only. The main reason is that the right-hand side of (4.2) is a *continuous* set-valued map, and in this case it is not restrictive to consider the Clarke gradient-based condition (cfr. Section 2.4 and Proposition 3.1).

In the following subsection, we provide a geometric interpretation of the set-valued Lie derivatives for max-min functions.

**4.3.3 Geometry of  $\dot{\bar{V}}_F(x)$** 

The developments in this subsection are of independent interest and provide insight into the stability conditions given in the previous section. To avoid excessive notation we consider the case  $\mathcal{O} = \mathbb{R}^n$ . Given  $V_1, \dots, V_K \in \mathcal{C}^1(\mathbb{R}^n, \mathbb{R})$ , consider a function  $V \in \mathbf{Mm}(V_1, \dots, V_K)$  and a point  $\tilde{x}$  where  $V$  is not differentiable. Denote  $\alpha_V(\tilde{x}) = \{\ell_1, \dots, \ell_p\}$ . It follows from (4.11) that

$$\tilde{x} \in \mathcal{S} := \{x \in \mathbb{R}^n \mid V_{\ell_1}(x) = \dots = V_{\ell_p}(x)\}. \quad (4.21)$$

Consider the map  $G : \mathbb{R}^n \rightarrow \mathbb{R}^{p-1}$  defined by

$$G(x) := \begin{pmatrix} V_{\ell_1}(x) - V_{\ell_2}(x) \\ V_{\ell_1}(x) - V_{\ell_3}(x) \\ \vdots \\ V_{\ell_1}(x) - V_{\ell_p}(x) \end{pmatrix} \quad (4.22)$$

then  $\tilde{x} \in \mathcal{S}$  in (4.21) is equivalent to the condition  $G(\tilde{x}) = 0 \in \mathbb{R}^{p-1}$ . In order to apply the inverse function theorem we suppose that the Jacobian matrix of  $G$  at  $\tilde{x}$  has maximal rank, that is,

$$\text{rank}(J_G(\tilde{x})) = \text{rank} \begin{pmatrix} \nabla V_{\ell_1}(\tilde{x}) - \nabla V_{\ell_2}(\tilde{x}) \\ \nabla V_{\ell_1}(\tilde{x}) - \nabla V_{\ell_3}(\tilde{x}) \\ \vdots \\ \nabla V_{\ell_1}(\tilde{x}) - \nabla V_{\ell_p}(\tilde{x}) \end{pmatrix} = p - 1,$$

namely  $\tilde{x}$  is a *regular point* of  $G$ . By [13, Corollary 1.3.7], if  $\tilde{x}$  is a regular point, locally around  $\tilde{x}$  the set  $\mathcal{S}$  is a  $(n - p + 1)$ -dimensional  $\mathcal{C}^1$ -manifold, and then the tangent space  $T_{\mathcal{S}}(\tilde{x})$  is well defined.

#### Proposition 4.11

Given  $V_1, \dots, V_K \in \mathcal{C}^1(\mathbb{R}^n, \mathbb{R})$ , let us consider a function  $V \in \mathbf{Mm}(V_1, \dots, V_K)$ . If  $\tilde{x}$  is a regular point of  $G : \mathbb{R}^n \rightarrow \mathbb{R}^{p-1}$  given in (4.22) with  $p \leq K$ , then the Lie derivative in (4.15) satisfies

$$\dot{\bar{V}}_F(\tilde{x}) = \{\langle \nabla V_{\ell}(\tilde{x}), f \rangle \mid \forall \ell \in \alpha_V(\tilde{x}), \forall f \in T_{\mathcal{S}}(\tilde{x})\}, \quad (4.23)$$

where  $\mathcal{S}$  is given in (4.21).

*Proof* : Using Proposition 4.7, it is enough to show equivalence between (4.15) and (4.23). To this end we need to show that a vector  $f \in T_{\mathcal{S}}(\tilde{x})$  if and only if

$$\nabla V_{\ell_1}(\tilde{x})^\top f = \dots = \nabla V_{\ell_p}(\tilde{x})^\top f. \quad (4.24)$$

By definition of the tangent space,  $f \in T_{\mathcal{S}}(\tilde{x})$  if and only if there exists a differentiable curve  $\phi : (-1, 1) \rightarrow \mathbb{R}^n$  with  $\phi(t) \in \mathcal{S}$  for all  $t \in (-1, 1)$  such that  $\dot{\phi}(0) = f$ ,  $\phi(0) = \tilde{x}$ . From the definition of  $\mathcal{S}$  in (4.21), we have  $G(\phi(t)) = 0$  for all  $t \in (-1, 1)$ , therefore

$$\begin{aligned} 0 &= \frac{d}{dt} (V_{\ell_j}(\phi(t)) - V_{\ell_k}(\phi(t))) \Big|_{t=0} \\ &= \nabla V_{\ell_j}(\phi(0))^\top \dot{\phi}(0) - \nabla V_{\ell_k}(\phi(0))^\top \dot{\phi}(0) \\ &= \nabla V_{\ell_j}(\tilde{x})^\top f - \nabla V_{\ell_k}(\tilde{x})^\top f, \end{aligned}$$

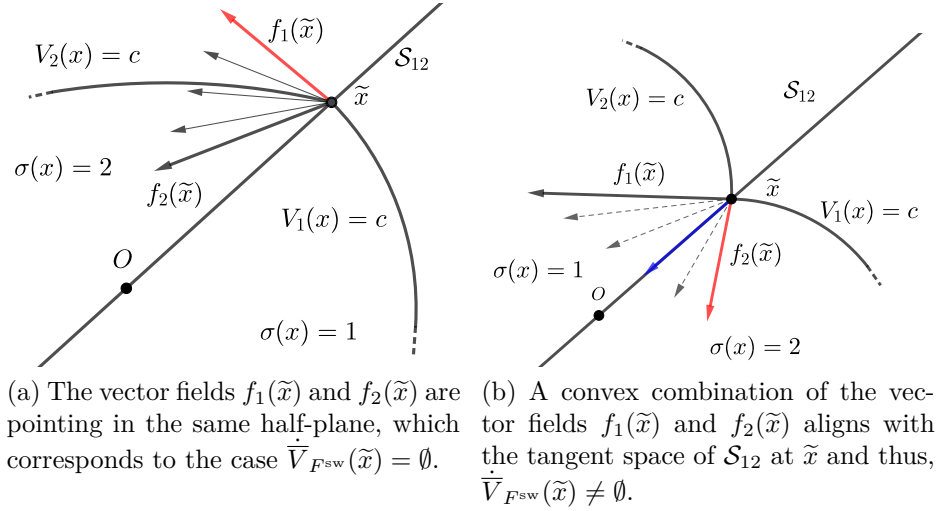


Figure 4.3: A geometric interpretation of the set  $\dot{V}_{F^{sw}}(\tilde{x})$  in  $\mathbb{R}^2$ .

for all  $\ell_j, \ell_k \in \alpha_V(x)$ , and hence (4.24) holds.  $\diamond$

Summarizing, given a point  $x \in \mathbb{R}^n$  where the function  $V \in \mathbf{Mm}(V_1, \dots, V_K)$  is not differentiable, and under the assumption that  $x$  is a regular point, this proposition gives us an intuitive criterion to establish if the set  $\dot{V}_F(x)$  is empty or not. Roughly speaking  $\dot{V}_F(x)$  is nonempty if and only if there exists at least one vector  $f \in F(x)$  that points inside the surface  $\mathcal{S}$  in (4.21), where  $V$  is not differentiable.

In Figure 4.3 we propose a graphic representation in the plane of this result in the state-dependent switching case.

#### 4.3.4 Proof of Lemma 4.9

Lemma 4.9 is the key result used in the proof of Theorem 4.8, establishing properties of the directional derivative of  $V \in \mathbf{Mm}(V_1, \dots, V_K)$  along the solutions of (4.1). In its proof we will use the following result.

**Claim 1** *Given functions  $\xi_1, \dots, \xi_J : \mathbb{R} \rightarrow \mathbb{R}$  continuous at 0, we have that*

$$\lim_{h \rightarrow 0} \min_{j \in \{1, \dots, J\}} \xi_j(h) = \min_{j \in \{1, \dots, J\}} \lim_{h \rightarrow 0} \xi_j(h).$$

*Proof :* [Proof of Claim 1] Define  $\xi(h) := \min_{j \in \{1, \dots, J\}} \xi_j(h)$  for all  $h \in \mathbb{R}$ ;  $\xi$  is continuous at 0 since it is the pointwise minimum of continuous functions. We have

$$\begin{aligned} \lim_{h \rightarrow 0} \min_{j \in \{1, \dots, J\}} \xi_j(h) &= \lim_{h \rightarrow 0} \xi(h) = \xi(0) \\ &= \min_{j \in \{1, \dots, J\}} \xi_j(0) = \min_{j \in \{1, \dots, J\}} \lim_{h \rightarrow 0} \xi_j(h), \end{aligned}$$

thus concluding the proof.  $\diamond$

*Proof :* [Proof of Lemma 4.9] Recalling that  $\phi(\cdot)$  is an absolutely continuous solution of the differential inclusion (4.1) and that  $V$  is a locally Lipschitz function, the function  $V \circ \phi : [0, T) \rightarrow \mathbb{R}$  is absolutely continuous, and hence  $\frac{d}{dt} V(\phi(t))$  exists

almost everywhere in  $[0, T)$ , proving (4.17a). Moreover, there exists a set  $\mathcal{N}_0$  of measure zero such that, for every  $t \in [0, T) \setminus \mathcal{N}_0$ , both  $\dot{\phi}(t)$  and  $\frac{d}{dt}V(\phi(t))$  exist, and  $\dot{\phi}(t) \in F(\phi(t))$ .

To prove (4.17b), from Proposition 4.3, we use the equivalent representation (4.8b) of  $V \in \mathbf{Mm}(V_1, \dots, V_K)$ , dropping the superscript “ $\star$ ” for notational simplicity, that is

$$V(x) := \min_{j \in \{1, \dots, J\}} \left\{ \max_{\ell \in S_j} \{V_\ell(x)\} \right\},$$

where  $J \geq 0$  and  $S_1, \dots, S_J$  are non-empty subsets of  $\{1, \dots, K\}$ . By Lemma 4.6, for each  $t \in [0, T)$ , and for each  $x$  in a neighborhood of  $\phi(t)$ ,  $V(x)$  can be expressed as

$$V(x) := \min_{j \in \{1, \dots, J\}} \left\{ \max_{\ell \in S_j \cap \alpha_V(\phi(t))} \{V_\ell(x)\} \right\};$$

namely, only the essentially active indexes in  $\alpha_V(\phi(t))$  play a role (possibly ruling out the sets  $S_j$  for which  $S_j \cap \alpha_V(\phi(t)) = \emptyset$ ). Let us introduce the notation

$$\bar{V}_j(x) := \max_{\ell \in S_j \cap \alpha_V(\phi(t))} \{V_\ell(x)\}. \quad (4.25)$$

To proceed in a constructive manner, consider the set  $\mathbf{M}(V_1, \dots, V_K)$  containing all the functions obtained by max (and only max) combination over  $V_1, \dots, V_K$ . The cardinality of  $\mathbf{M}(V_1, \dots, V_K)$  is finite and equal to  $N_K := 2^K - 1$  and we can denote its elements by  $W_k$ , for  $k \in \{1, \dots, N_K\}$ . Reasoning as before, for each  $k$  define  $\mathcal{N}_k$  as the subset of  $[0, T)$  where  $W_k \circ \phi$  is not differentiable. Since  $W_k$  are locally Lipschitz, then each  $\mathcal{N}_k$  has measure zero. Fix any  $t \in [0, T) \setminus (\bigcup_{k \in \{0, \dots, N_K\}} \mathcal{N}_k)$ . From the fact that  $\bar{V}_j$  in (4.25) is locally Lipschitz for each  $j \in \{1, \dots, J\}$ , we obtain

$$\frac{d}{dt} \bar{V}_j(\phi(t)) = \lim_{h \rightarrow 0} \frac{\bar{V}_j(\phi(t) + h\dot{\phi}(t)) - \bar{V}_j(\phi(t))}{h}, \quad (4.26)$$

where the limit exists because  $t \notin \bigcup_{k \in \{1, \dots, N_K\}} \mathcal{N}_k$ . The functions  $\bar{V}_j$  in (4.25) are regular (Definition 2.14). We can follow the idea of [9, Lemma 1]: by letting  $h$  go to zero from the right, recalling inclusion (4.11), we get

$$\frac{d}{dt} \bar{V}_j(\phi(t)) = \max_{\ell \in S_j \cap \alpha_V(\phi(t))} \left\{ \nabla V_\ell(\phi(t))^\top \dot{\phi}(t) \right\}. \quad (4.27)$$

Similarly, by letting  $h$  go to zero from the left in (4.26), we get

$$\frac{d}{dt} \bar{V}_j(\phi(t)) = \min_{\ell \in S_j \cap \alpha_V(\phi(t))} \left\{ \nabla V_\ell(\phi(t))^\top \dot{\phi}(t) \right\}. \quad (4.28)$$

Since  $\frac{d}{dt} \bar{V}_j(\phi(t))$  exists, we have (4.27)=(4.28), and thus for each  $j \in \{1, \dots, J\}$  we can write, for all  $\ell \in S_j \cap \alpha_V(\phi(t))$ ,

$$\frac{d}{dt} \bar{V}_j(\phi(t)) = \nabla V_\ell(\phi(t))^\top \dot{\phi}(t) =: a_j(t). \quad (4.29)$$

Now consider the function  $V(x) = \min_{j \in \{1, \dots, J\}} \{\bar{V}_j(x)\}$ , for  $x$  in some neighborhood of  $\phi(t)$ . For all  $h > 0$ , we use the fact that  $\bar{V}_j(\phi(t)) = V(\phi(t))$  for all  $j \in \{1, \dots, J\}$ , to obtain

$$\begin{aligned} \xi(h) &:= \frac{V(\phi(t) + h\dot{\phi}(t)) - V(\phi(t))}{h} \\ &= \frac{\min_j \{\bar{V}_j(\phi(t) + h\dot{\phi}(t))\} - V(\phi(t))}{h} \\ &= \min_{j \in \{1, \dots, J\}} \left\{ \frac{\bar{V}_j(\phi(t) + h\dot{\phi}(t)) - \bar{V}_j(\phi(t))}{h} \right\} =: \min_{j \in \{1, \dots, J\}} \xi_j(h). \end{aligned}$$

Then, applying Claim 1 and (4.29) we have

$$\begin{aligned} \frac{d}{dt}V(\phi(t)) &= \lim_{h \rightarrow 0^+} \min_{j \in \{1, \dots, J\}} \{\xi_j(h)\} = \min_{j \in \{1, \dots, J\}} \left\{ \lim_{h \rightarrow 0^+} \xi_j(h) \right\} \\ &= \min_{j \in \{1, \dots, J\}} \left\{ \frac{d}{dt} \bar{V}_j(\phi(t)) \right\} = \min_{j \in \{1, \dots, J\}} \{a_j(t)\}. \end{aligned} \quad (4.30)$$

Using again Claim 1, we can also write

$$\begin{aligned} \frac{d}{dt}V(\phi(t)) &= \lim_{h \rightarrow 0^-} \frac{V(\phi(t) + h\dot{\phi}(t)) - V(\phi(t))}{h} \\ &= - \lim_{h \rightarrow 0^-} \left( \min_{j \in \{1, \dots, J\}} \left\{ \frac{\bar{V}_j(\phi(t) + h\dot{\phi}(t)) - \bar{V}_j(\phi(t))}{-h} \right\} \right) \\ &= - \min_{j \in \{1, \dots, J\}} \left\{ \lim_{h \rightarrow 0^-} \frac{\bar{V}_j(\phi(t) + h\dot{\phi}(t)) - \bar{V}_j(\phi(t))}{-h} \right\} \\ &= - \min_{j \in \{1, \dots, J\}} \{-a_j(t)\} = \max_{j \in \{1, \dots, J\}} \{a_j(t)\}. \end{aligned} \quad (4.31)$$

Summarizing, from (4.30) and (4.31), it follows that  $a_1(t) = \dots = a_J(t) := \bar{a}(t)$ . Therefore, from (4.29) we get, for each  $j \in \{1, \dots, J\}$ , that  $\ell \in S_j \cap \alpha_V(\phi(t))$  implies  $\nabla V_\ell(\phi(t))^\top \dot{\phi}(t) = \bar{a}(t)$ . Finally, recalling that  $\alpha_V(\phi(t)) = \bigcup_j S_j \cap \alpha_V(\phi(t))$ , we have

$$\nabla V_\ell(\phi(t))^\top \dot{\phi}(t) = \bar{a}(t), \quad \forall \ell \in \alpha_V(\phi(t)).$$

From (4.15), it follows that  $\bar{a}(t) \in \dot{\bar{V}}_F(\phi(t))$ , which then implies (4.17b).  $\diamond$

## 4.4 Application to Switched Systems

We now focus our attention on system (4.3), and more specifically we will study the arising Filippov regularization. We briefly recall here the material presented in Section 2.2:

Consider a family of vector fields  $\{f_1, \dots, f_M\} \subset \mathcal{C}^1(\mathbb{R}^n, \mathbb{R}^n)$  in (4.3). A switching signal  $\sigma : \mathbb{R}^n \rightarrow \mathcal{I} := \{1, \dots, M\}$  switching signal is defined starting with a *proper partition*  $\mathcal{X} = \{X_i, \mathcal{O}_i\}_{i \in \{1, \dots, M\}}$  of the state space, as introduced in Definition 2.4. We recall here the formal definition:

**Definition 4.12: State-Dependent Switching Signal**

Given  $\mathcal{X} = \{X_i, \mathcal{O}_i\}_{i \in \mathcal{I}}$  a proper partition of  $\mathbb{R}^n$ , a *state-dependent switching signal* associated to  $\mathcal{X}$  is a function  $\sigma : \mathbb{R}^n \rightarrow \mathcal{I}$  such that

$$\sigma(x) = i, \quad \text{if } x \in \text{int}(X_i), \quad (4.32)$$

Note that, in Definition 4.12 the value of  $\sigma$  remains unspecified on  $\partial X_i$ , i.e. the boundaries of  $X_i$ ,  $i = \{1, \dots, M\}$ . Since  $\partial X_i$  by definition has zero Lebesgue measure, this ambiguity will not affect the solution set of (4.3).

Given  $f_1, \dots, f_M \in \mathcal{C}^1(\mathcal{O}_i, \mathbb{R}^n)$  and  $\sigma : \mathbb{R}^n \rightarrow \{1, \dots, M\}$  as in Definition 4.12, we define  $f^{\text{sw}} : \mathbb{R}^n \rightarrow \mathbb{R}^n$ , as

$$f^{\text{sw}}(x) := f_{\sigma(x)}(x). \quad (4.33)$$

Because the vector field in (4.33) is in general discontinuous, we define an appropriate notion of solution of (4.33), arising from the Filippov regularization.

**Definition 4.13: Filippov Regularization, [54]**

Given  $f^{\text{sw}} : \mathbb{R}^n \rightarrow \mathbb{R}^n$  in (4.33), and the system

$$\dot{x}(t) = f^{\text{sw}}(x(t)), \quad (4.34)$$

define the set-valued *Filippov regularization*

$$\dot{x} \in F^{\text{sw}}(x) := \bigcap_{\delta > 0} \bigcup_{\mu_L(S)=0} \overline{\text{co}}\{f_{\sigma(y)}(y) \mid y \in \mathbb{B}(x, \delta) \setminus S\}. \quad (4.35)$$

For the vector field  $f^{\text{sw}}$  in (4.33), the computation of  $F^{\text{sw}}$  is simplified as observed in Section 2.2 and summarized again here:

**Proposition 4.14**

Consider the vector field  $f^{\text{sw}}$  in (4.33) with  $\sigma$  as in Definition 4.12. Introduce again, the set-valued map  $\mathcal{I}_{\mathcal{X}} : \mathbb{R}^n \rightrightarrows \{1, \dots, M\}$  as

$$\mathcal{I}_{\mathcal{X}}(x) := \{i \mid x \in X_i\} \quad (4.36)$$

then  $F^{\text{sw}}$  in (4.35) satisfies

$$F^{\text{sw}}(x) = \text{co}\{f_i(x) \mid i \in \mathcal{I}_{\mathcal{X}}(x)\}. \quad (4.37)$$

We underline again that the Filippov regularization  $F^{\text{sw}}$  is an outer semicontinuous map with  $F^{\text{sw}}(x)$  being nonempty, compact, and convex for each  $x \in \mathbb{R}^n$ . Thus, we can study stability of switched systems in (4.33) using the results developed in Section 4.3. Defining  $\dot{V}_{F^{\text{sw}}}(x)$  as in (4.15) with  $F$  replaced by  $F^{\text{sw}}$ , Theorem 4.8 leads to the following statement in the context of switched systems.

**Theorem 4.15**

Consider system (4.3), and a switching law  $\sigma : \mathbb{R}^n \rightarrow \{1, \dots, M\}$  as in Definition 4.12. Consider an open and connected set  $\mathcal{O} \subset \mathbb{R}^n$  such that  $0 \in \mathcal{O}$  and  $K$  positive-definite functions  $V_1, \dots, V_K \in \mathcal{C}^1(\mathcal{O}, \mathbb{R})$ . If, for a max-min function  $V \in \mathbf{Mm}\{V_1, \dots, V_K\}$ , there exists  $\gamma \in \mathcal{PD}$  such that

$$\max \dot{\bar{V}}_{F^{\text{sw}}}(x) \leq -\gamma(|x|), \quad \forall x \in \mathcal{O}, \quad (4.38)$$

then the origin of (4.35) is AS. If  $\mathcal{O} = \mathbb{R}^n$ , and each  $V_j$ ,  $j \in \{1, \dots, K\}$ , is radially unbounded, then (4.35) is GAS.

Theorem 4.15 simultaneously accounts for points  $x$  where  $\mathcal{I}_{\mathcal{X}}(x)$  (associated to  $\sigma$ ), and/or points where  $\alpha_V(x)$  (associated to  $V$ ) are multivalued. Interesting things happen when these points coincide, namely when  $V$  mimics the patchy shape of  $F^{\text{sw}}$ .

As an example, consider the simplest non-trivial case, taking an  $\tilde{x} \in \mathcal{O}$  such that  $\mathcal{I}_{\mathcal{X}}(\tilde{x}) = \{1, 2\}$  and  $\alpha_V(\tilde{x}) = \{\ell_1, \ell_2\}$ , for some  $\ell_1, \ell_2 \in \{1, \dots, K\}$ . We may give a geometric interpretation of (4.38). Parameterizing an  $f \in F^{\text{sw}}(\tilde{x})$  with  $f = \lambda f_1(\tilde{x}) + (1 - \lambda)f_2(\tilde{x})$  in expression (4.15), we have that  $\dot{\bar{V}}_{F^{\text{sw}}}(\tilde{x}) \neq \emptyset$ , if and only if there exists  $\lambda \in [0, 1]$  such that (we omit the argument  $\tilde{x}$  of the gradients to simplify the notation),

$$\lambda(\nabla V_{\ell_1} - \nabla V_{\ell_2})^\top f_1(\tilde{x}) = -(1 - \lambda)(\nabla V_{\ell_1} - \nabla V_{\ell_2})^\top f_2(\tilde{x}),$$

which holds only if

$$\left( (\nabla V_{\ell_1} - \nabla V_{\ell_2})^\top f_1(\tilde{x}) \right) \left( (\nabla V_{\ell_1} - \nabla V_{\ell_2})^\top f_2(\tilde{x}) \right) \leq 0.$$

It follows that  $\dot{\bar{V}}_{F^{\text{sw}}}(\tilde{x}) \neq \emptyset$  only if the vector fields  $f_1(\tilde{x})$  and  $f_2(\tilde{x})$  are such that the inner product of their respective components, normal to the hypersurface  $\mathcal{S}_{12} = \{x \in \mathbb{R}^n \mid V_{\ell_1}(x) = V_{\ell_2}(x)\}$  is negative, namely they do not point both on the same side of  $\mathcal{S}_{12}$ . Figure 4.3 provides an illustration of this fact in the planar case.

In Example 4.2, an illustration of this idea is provided.

**Example 4.2** We consider a system of the form (4.3) and analyze its stability using Theorem 4.15. Given  $A_1 = \begin{bmatrix} -0.1 & 1 \\ -5 & -0.1 \end{bmatrix}$ ,  $A_2 = \begin{bmatrix} -0.1 & -5 \\ 1 & -0.1 \end{bmatrix}$  and  $Q = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ , consider the switched system

$$\dot{x} = \begin{cases} f_1(x) := A_1 x - b\tilde{g}(x), & \text{if } x^\top Q x < 0, \\ f_2(x) := A_2 x - b\tilde{g}(x), & \text{if } x^\top Q x > 0, \end{cases} \quad (4.39)$$

where  $b \geq 0$ , and function  $\tilde{g} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  is defined as

$$\tilde{g}(x_1, x_2) = \begin{bmatrix} g(x_1) \\ g(x_2) \end{bmatrix} = \begin{bmatrix} \arctan(x_1) \\ \arctan(x_2) \end{bmatrix}.$$



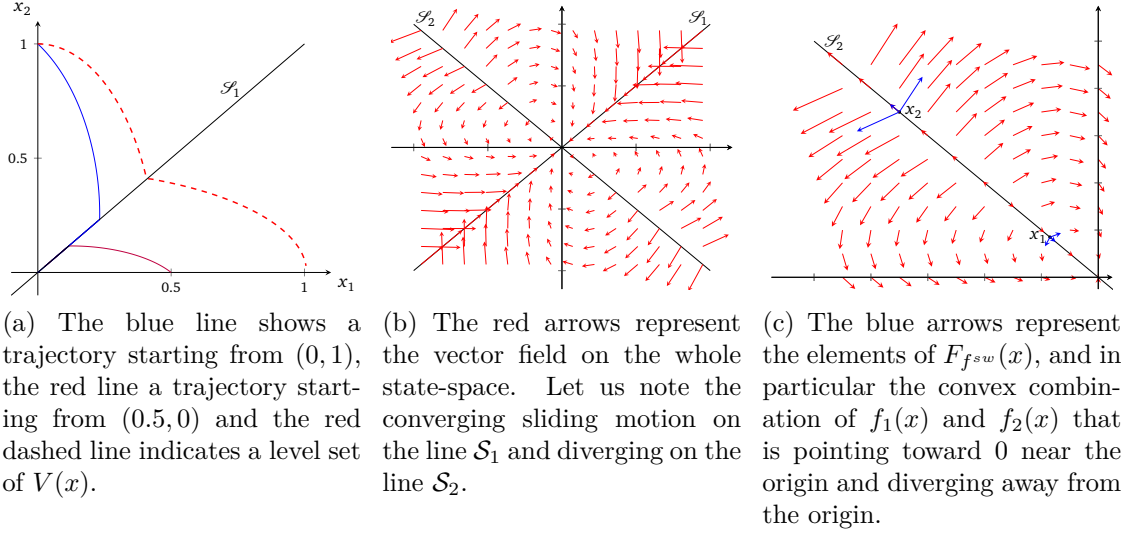


Figure 4.4: Trajectories of switched system (4.39) in Example 4.2.

It is clear the the proper partition defining the switching rule in (4.39) arises from the proper partition  $X_1 := \{x \in \mathbb{R}^2 \mid x^\top Qx \leq 0\}$ ,  $X_2 := \{x \in \mathbb{R}^2 \mid x^\top Qx \geq 0\}$  and  $\mathcal{O}_1, \mathcal{O}_2 = \mathbb{R}^2$ . Consider now  $P_1 = \begin{bmatrix} 5 & 0 \\ 0 & 1 \end{bmatrix}$ ,  $P_2 = \begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix}$ , we prove that

$$V(x) = \min\{x^\top P_1 x, x^\top P_2 x\}$$

is a Lyapunov function in the sense of Theorem 4.15. Noting that  $P_1 - P_2 = 4Q$ , we can say that the points where  $V$  is not differentiable coincide with the points where  $\sigma$  is not continuous. To show inequality (4.38), we proceed in three steps:

Step 1: Each subsystem is GAS. Analyzing each subsystem where  $V$  is differentiable, it can be shown that

$$\nabla V(x)^\top f \leq -0.1|x|^2, \quad \forall f \in F^{\text{sw}}(x), \quad \text{if } x^\top Qx \neq 0.$$

The next step is to check the inequality (4.38) where  $V$  is not differentiable, that is on the lines  $\mathcal{S}_1 := \{x \in \mathbb{R}^2 \mid x_2 = x_1\}$ , and  $\mathcal{S}_2 := \{x \in \mathbb{R}^2 \mid x_2 = -x_1\}$ , so that  $\mathcal{S}_1 \cup \mathcal{S}_2$  is the set where  $x^\top Qx = 0$ .

Step 2: Line  $\mathcal{S}_1$  with converging sliding motion. We compute the set-valued derivative  $\dot{\bar{V}}_{F^{\text{sw}}}(x)$  for a point  $x \in \mathcal{S}_1$ . Based on (4.15), it is seen that

$$\lambda x^\top (P_1 - P_2) f_1(x) + (1 - \lambda) x^\top (P_1 - P_2) f_2(x) = 0 \quad (4.40)$$

holds with  $\lambda = 0.5$ , for every  $x \in \mathcal{S}_1$ . Consequently, for each  $x \in \mathcal{S}_1$ , we have  $\dot{\bar{V}}_{F^{\text{sw}}}(x) = \left\{ 2x^\top P_1 \left( \frac{1}{2} f_1(x) + \frac{1}{2} f_2(x) \right) \right\} = \left\{ x^\top P_1 (A_1 x + A_2 x) - 2b x^\top P_1 \tilde{g}(x) \right\}$ . By construction, the same singleton would be obtained if we replaced  $P_1$  by  $P_2$ . Substituting the values of  $A_i$  and  $P_i$ ,  $i = 1, 2$ , it thus follows that  $\max \dot{\bar{V}}_{F^{\text{sw}}}(x) < -\frac{25}{2}|x|^2$ ,  $\forall x \in \mathcal{S}_1$ . In Figure 4.4(a), we have plotted two converging “sliding” solutions.

Step 3: Line  $\mathcal{S}_2$  with diverging sliding motion. Choosing  $x \in \mathcal{S}_2$ , and following the same reasoning as in Step 2, it is seen that the set  $\dot{\bar{V}}_{F^{\text{sw}}}(x)$  is nonempty because (4.40) holds with  $\lambda = 0.5$ , for every  $x \in \mathcal{S}_2$ . As a result,  $\dot{\bar{V}}_{F^{\text{sw}}}(x) =$

$\{x^\top P_1(A_1x + A_2x) + 2bx^\top P_1\tilde{g}(x)\}$ . Analyzing the linear term, we have  $x^\top P_1(A_1x + A_2x) = 22.8x_1^2$ ; for the nonlinear term, for each  $x \in \mathcal{S}_2$ , we have

$$-2bx^\top P_1\tilde{g}(x) = -12bx_1g(x_1).$$

For  $x_1$  small enough, we see that  $x_1g(x_1) = x_1 \arctan(x_1) = x_1^2 + o(x_1^2)$  where  $\lim_{x_1 \rightarrow 0} \frac{o(x_1^2)}{x_1^2} = 0$ . Thus, for sufficiently large values of  $b > 0$ , there exists a  $\delta > 0$  such that

$$\dot{V}_{F^{\text{sw}}}(x) = 22.8x_1^2 - 12bx_1^2 + o(x_1^2) < -0.1|x|^2, \quad (4.41)$$

if  $x \in \mathcal{S}_2$  and  $|x| < \delta$ .

Combining the three steps we proved (4.38) (with  $\gamma(|x|) = 0.1|x|^2$ ) for a small open neighborhood  $\mathcal{O} = \mathbb{B}(0, \delta)$  of the origin, and Theorem 4.15 establishes local asymptotic stability of the origin by using the minimum of two quadratics as a Lyapunov function. Condition (4.38) fails to be true on the line  $\mathcal{S}_2$ , away from the origin regardless of the selection of  $b > 0$ . Hence, there exist Filippov solutions, starting in  $\mathcal{S}_2$  with large enough initial condition that stay in  $\mathcal{S}_2$  and diverge; see Figure 4.4 for a graphical illustration. We want to underline that, since  $x^\top (A_1^\top P_2 + A_1 P_2)x > 0$  for all  $x \in \mathcal{S}_2$ , recalling (4.14), it holds that  $\max \dot{V}_F(x) > 0, \forall x \in \mathcal{S}_2$ . This observation again shows the utility of using Lie derivative compared to Clarke derivative which does not allow establishing asymptotic stability of the origin in this case. It is also interesting to note that the set of initial values leading to instable behavior (the line  $\mathcal{S}_2$ ) has measure zero, and thus the system can be consider as ‘‘GAS’’ almost everywhere. Moreover, since on  $\mathcal{S}_2$  we have repelling sliding, for all the initial condition in  $\mathcal{S}_2$  there exists at least one solution (in fact two) that is converging. This interesting property is deeply analyze in the recent paper [71], where a new concept of solutions avoiding this phenomenon is introduced.  $\triangle$

## 4.5 Linear Switched Systems and Quadratic Basis

We are now interested in applying Theorem 4.15 to switched systems (4.33) with *linear* vector fields and a partition given by symmetric cones. More precisely, given  $A_1, \dots, A_M \in \mathbb{R}^{n \times n}$ , we consider the differential inclusion

$$\dot{x} \in F_{\text{lin}}^{\text{sw}}(x) := \text{co}\{A_i x \mid i \in \mathcal{I}_{\mathcal{X}}(x)\}. \quad (4.42)$$

The set valued map  $\mathcal{I}_{\mathcal{X}} : \mathbb{R}^n \rightrightarrows \{1, \dots, M\}$  arises from a switching function  $x \mapsto \sigma(x)$  as in Definition 4.12, where the sets  $X_1, \dots, X_M \subset \mathbb{R}^n$  are defined by

$$X_i := \overline{\{x \in \mathbb{R}^n \mid x^\top Q_i x > 0\}}, \quad (4.43)$$

with properly chosen symmetric matrices  $Q_i \in \text{Sym}(\mathbb{R}^n) := \{R \in \mathbb{R}^{n \times n} \mid R^\top = R\}$  and  $Q_i$  not negative semidefinite for each  $i \in \{1, \dots, M\}$ . The sets  $X_i$  in (4.43) are *symmetric closed cones* (if  $x \in X_i$  then  $\lambda x \in X_i$  for all  $\lambda \in \mathbb{R} \setminus \{0\}$ ). The map  $\mathcal{I}_{\mathcal{X}} : \mathbb{R}^n \rightrightarrows \{1, \dots, M\}$  in (4.36), can be rewritten in this context as follows:

$$\mathcal{I}_{\mathcal{X}}(x) := \{i \in \{1, \dots, M\} \mid x^\top Q_i x \geq 0\}.$$

Indeed  $Q_i$  not negative semidefinite implies  $X_i = \{x \in \mathbb{R}^n \mid x^\top Q_i x \geq 0\}$ .

#### Remark 4.16

Another possible kind of partition of the state space arises by considering *polyhedral cones* (with a common vertex at the origin), that is sets  $X_1, \dots, X_M \subset \mathbb{R}^n$  defined by linear inequalities  $X_i := \{x \in \mathbb{R}^n \mid K_i x \geq_c 0\}$ , where  $K_i \in \mathbb{R}^{k_i \times n}$ , for all  $i \in \{1, \dots, M\}$  and  $\geq_c$  denotes the component-wise relation. The techniques employed in what follows could be adapted also to this case.

We restrict our attention to Lyapunov functions *homogeneous of degree 2*, considering max-min functions obtained from quadratic forms. This choice is motivated by the fact that, as proved in [68], max of quadratics Lyapunov functions are universal (existence is sufficient *and necessary*) for GAS of linear differential inclusions (LDI). For linear state-dependent switched systems (4.42), as we noted, non-convex (but still homogeneous) Lyapunov functions are required, and thus the *min*-operator was added to have this flexibility. The study of universality for max-min of quadratics for (4.42) is open for further research. The construction of “piecewise” quadratic Lyapunov functions, in similar settings, is studied also in [77], [59], and references therein.

#### Definition 4.17

Given  $K$  distinct, symmetric and positive definite matrices  $P_1, \dots, P_K \in \mathbb{R}^{n \times n}$ , a *max-min of quadratics* is denoted by  $V \in \mathbf{Mmq}(P_1, \dots, P_K)$ , and is defined as

$$V(x) = \max_{j \in \{1, \dots, J\}} \left\{ \min_{k \in S_j} \{x^\top P_k x\} \right\}, \quad (4.44)$$

where  $J \geq 1$  and for each  $j \in \{1, \dots, J\}$ , the set  $S_j \subset \{1, \dots, K\}$  is nonempty.

#### Remark 4.18: Homogeneity

Since the sets  $X_i$  are symmetric cones, the set-valued map in (4.42) is *homogeneous of degree 1*, in the sense that  $F_{\text{lin}}^{\text{sw}}(\lambda x) = \lambda F_{\text{lin}}^{\text{sw}}(x)$ ,  $\forall x \in \mathbb{R}^n, \forall \lambda \in \mathbb{R}$ . Similarly, a max-min of quadratics function defined as in (4.44) is *homogeneous of degree 2*, that is  $V(\lambda x) = \lambda^2 V(x)$ ,  $\forall x \in \mathbb{R}^n, \forall \lambda \in \mathbb{R}$ , and  $\alpha_V$  is constant along rays emanating from the origin, that is  $\alpha_V(\lambda x) = \alpha_V(x)$ ,  $\forall x \in \mathbb{R}^n, \forall \lambda \in \mathbb{R} \setminus \{0\}$ .

### 4.5.1 Stability Conditions with Set-Valued Lie Derivative

We first specialize the conditions of Theorem 4.15 for system (4.42) with  $V$  of the form (4.44). To this end, points  $x \in \mathbb{R}^n$  where  $\alpha_V(x) = \{\ell(x)\}$  is a singleton are easily characterized because they satisfy  $x \in \text{int}(C_{\ell(x)})$ . Instead, consider any  $x \in \mathbb{R}^n$ , such that  $\alpha_V(x) = \{\ell_1, \dots, \ell_p\}$  with  $p > 1$ , namely any point  $x$  where the locally Lipschitz function  $V$  is not continuously differentiable. Define now the

probability simplex of dimension  $m$  as

$$\Lambda_0^m := \{\lambda \in \mathbb{R}_{\geq 0}^m \mid \sum_{j=1}^m \lambda_j = 1\}.$$

Denoting  $\mathcal{I}_{\mathcal{X}}(x) = \{i_1, \dots, i_m\} \subseteq \{1, \dots, M\}$  by (4.15) we have that  $\dot{V}_{F_{\text{lin}}^{\text{sw}}}(x) \neq \emptyset$  if and only if there exist  $\lambda = (\lambda_1, \dots, \lambda_m) \in \Lambda_0^m$  such that

$$\nabla V_{\ell_{k+1}}(x)^\top \left( \sum_{j=1}^m \lambda_j A_{i_j} x \right) = \nabla V_{\ell_k}(x)^\top \left( \sum_{j=1}^m \lambda_j A_{i_j} x \right), \quad (4.45)$$

for each  $k \in \{1, \dots, p-1\}$ . Based on (4.45), define the set  $\Lambda(x, \{A_i\}_{i \in \mathcal{I}_{\mathcal{X}}(x)}) \subset \Lambda_0^m$  as

$$\lambda \in \Lambda(x, \{A_i\}_{i \in \mathcal{I}_{\mathcal{X}}(x)}) \Leftrightarrow \begin{cases} \sum_{j=1}^m \lambda_j x^\top (P_{\ell_2} - P_{\ell_1}) A_{i_j} x = 0, \\ \vdots \\ \sum_{j=1}^m \lambda_j x^\top (P_{\ell_p} - P_{\ell_{p-1}}) A_{i_j} x = 0, \end{cases} \quad (4.46)$$

where  $\lambda = (\lambda_1, \dots, \lambda_m) \in \Lambda_0^m$ . Then, recalling (4.15), we have

$$\ell \in \alpha_V(x) \Rightarrow \dot{V}_{F_{\text{lin}}^{\text{sw}}}(x) = \left\{ \begin{array}{l} 2x^\top P_\ell (\lambda_1 A_{i_1} + \dots + \lambda_m A_{i_m}) x : \\ (\lambda_1, \dots, \lambda_m) \in \Lambda(x, \{A_i\}_{i \in \mathcal{I}_{\mathcal{X}}(x)}) \end{array} \right\}. \quad (4.47)$$

The characterization provided by (4.47) is used to prove the next corollary of Theorem 4.15.

#### Corollary 4.19

Consider system (4.42) and a max-min of quadratics  $V \in \mathbf{Mm}q(P_1, \dots, P_K)$ , where  $P_1, \dots, P_K$  are symmetric, positive-definite, and pairwise distinct matrices. Suppose that there exists  $\varepsilon > 0$  such that

- (i) For each  $x \in \mathbb{R}^n$  with  $\alpha_V(x) = \{\ell\}$  and  $\mathcal{I}_{\mathcal{X}}(x) = \{i\}$  being singletons, it holds that

$$x^\top (A_i^\top P_\ell + P_\ell A_i) x \leq -\varepsilon |x|^2. \quad (4.48)$$

- (ii) For each  $x \in \mathbb{R}^n$  satisfying  $\alpha_V(x) = \{\ell_1, \dots, \ell_p\} \subset \{1, \dots, K\}$ , with  $p > 1$ , and  $\mathcal{I}_{\mathcal{X}}(x) = \{i_1, \dots, i_m\} \subset \{1, \dots, M\}$  with  $m > 1$ , there exists  $\lambda \in \alpha_V(x)$  such that

$$\sum_{i \in \mathcal{I}_{\mathcal{X}}(x)} \lambda_i x^\top (P_\ell A_i + A_i^\top P_\ell) x \leq -\varepsilon |x|^2, \quad (4.49)$$

for all  $(\lambda_1, \dots, \lambda_m) \in \Lambda(x, \{A_i\}_{i \in \mathcal{I}_{\mathcal{X}}(x)})$ .

Then the origin of (4.42) is GAS.

*Proof :* It follows from Theorem 4.15 that the origin of (4.42) is GAS if (4.38) holds for all  $x \in \mathbb{R}^n$ . We will proceed by analyzing four cases, depending on whether the sets  $\mathcal{I}_{\mathcal{X}}(x)$  and  $\alpha_V(x)$  are singletons or not.

First, consider  $x$  such that  $\alpha_V(x) = \{\ell\}$  and  $\mathcal{I}_X(x) = \{i\}$  are singletons. In this case,

$$\dot{\bar{V}}_{F_{\text{lin}}^{\text{sw}}}(x) = \left\{ x^\top (A_i^\top P_\ell + P_\ell A_i) x \right\} \leq -\varepsilon |x|^2,$$

where the inequality is due to condition (i).

Secondly, for a point  $x$  with  $\alpha_V(x) = \{\ell_1, \dots, \ell_p\}$ , with  $p > 1$ , and  $\mathcal{I}_X(x) = \{i_1, \dots, i_m\}$  with  $m > 1$ , it follows from (4.47) and condition (ii) that  $\max \dot{\bar{V}}_{F_{\text{lin}}^{\text{sw}}}(x) \leq -\varepsilon |x|^2$ .

Next, consider the case where  $\alpha_V(x) = \{\ell\}$  is a singleton and  $\mathcal{I}_X(x) = \{i_1, \dots, i_m\}$  with  $m > 1$ , that is a point where  $V$  is continuously differentiable and the set  $F_{\text{lin}}^{\text{sw}}(x)$  in (4.42) is multivalued. We thus have  $\partial V(x) = \{\nabla V_\ell(x)\}$ , and from linearity we have

$$\max \dot{\bar{V}}_{F_{\text{lin}}^{\text{sw}}}(x) \leq \max_{\lambda \in \Lambda_0^m} \sum_{j=1}^m 2\lambda_j x^\top P_\ell A_{i_j} x = 2x^\top P_\ell A_{i^*} x, \quad (4.50)$$

where  $i^* \in \arg \max_{i=i_1, \dots, i_m} 2x^\top P_\ell A_i x$ . Since  $i^* \in \mathcal{I}_X(x)$ , by (4.36)  $x \in X_{i^*}$ ; from item (i) we have

$$x_k^\top (A_{i^*}^\top P_\ell + P_\ell A_{i^*}) x_k \leq -\varepsilon |x_k|^2.$$

for some sequence  $x_k \rightarrow x$  with  $x_k \in \text{int}(X_{i^*}) \cap \text{int}(C_\ell)$ ,  $\forall k \in \mathbb{N}$ . By continuity we thus have  $x^\top (A_{i^*}^\top P_\ell + P_\ell A_{i^*}) x \leq -\varepsilon |x|^2$ , and from (4.50) we have  $\max \dot{\bar{V}}_{F_{\text{lin}}^{\text{sw}}}(x) \leq -\varepsilon |x|^2$ .

Finally, we consider the case  $\alpha_V(x) = \{\ell_1, \dots, \ell_p\}$  with  $p > 1$  and  $\mathcal{I}_X(x) = \{i\}$ , namely a point where the function  $V$  is not continuously differentiable and the set  $F_{\text{lin}}^{\text{sw}}(x)$  is a singleton, since  $x \in \text{int}(X_i)$ . If  $\dot{\bar{V}}_{F_{\text{lin}}^{\text{sw}}}(x) = \emptyset$  we are done. Otherwise, in view of (4.46),  $\dot{\bar{V}}_{F_{\text{lin}}^{\text{sw}}}(x) \neq \emptyset$  implies

$$\{2x^\top P_{\ell_1} A_i x\} = \dots = \{2x^\top P_{\ell_p} A_i x\} = \dot{\bar{V}}_{F_{\text{lin}}^{\text{sw}}}(x). \quad (4.51)$$

Considering, without loss of generality, the index  $\ell_1 \in \alpha_V(x)$ , by Definition 4.4 we can consider a sequence  $x_k \rightarrow x$  such that  $x_k \in \text{int}(X_i) \cap \text{int}(C_{\ell_1})$ , for all  $k \in \mathbb{N}$ . By condition (i) we have

$$x_k^\top (A_i^\top P_{\ell_1} + P_{\ell_1} A_i) x_k \leq -\varepsilon |x_k|^2, \quad \forall k \in \mathbb{N}.$$

By continuity we have  $x^\top (A_i^\top P_{\ell_1} + P_{\ell_1} A_i) x \leq -\varepsilon |x|^2$ ; recalling (4.51), it implies that  $\max \dot{\bar{V}}_{F_{\text{lin}}^{\text{sw}}}(x) \leq -\varepsilon |x|^2$ .

Having analyzed all the cases, we conclude that (4.38) holds for all  $x \in \mathbb{R}^n$  and the assertion follows from Theorem 4.15.  $\diamond$

## 4.5.2 Checking Item (i) of Corollary 4.19

In this section, we exploit the properties of system (4.42) and the family of candidate max-min Lyapunov functions in (4.44) to computationally check condition (i) of Corollary 4.19. We do so by following two steps: first, fixing  $K \geq 1$ ,  $J \geq 1$ , nonempty subsets  $S_1, \dots, S_J \subset \{1, \dots, K\}$ , and hence the corresponding max-min combination in (4.44), we construct an auxiliary function  $\Phi$ , which characterizes

Algorithm 0: The function  $\Phi : \mathbb{S}_K \rightarrow \{1, \dots, K\}$ .

**Data:**  $K \in \mathbb{N}$ ,  $J \geq 1$ ,  $S_1, \dots, S_J \subset \{1, \dots, K\}$   
**Input:**  $\rho = (\rho_1, \dots, \rho_K) \in \mathbb{S}_K$   
**Output:**  $\text{out} = \Phi_{K,J,S_1,\dots,S_J}(\rho)$   
**Function**  $\Phi_{K,J,S_1,\dots,S_J}(\rho)$ :  
 Set:  $\text{out} = 0$ ,  $S_{\min} = \emptyset$ ,  
 for ( $j = 1, j \leq J, j = j + 1$ ) do  
 | for ( $i = 1, i \leq K, i = i + 1$ ) do  
 | | if  $\rho_i \in S_j$  then  
 | | | Add  $\rho_i$  to  $S_{\min}$ , **break**  
 | | end  
 | end  
 end  
 for ( $j = J, j \geq 0, j = j - 1$ ) do  
 | if  $\rho_j \in S_{\min}$  then  
 | |  $\text{out} = \rho_j$ , **break**  
 | end  
 end  
 return out  
**End Function**

the regions where  $\alpha_V : \mathbb{R}^n \rightrightarrows \{1, \dots, K\}$  is single-valued. Notably, this function is independent of  $P_1, \dots, P_K$ . Secondly, we use  $\Phi$  to compute matrices  $P_1, \dots, P_K$  satisfying item (i) of Corollary 4.19 by only checking the feasibility of a finite set of matrix inequalities. The details of implementing these two steps now follow:

**Step 0** Consider the symmetric group of order  $K$  denoted by  $\mathbb{S}_K$ , which is the group of all possible permutations of the first  $K$  positive integers. Given any  $K$  pairwise distinct quadratic functions associated to some  $P_1, \dots, P_K > 0$ , for any  $\rho = (\rho_1, \dots, \rho_K) \in \mathbb{S}_K$ , define the open set

$$E_\rho := \{x \in \mathbb{R}^n \mid x^\top P_{\rho_1} x < \dots < x^\top P_{\rho_K} x\}, \quad (4.52)$$

which is a cone (possibly empty) where a strict ordering among the  $K$  quadratic functions holds. For a given max-min combination in (4.44), namely given  $J \geq 1$  and nonempty sets  $S_j \subset \{1, \dots, K\}$ ,  $\forall j \in \{1, \dots, J\}$ , in each  $E_\rho$  the function  $\alpha_V : \mathbb{R}^n \rightrightarrows \{1, \dots, K\}$  defined in (4.10) is constant and single valued; let us denote it by  $\Phi(\rho) := \alpha_V(E_\rho) \in \{1, \dots, K\}$ .  $\triangle$

In Algorithm 0, we present how to numerically construct  $\Phi : \mathbb{S}_K \rightarrow \{1, \dots, K\}$ , independently of matrices  $(P_1, \dots, P_K)$ .

Algorithm 1: Lyapunov conditions: Differentiable case.

**Data:**  $A_1, \dots, A_M \in \mathbb{R}^{n \times n}$ ,  $Q_1, \dots, Q_M \in \text{Sym}(\mathbb{R}^n)$ .

INITIALIZATION: Choose the max-min structure:

Take  $K \in \mathbb{N}$ ,  $J \geq 1$ ,  $S_1, \dots, S_J \subset \{1, \dots, K\}$ , construct  $\Phi : \mathbb{S}_K \rightarrow \{1, \dots, K\}$  (Algorithm 0).

LYAPUNOV CONDITIONS ON  $E_\rho$ ,  $\forall \rho \in \mathbb{S}_K$ :

(Step 1) Check the feasibility of

$$\begin{aligned} A_i^\top P_{\Phi(\rho)} + P_{\Phi(\rho)} A_i + \sum_{k=1}^{K-1} \tau_{i,k}(\rho)(P_{\rho_{k+1}} - P_{\rho_k}) + \beta_i(\rho)Q_i < 0, \\ P_1, \dots, P_K > 0, \beta_i(\rho), \tau_{i,k}(\rho) \geq 0, \forall \rho = (\rho_1, \dots, \rho_K) \in \mathbb{S}_K, \\ k \in \{1, \dots, K-1\}, i \in \{1, \dots, M\}. \end{aligned} \quad (4.54)$$

**if** (4.54) are feasible **then**

| **Output:** Matrices  $(P_1, \dots, P_K)$

**else**

| **Output:**  $\emptyset$

**end**

#### Remark 4.20

We emphasize that the function  $\Phi$  is independent of  $P_1, \dots, P_K$ , but only depends on the max-min policy defined by sets  $S_1, \dots, S_J$ . As an example, considering  $J = K$  and  $S_j = \{j\}$ , the max-min combination (4.44) coincides with the maximum of the  $K$  quadratic functions. In this case,  $\Phi$  will be defined as  $\Phi((\rho_1, \dots, \rho_K)) = \rho_K$ ,  $\forall \rho = (\rho_1, \dots, \rho_K) \in \mathbb{S}_K$ , because of (4.52). Also, to relate  $\Phi$  with  $\alpha_V$ , it is seen that for any  $K$  base quadratics defined by  $(P_1, \dots, P_K)$  with a specific max-min combination determined by  $V$ , the mapping  $\alpha_V$  in (4.10) corresponds to  $\alpha_V(x) = \bigcap_{\epsilon > 0} \{\Phi(\rho) \mid E_\rho \cap \mathbb{B}(x, \epsilon) \neq \emptyset\}$ .

Next, in Step 1, we use the function  $\Phi$  to check condition (i) of Corollary 4.19:

**Step 1 (Conditions on  $E_\rho$ )** Consider system (4.42), and take  $K \in \mathbb{N}$ ,  $J \geq 1$  and  $S_1, \dots, S_J \subset \{1, \dots, K\}$  nonempty sets. Find  $P_1, \dots, P_K > 0$ ,  $\beta_i(\rho) \geq 0$ ,  $\tau_{i,k}(\rho) \geq 0$ ,  $\forall \rho = (\rho_1, \dots, \rho_K) \in \mathbb{S}_K$ ,  $\forall k \in \{1, \dots, K-1\}$ , and  $\forall i \in \{1, \dots, M\}$ , such that

$$A_i^\top P_{\Phi(\rho)} + P_{\Phi(\rho)} A_i + \sum_{k=1}^{K-1} \tau_{i,k}(\rho)(P_{\rho_{k+1}} - P_{\rho_k}) + \beta_i(\rho)Q_i < 0. \quad (4.53)$$

In Proposition 4.21 below, we prove that the feasibility of Step 1 yields  $K$  matrices such that condition (i) of Corollary 4.19 holds, while in Algorithm 1 we formalize this step of computationally checking condition (4.53).

**Proposition 4.21**

Consider  $K \in \mathbb{N}$ ,  $J \geq 1$ ,  $S_1, \dots, S_J \subset \{1, \dots, K\}$  non-empty,  $(P_1, \dots, P_K)$  positive definite matrices and  $V$  defined as in (4.44). If, for any  $\rho = (\rho_1, \dots, \rho_K) \in \mathbb{S}_K$ , any  $i \in \{1, \dots, M\}$  and any  $k = \{1, \dots, K-1\}$ , there exist  $\beta_i(\rho) \geq 0$ ,  $\tau_{i,k}(\rho) \geq 0$  such that (4.53) holds, then item (i) of Corollary 4.19 holds.

*Proof :* The set  $E_\rho \cap \text{int}(X_i)$  can be written as

$$E_\rho \cap \text{int}(X_i) = \left\{ x \in \mathbb{R}^n \left| \begin{array}{l} x^\top Q_i x > 0 \wedge x^\top (P_{\rho_{k+1}} - P_{\rho_k}) x > 0, \\ \forall k \in \{1, \dots, K-1\} \end{array} \right. \right\}.$$

If (4.53) holds, due to the strict inequality, there exists  $\varepsilon_{i,\rho} > 0$  such that

$$x^\top (A_i^\top P_{\Phi(\rho)} + P_{\Phi(\rho)} A_i) x \leq -\varepsilon_{i,\rho} |x|^2, \quad \forall x \in X_i \cap E_\rho. \quad (4.55)$$

By Step 0 we have  $\alpha_V(x) = \{\Phi(\rho)\}$  and, by (4.43),  $\mathcal{I}_X(x) = \{i\}$  for all  $\text{int}(X_i) \cap E_\rho$ , and thus (4.55) implies that (4.48) holds for all  $x \in \text{int}(X_i) \cap E_\rho$ . Defining  $\varepsilon := \min_{i,\rho} \varepsilon_{i,\rho}$  we have that (4.48) holds for each  $x \in \mathbb{R}^n$  with  $\alpha_V(x)$  and  $\mathcal{I}_X(x)$  being singletons, thus concluding the proof.  $\diamond$

**Remark 4.22: Polyhedral cones**

Consider again the alternative state-space partition discussed in Remark 4.16. More precisely, consider polyhedral cones  $D_1, \dots, D_M \subset \mathbb{R}^n$  defined by  $X_i := \{x \in \mathbb{R}^n \mid K_i x \geq_c \mathbf{0}\}$ , where, for each  $i \in \{1, \dots, M\}$   $K_i \in \mathbb{R}^{k_i \times n}$ , for some  $k_i \in \mathbb{N}$ . Equivalently, the sets  $X_i$  can be represented by  $X_i = \text{cone}(v_i)_{i=1}^{M_i}$ , where  $v_1, \dots, v_{M_i} \in \mathbb{R}^n$  are the rays of the cone  $X_i$ . Let us call by  $R_i \in \mathbb{R}^{n \times M_i}$  the matrix whose columns are the vectors  $v_i$ . As presented in [70, Lemma 1] we have that, given any symmetric matrix  $S \in \mathbb{R}^{n \times n}$ , if there exists a symmetric and entry-wise positive matrix  $N_i \in \mathbb{R}^{n \times n}$  such that  $R_i^\top S R_i + N_i \leq 0$  then  $x^\top S x < 0$ ,  $\forall x \in X_i$ . Using this result, the procedure presented in Step 1 and Proposition 4.21 can be adapted to the polyhedral cones case by requiring that for any  $\rho = (\rho_1, \dots, \rho_K) \in \mathbb{S}_K$ , any  $i \in \{1, \dots, M\}$  and any  $k = \{1, \dots, K-1\}$ , there exist  $\tau_{i,k}(\rho) \geq 0$  and a symmetric entry-wise positive matrix  $N_i(\rho)$  such that

$$R_i^\top S(\rho) R_i + N_i(\rho) \leq 0,$$

with  $S(\rho) := A_i^\top P_{\Phi(\rho)} + P_{\Phi(\rho)} A_i + \sum_{k=1}^{K-1} \tau_{i,k}(\rho) (P_{\rho_{k+1}} - P_{\rho_k})$ .



**Remark 4.23: Computational burden**

It is noted that, in general, since  $|\mathbb{S}_K| = K!$ , Algorithm 1 requires studying the feasibility of  $M \cdot K!$  inequalities, which involve  $MKK!$  non-negative scalars and  $K$  symmetric positive-definite matrices. It is clear that the computational burden grows quickly as a function of the number  $K$  of the chosen base-quadratics. However, fixing  $J \geq 1$ ,  $S_1, \dots, S_J \subset \{1, \dots, K\}$  in (4.44) (thus fixing a particular max-min structure) the computational burden can be reduced. In Appendix A, Section A.2, we show how the number of required inequalities depends on the choice of sets  $S_j$  in the case of three quadratics, i.e.  $K = 3$ .

**Example (4.1 - Continued: Item (i))** *We have already proved that there does not exist a convex Lyapunov function for system (4.5). We will construct a max-min of quadratics Lyapunov function  $V$  of the form (4.4). In other words, we have fixed  $K = 3$ ,  $J = 2$ ,  $S_1 = \{1, 2\}$  and  $S_2 = \{3\}$ . Using Algorithm 0 we construct the function  $\Phi$  that reads  $\Phi(\rho_1) = \Phi(\rho_2) = \Phi(\rho_3) = \Phi(\rho_4) = 3$ , where  $\rho_1 = (1, 2, 3)$ ,  $\rho_2 = (1, 3, 2)$ ,  $\rho_3 = (2, 1, 3)$ ,  $\rho_4 = (2, 3, 1)$ ; and  $\Phi(\rho_5) = 1$ , where  $\rho_5 = (3, 1, 2)$ ; and  $\Phi(\rho_6) = 2$ , where  $\rho_6 = (3, 2, 1)$ . In these cases, the matrix inequalities of Algorithm 1 (after the reductions outlined in Remark 4.23) read*

$$\begin{aligned} A_2^\top P_2 + P_2 A_2 + \tau_1(P_2 - P_3) + \tau_2(P_1 - P_2) + \beta_1 Q_2 &< 0, \\ A_1^\top P_1 + P_1 A_1 + \tau_3(P_1 - P_3) + \tau_4(P_2 - P_1) + \beta_2 Q_1 &< 0, \\ A_3^\top P_3 + P_3 A_3 + \tau_5(P_3 - P_1) + \beta_3 Q_3 &< 0, \\ A_3^\top P_3 + P_3 A_3 + \tau_6(P_3 - P_2) + \tau_7(P_1 - P_3) + \beta_4 Q_4 &< 0, \\ \tau_k \geq 0, \forall k \in \{1, \dots, 7\}, \beta_i \geq 0, \forall i \in \{1, \dots, 4\}, P_1, P_2, P_3 &> 0. \end{aligned}$$

Using numerical solvers, it follows that these inequalities are feasible, and in particular they are satisfied by

$$P_1 = \begin{bmatrix} 5 & 0 \\ 0 & 1 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix}, \quad P_3 = \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix}, \quad (4.56)$$

$\tau = (0.258, 0.102, 0.258, 0.102, 0.284, 0.193, 0.090)$  and  $\beta_i = 0, \forall i \in \{1, \dots, 4\}$ . A level set of  $V$  is plotted in Fig. 4.1. This proves that  $V$  in (4.4) with  $P_i$  as in (4.56) satisfies item (i) of Corollary 4.19.

In general, finding a function  $V$  as in (4.44) that satisfies only condition (i) of Corollary 4.19 is not enough to conclude that the system (4.42) is GAS. On the other hand, if we consider a *linear differential inclusion* (LDI), this would be sufficient, as explained in the following remark.

**Remark 4.24: Linear Differential Inclusions**

The construction of max-min Lyapunov functions is also tailored for linear differential inclusions (LDI), that is

$$\dot{x} \in \text{co}\{A_i x \mid i \in \{1, \dots, M\}\}. \quad (4.57)$$

for some matrices  $A_1, \dots, A_M \in \mathbb{R}^{n \times n}$ . Remarkably, since system (4.57) is a *continuous* differential inclusion, checking the Lie's conditions on the set of non differentiability of the candidate Lyapunov function  $V$  is not needed, see also [59]. In other words, for *continuous* differential inclusions it will be sufficient to have an output  $(P_1, \dots, P_K)$  of Algorithm 1 to conclude global asymptotic stability.

**4.5.3 Checking item (ii) of Corollary 4.19 in  $\mathbb{R}^2$ .**

To study GAS of system (4.42), we also need to check item (ii) of Corollary 4.19, which is computationally harder than item (i). We now discuss how this condition simplifies in the planar case, that is when  $n = 2$ . To do so, let us analyze the geometry of the switching rule proposed in (4.43). To non-trivially satisfy Definition 4.12, we will suppose that the matrices  $Q_1, \dots, Q_M \in \text{Sym}(\mathbb{R}^2)$  are sign indefinite. We will characterize the sets  $X_i$  in (4.43) using the following result.

**Lemma 4.25**

Given any sign indefinite matrix  $Q \in \text{Sym}(\mathbb{R}^2)$ , there exist  $\theta_1, \theta_2 \in \mathbb{R}^2 \setminus \{0\}$ ,  $\theta_2 \notin \text{span}(\theta_1)$  such that

$$Q = \theta_1 \theta_2^\top + \theta_2 \theta_1^\top. \quad (4.58)$$

*Proof* : [Sketch of the proof] Let us denote by  $\lambda_- < 0 < \lambda_+$  the eigenvalues of  $Q$ , and with  $v_-, v_+ \in \mathbb{R}^2$  the corresponding unit eigenvectors ( $|v_-| = |v_+| = 1$ ). By the spectral decomposition we have that  $Q = \lambda_+ v_+ v_+^\top + \lambda_- v_- v_-^\top$ . Let us call  $\eta = \sqrt{\frac{-\lambda_-}{\lambda_+ - \lambda_-}} > 0$  and  $\kappa = \sqrt{\frac{2}{\lambda_+ - \lambda_-}} > 0$ , then by choosing  $\theta_1 = \kappa [\sqrt{1 - \eta^2} v_+ - \eta v_-]$  and  $\theta_2 = \kappa [\sqrt{1 - \eta^2} v_+ + \eta v_-]$ , it is seen that (4.58) holds.  $\diamond$

Lemma 4.25 allows checking algorithmically condition (ii) of Corollary 4.19 in the planar case. This is done in two steps.

**Step 2a** Given  $M$  sign indefinite matrices  $Q_1, \dots, Q_M \in \text{Sym}(\mathbb{R}^2)$  such that the corresponding switching signal satisfies Definition 4.12, then matrices  $Q_i$ ,  $i = 1, \dots, M$ , decomposed as in (4.58), can be suitably ordered<sup>1</sup> in such a way that

$$\begin{aligned} Q_i &= \theta_i \theta_{i+1}^\top + \theta_{i+1} \theta_i^\top \quad \text{for } i = 1, \dots, M-1, \\ Q_M &= \theta_M (-\theta_1)^\top + (-\theta_1) \theta_M^\top, \end{aligned} \quad (4.59)$$

<sup>1</sup>For the ordering of matrices  $Q_i$ , via vectors  $\theta_i$  in (4.58), we can associate an angle with each one of the lines  $\theta_i$ ,  $i = 1, \dots, M$ , using the *atan2* function.

for some suitable selections of linear independent vectors  $\theta_1, \dots, \theta_M \in \mathbb{R}^2 \cap \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 \geq 0\}$ .

For each  $i \in \{1, \dots, M\}$ , take  $v_i \in \mathbb{R}^2$  as a unit vector generating the subspace  $\theta_i^\perp := \{x \in \mathbb{R}^2 \mid \theta_i^\top x = 0\}$ .  $\triangle$

**Step 2b** Consider  $V \in \mathbf{Mmq}(P_1, \dots, P_K)$  satisfying condition (i) of Corollary 4.19. For every  $v_i$  such that  $\alpha_V(v_i) = \{\ell_1^i, \ell_2^i\}$  is multivalued, solve the system

$$\begin{cases} 0 \leq \lambda \leq 1, \\ \lambda v_i^\top (P_{\ell_2^i} - P_{\ell_1^i}) A_{i-1} v_i + (1 - \lambda) v_i^\top (P_{\ell_2^i} - P_{\ell_1^i}) A_i v_i = 0, \end{cases} \quad (4.60)$$

(with  $i - 1 = M$  if  $i = 1$ ), and denote by  $\Lambda^i \subset [0, 1]$  the set of solutions of (4.60) for  $v_i$  (possibly empty).  $\triangle$

In the following we formally prove the effectiveness of Steps 2a and 2b.

#### Proposition 4.26

Consider  $A_1, \dots, A_M \in \mathbb{R}^{2 \times 2}$  and  $M$  indefinite matrices  $Q_1, \dots, Q_M \in \text{Sym}(\mathbb{R}^2)$  such that the corresponding switching signal satisfy Definition 4.12. Consider the parametrization of  $Q_1, \dots, Q_M$  presented in (4.59). Suppose that there exist  $P_1, \dots, P_K > 0$  such that  $V \in \mathbf{Mmq}(P_1, \dots, P_K)$  satisfies condition (i) of Corollary 4.19. If, for all  $i \in \{1, \dots, M\}$  such that  $\alpha_V(v_i)$  is multivalued, we have

$$\lambda v_i^\top (P_{\ell_1^i} A_{i-1} + A_{i-1}^\top P_{\ell_1^i}) v_i + (1 - \lambda) v_i^\top (P_{\ell_1^i} A_i + A_i^\top P_{\ell_1^i}) v_i < 0, \quad (4.61)$$

for all  $\lambda \in \Lambda^i$ , then item (ii) of Corollary 4.19 holds.

*Proof* : Recalling (4.43), the parametrization in (4.59) characterizes the points  $x$  where the map  $\mathcal{I}_X(x)$  is multivalued. From (4.59), we have that  $\overline{D}_i \cap \overline{D}_{i+1} = \theta_{i+1}^\perp$ , for all  $i = 1, \dots, M - 1$  and  $\overline{D}_M \cap \overline{D}_1 = \theta_1^\perp$ . Thus

$$\mathcal{I}_X(x) = \begin{cases} \{i, i + 1\}, & \text{if } x \in \theta_{i+1}^\perp, \ i = 1, \dots, M - 1, \\ \{1, M\}, & \text{if } x \in \theta_1^\perp, \\ \{i\}, & \text{if } x^\top Q_i x > 0. \end{cases} \quad (4.62)$$

Let us now consider a function  $V \in \mathbf{Mmq}(P_1, \dots, P_K)$  that satisfies condition (i) of Corollary 4.19. From Remark 4.18, for any max-min function  $V \in \mathbf{Mmq}(P_1, \dots, P_K)$ , the value of the map  $\alpha_V : \mathbb{R}^2 \rightrightarrows \{1, \dots, K\}$  has *at most* 2 elements. To check item (ii) of Corollary 4.19 we must consider all the points  $x \in \mathbb{R}^2$  such that  $\mathcal{I}_X(x)$  and  $\alpha_V(x)$  are multivalued. As shown in (4.62), the set of points where the map  $I$  is multivalued coincides with the union of the  $M$  lines  $\theta_1^\perp, \dots, \theta_M^\perp$ . From Remark 4.18, the homogeneity of  $F_{\text{lin}}^{\text{sw}}$  and  $V \in \mathbf{Mmq}(P_1, \dots, P_K)$  implies that it is sufficient to check condition (ii) of Corollary 4.19 only for the chosen unit vectors  $v_1, \dots, v_M$  which span  $\theta_1^\perp, \dots, \theta_M^\perp$  respectively. We can conclude noting that, for each  $i \in \{1, \dots, M\}$  such that  $\alpha_V(v_i)$  is multivalued, system (4.60) corresponds to (4.46),

and equation (4.49) follows from (4.61) selecting a small enough  $\varepsilon > 0$ .  $\diamond$

Proposition 4.26 shows that for a *planar* linear switched system (4.42), (4.43) involving  $M$  subsystems, it is sufficient to identify unit vectors  $v_i$ ,  $i = 1, \dots, M$  generating the switching lines, and verify inequality (4.49) for these  $M$  points. Item (ii) of Corollary 4.19 then follows from homogeneity. This result allows concluding the analysis of Example 4.1.

**Example (4.1 - Continued: Item (ii))** *As a last step to show that the origin of (4.5), (4.6) is GAS, we have to ensure the condition (ii) of Corollary 4.19. Since the signal (4.6) can be rewritten in the form (4.59), we can follow Steps 2a and 2b, taking  $v_1 \in \mathcal{S}_{13}$ ,  $v_2 \in \mathcal{S}_{21}$ ,  $v_3 \in \mathcal{S}_{32}$  such that  $|v_j| = 1$ , for all  $j \in \{1, 2, 3\}$ . Considering system (4.60), it is easily checked that  $\Lambda^j = \emptyset$ ,  $\forall j \in \{1, 2, 3\}$ . Recalling (4.47),  $\bar{V}_F(v_j) = \emptyset$ , for  $j = 1, 2, 3$ . Then by Proposition 4.26 the function  $V$  in (4.4) is a Lyapunov function for system (4.5) which certifies GAS.*

#### Remark 4.27

In Example 4.1, it can be shown that  $V$  in (4.4) does not satisfy the conditions  $\dot{V}_{F^{\text{sw}}}(x) < 0$  for some  $x \in \mathbb{R}^2$ : consider the point  $v_1 \in \mathcal{S}_{13}$ , where we have shown  $\bar{V}_{F^{\text{sw}}}(v_1) = \emptyset$ . Since  $v_1 \in \mathcal{S}_{13}$ , then  $\partial V(v_1) = \text{co}\{2P_1v_1, 2P_3v_1\}$  and  $F^{\text{sw}}(z_0) = \text{co}\{A_1v_1, A_3v_1\}$ . Straightforward computations yield  $v_1^\top(P_3A_1 + A_1^\top P_3)v_1 = 8.65 > 0$ , and thus  $\exists w \in \partial V(v_1)$  and  $f \in F^{\text{sw}}(v_1)$  such that  $0 < w^\top f \in \dot{V}_{F^{\text{sw}}}(v_1)$ , which implies that conditions relying on the Clarke derivative concept would not prove stability in this case. In that sense, we Corollary 4.19 provides less conservative conditions.

#### 4.5.4 Checking item (ii) of Corollary 4.19 in $\mathbb{R}^n$ with 2 modes.

The main difficulty in checking item (ii) of Corollary 4.19 in higher dimensions is that the set  $\Lambda(x, \{A_i\}_{i \in \mathcal{I}_X(x)})$  of (4.46) cannot be finitely parameterized. In this section, we impose a structure on (4.42) which allows us to check this condition without explicitly computing  $\Lambda(x, \{A_i\}_{i \in \mathcal{I}_X(x)})$ . The idea is to rule out the motion on switching surfaces, in which case negative definiteness of  $\bar{V}_{F^{\text{sw}}}$  on the switching surface can be established by continuity arguments. More precisely, we consider a 2-mode  $n$ -dimensional switched system, i.e.  $M = 2$  in (4.42), and the sets  $X_1, X_2 \subset \mathbb{R}^n$  are defined by

$$\begin{aligned} X_1 &= \overline{\{x \in \mathbb{R}^n \mid x^\top Q_1 x = x^\top Q x > 0\}}, \\ X_2 &= \overline{\{x \in \mathbb{R}^n \mid x^\top Q_2 x = -x^\top Q x > 0\}}, \end{aligned} \quad (4.63)$$

where  $Q \in \text{Sym}(\mathbb{R}^n)$  is invertible. In other words, we are considering a proper partition of  $\mathbb{R}^n$  (in the sense of Definition 2.4), which comprises two symmetric cones  $D_1, D_2 \subset \mathbb{R}^n$ . We will denote the boundary of these cones (also called the *switching surface*) with  $\mathcal{Q} := \{x \in \mathbb{R}^n \mid x^\top Q x = 0\}$ .

A computationally attractive way to avoid sliding motion, is to follow a preliminary step, presented in what follows.

**Step 2\*** (Ruling out motion on the switching surface) For every  $z \in \mathbb{R}^n$  such that  $|z| = 1$ , check if the implication

$$z^\top Qz = 0 \Rightarrow (z^\top QA_1z)(z^\top QA_2z) > 0 \quad (4.64)$$

is satisfied.  $\triangle$

Condition (4.64) intuitively means that, given a unit vector  $z \in \mathcal{Q}$ , the vectors  $A_1z$  and  $A_2z$  are both pointing inside (or outside) the cone  $D_1$ , and thus it rules out the possibility of having solutions sliding along  $\mathcal{Q}$ , which can be consider a “crossing boundary” (cft. [71]). A viable way to check condition (4.64) is to consider the decomposition of  $Q$  as  $Q = V\Lambda V^\top$ , where invertibility of  $Q$  implies that  $\Lambda$  is a diagonal matrix with only 1 and  $-1$  diagonal elements and then check the simpler implication

$$\bar{z}^\top \Lambda \bar{z} = 0 \Rightarrow (\bar{z}^\top V^{-\top} Q A_1 V^{-1} \bar{z})(\bar{z}^\top V^{-\top} Q A_2 V^{-1} \bar{z}) > 0$$

for all  $|\bar{z}| = 1$ . To simplify the discussion, consider max-min combination over  $K$  quadratics defined by  $K$  symmetric and positive definite matrices  $P_1, \dots, P_K \in \mathbb{R}^{n \times n}$  satisfying:

$$\text{Rank}(P_{j_1} - P_{j_2}) = n, \quad \forall j_1, j_2 \in \{1, \dots, K\}, j_1 \neq j_2, \quad (4.65)$$

which is not too restrictive since full-rank matrices are dense in  $\mathbb{R}^{n \times n}$ .

#### Proposition 4.28

Consider a 2-mode linear switched system (4.42) with  $D_1, D_2 \subset \mathbb{R}^n$  as in (4.63) and  $Q \in \text{Sym}(\mathbb{R}^n)$  invertible. Suppose that for all  $z \in \mathbb{R}^n$ ,  $|z| = 1$ , the implication (4.64) in Step 2\* holds. If there exist  $P_1, \dots, P_K > 0$  satisfying (4.65), and  $V \in \mathbf{Mmq}(P_1, \dots, P_K)$  satisfying condition (i) of Corollary 4.19, then item (ii) holds and system (4.42) is GAS.

*Proof :* To check item (ii) of Corollary 4.19, consider any  $x \in \mathbb{R}^n$  such that  $x^\top Qx = 0$ , i.e.  $\mathcal{I}_X(x) = \{1, 2\}$ , and  $\alpha_V(x) = \{\ell_1, \dots, \ell_p\} \subset \{1, \dots, K\}$  with  $p > 1$ . We consider 2 cases:

*Case 1:* Suppose there exist  $\ell', \ell'' \in \alpha_V(x)$ ,  $\ell' \neq \ell''$  such that  $x^\top (P_{\ell'} - P_{\ell''}) = \tau x^\top Q$ , for some  $\tau \in \mathbb{R} \setminus \{0\}$ .

Then the equation (resembling (4.45),)

$$x^\top P_{\ell'} (\lambda A_1 x + (1 - \lambda) A_2 x) = x^\top P_{\ell''} (\lambda A_1 x + (1 - \lambda) A_2 x),$$

has solutions  $\lambda \in [0, 1]$  if and only if there exists  $\lambda \in [0, 1]$  such that

$$x^\top Q (\lambda A_1 x + (1 - \lambda) A_2 x) = 0. \quad (4.66)$$

We have supposed that (4.64) in Step 2\* holds for  $z = \frac{x}{|x|}$ , thus by homogeneity of  $F_{\text{lin}}^{\text{sw}}(x)$  equation (4.66) has no solution  $\lambda \in [0, 1]$  since the scalars  $x^\top Q A_1 x$  and  $x^\top Q A_2 x$  have the same sign (and are not zero). Recalling equations (4.46) and (4.47), this implies that  $\vec{V}_{F_{\text{lin}}^{\text{sw}}}(x) = \emptyset$ , ensuring (ii) of Corollary 4.19.

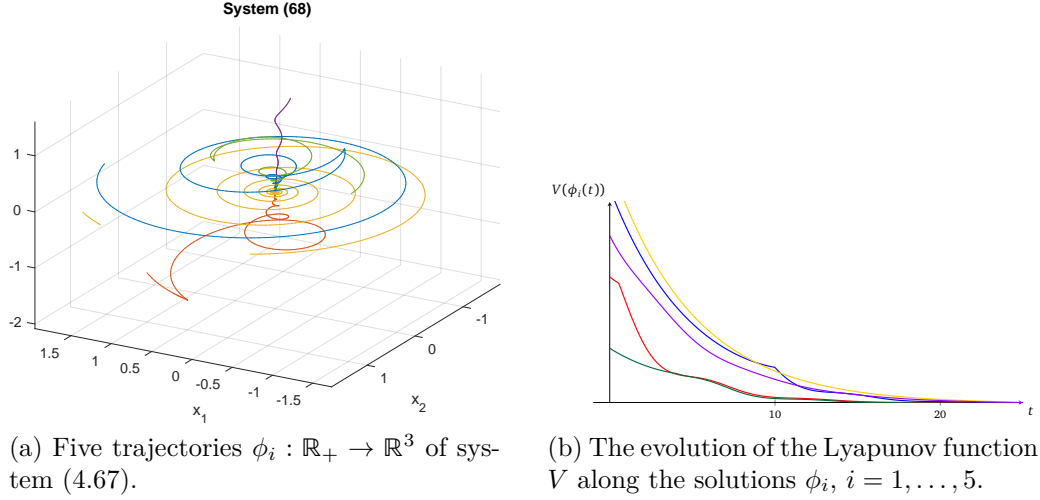


Figure 4.5: System (4.67) in Example 4.3

*Case 2: Suppose that  $\forall \ell', \ell'' \in \alpha_V(x), \ell' \neq \ell'', x^\top(P_{\ell'} - P_{\ell''}) \neq \tau x^\top Q_1$ , for all  $\tau \in \mathbb{R} \setminus \{0\}$ .*

In this case we show in Lemma A.1 in A.1 that there exists a sequence  $x_k \rightarrow x$  such that  $x_k \in \mathcal{Q}$ , (i.e.  $I(x_k) = \{1, 2\}$ ) and  $\alpha_V(x_k) = \{\bar{\ell}\}$ , for all  $k \in \mathbb{N}$ , for an  $\bar{\ell} \in \alpha_V(x)$ . By hypothesis,  $V$  satisfies item (i) of Corollary 4.19, implying by continuity that, for every  $k \in \mathbb{N}$ ,

$$x_k^\top P_{\bar{\ell}} A_1 x_k \leq -\varepsilon |x_k|^2, \text{ and } x_k^\top P_{\bar{\ell}} A_2 x_k \leq -\varepsilon |x_k|^2.$$

Since  $x_k \rightarrow x$  when  $k \rightarrow \infty$ , again by continuity we have

$$x^\top P_{\bar{\ell}} A_1 x \leq -\varepsilon |x|^2 \text{ and } x^\top P_{\bar{\ell}} A_2 x \leq -\varepsilon |x|^2.$$

Thus,  $\forall \lambda \in [0, 1]$  such that

$$x^\top P_{\bar{\ell}_1} (\lambda A_1 + (1 - \lambda) A_2) x = \dots = x^\top P_{\bar{\ell}_p} (\lambda A_1 x + (1 - \lambda) A_2 x),$$

we have  $x^\top P_{\bar{\ell}} (\lambda A_1 + (1 - \lambda) A_2) x \leq -\varepsilon |x|^2$ . Recalling (4.46) and (4.47) it implies that  $\max \dot{\bar{V}}_{F_{\text{lin}}^{\text{sw}}}(x) \leq -\varepsilon |x|^2$ .

Having proved item (ii) of Corollary 4.19 in both *Cases 1* and *2*, we can conclude.  $\diamond$

**Example 4.3** *Concluding this section, we present a switched system evolving in  $\mathbb{R}^3$  and we prove GAS using Proposition 4.28.*

*Let us consider the matrices,*

$$A_1 = \begin{bmatrix} -0.1 & -1 & 0 \\ 1 & -0.1 & 0 \\ 0 & 0 & 0.2 \end{bmatrix}, A_2 = \begin{bmatrix} -0.2 & 1 & 0.1 \\ -1 & -0.2 & 0 \\ 0.1 & 0 & -0.1 \end{bmatrix}, Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \quad (4.67)$$

*and  $Q_1 = Q, Q_2 = -Q$ . It is easy to see that they define a system of the form (4.42) and moreover  $Q$  is invertible. Parameterizing a generic  $x \in \mathcal{Q} = \{x \in \mathbb{R}^3 \mid x^\top Q x = 0\}$  as  $x = [x_1, x_2, \pm \sqrt{x_1^2 + x_2^2}]^\top$ , it can be seen that (4.64) in Step 2\* holds. Using the*

algorithms 0 and 1 of Section 4.5.2, we prove here that the max of 2 quadratics defined by

$$V(x) := \max\{x^\top P_1 x, x^\top P_2 x\}$$

with  $P_1 := \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix}$  and  $P_2 := \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}$  satisfies item (i) of Corollary 4.19. First of all, we have that  $P_1 - P_2 = Q$ , and thus the analysis outlined in Step 0 is simplified, since

$$\begin{aligned} E_{\rho_1} &= \{x \in \mathbb{R}^3 \mid x^\top Q_1 x > 0\} =: D_1 \text{ and } \Phi(\rho_1) = 1, \\ E_{\rho_2} &= \{x \in \mathbb{R}^3 \mid x^\top Q_2 x > 0\} =: D_2 \text{ and } \Phi(\rho_2) = 2, \end{aligned}$$

where  $\rho_1 = (1, 2)$  and  $\rho_2 = (2, 1)$  denote the two elements of  $\mathbb{S}_2$ . Following Step 1, item (i) of Corollary 4.19 holds, since  $P_1 A_1 + A_1^\top P_1 + \tau_1 Q_1 < 0$  and  $P_2 A_2 + A_2^\top P_2 + \tau_2 Q_2 < 0$  are satisfied choosing  $\tau_1 = 0.6, \tau_2 = 0$ . Since (4.64) and (4.65) hold, invoking Proposition 4.28 we have that item (ii) of Corollary 4.19 is satisfied, and  $V$  is a Lyapunov function proving GAS of system (4.67). In Figure 4.5, we have plotted the trajectories of 5 particular solutions  $\phi_i : \mathbb{R}_+ \rightarrow \mathbb{R}^3, i = 1, \dots, 5$ , of system (4.67) and the evolution of  $V$  along these solutions.

## 4.6 Summary

For the class of systems comprising differential inclusions, and state-dependent switched systems, we introduced a family of nonsmooth functions obtained by max-min combinations. Based on two notions of generalized directional derivatives, we proposed sufficient conditions for global asymptotic stability. For a class of systems with conic switching regions and linear dynamics within each of these regions, we studied some conditions under which a max-min condition can be obtained by solving matrix inequalities. A possible route for future research is the generalization of this approach to a wider class of systems, and develop further numerical tools for checking the proposed Lie derivative based conditions.

# 5

## Stability of Interconnections via Non-Pathological ISS-Lyapunov Functions

This chapter address robustness analysis for interconnections of dynamical systems (described by outer semicontinuous differential inclusions) using generalized notions of derivatives associated with locally Lipschitz Lyapunov functions obtained from a finite family of differentiable functions. We first provide sufficient conditions for input-to-state stability (ISS) for differential inclusions, using a class of non-smooth (but locally Lipschitz) candidate Lyapunov functions. In general our conditions are less conservative than the more common Clarke derivative based conditions. We apply our result to (perturbed) state-dependent switched systems, and to the interconnection of two differential inclusions. As an example, we propose an observer-based controller for certain nonlinear two-mode state-dependent switched systems. The content of this chapter is based on the work presented in [52].

### 5.1 Overview

For analyzing stability or performance of integrated or large-scale dynamical systems, it is natural to consider them as a collection of several subsystems of lower dimension/complexity. After a certain abstraction, the behavior of the overall system can be obtained either by switching among the constituent subsystems, or through certain interconnections of the underlying subsystems, or through a combination of these. This viewpoint of analyzing complex systems provides the motivation to consider stability and robustness analysis for interconnections of switched systems.

Given a family of vector fields  $\{f_1, \dots, f_K\} \subset \mathcal{C}^1(\mathbb{R}^n \times \mathbb{R}^m, \mathbb{R}^n)$  and a switching signal  $\sigma : \mathbb{R}^n \rightarrow \{1, \dots, K\}$ , we consider the system

$$\dot{x} = f_{\sigma(x)}(x, u). \quad (5.1)$$

For studying generalized solutions of such discontinuous systems, we extend the map  $f_{\sigma(\cdot)}(\cdot, u)$ , considering the Filippov regularization of (5.1) (see [54]). This leads to a differential inclusion of the form (see Section 5.4 for details)

$$\dot{x} \in F(x, u), \quad (5.2)$$

where  $F$  satisfies some regularity assumptions. Our first objective is to study asymptotic stability and robustness with respect to  $u$  for system (5.2). We then apply our



results to the analysis of interconnected systems of the form

$$\begin{cases} \dot{x}_1 \in F_1(x_1, x_2, u), \\ \dot{x}_2 \in F_2(x_1, x_2, u). \end{cases} \quad (5.3)$$

In the theory of nonlinear control systems, the concept of input-to-state stability (ISS), introduced in [115], has been widely used to study the robustness of dynamical subsystems to external disturbances. Because of its elegant characterization in terms of Lyapunov functions, ISS is now perceived as a textbook tool for analyzing the performance of nonlinear systems, [82]. For example, the ISS notion has been useful in analyzing interconnections of two dynamical systems, either in cascade form [117], or in feedback by using the so-called small-gain condition [76, 75, 72]. Moving away from the framework of conventional nonlinear systems, the ISS notion has been generalized to systems with continuous and discrete dynamics. In this regard, we find sufficient conditions in terms of slow switching for ISS of time-dependent switched systems in [126], characterization of ISS for hybrid systems with jump dynamics in [27], [28], or for a class of differential inclusions in [74]. More recently, we have seen ISS results for interconnections of hybrid systems [85, 108], and time-dependent switched systems [130, 132].

By and large, most of the aforementioned results in the literature deal with *smooth* Lyapunov functions. This is partially justified by the fact that the existence of a smooth Lyapunov function is not only sufficient but also necessary for asymptotic stability of the equilibrium [36], [122], [42], and for ISS with respect to external perturbations [86]. The recent survey on converse Lyapunov theorems [81] provides an insightful background on such developments. In the context of switched and hybrid systems, even though there are converse results proving the existence of smooth Lyapunov functions [58], [28], the lack of constructive elements in the proofs of converse results and the system structure itself provide the motivation to work with multiple Lyapunov functions [84, Chapter 3]. When dealing with time-dependent switched systems, these multiple Lyapunov functions can still be combined to get a smooth (with respect to state) common Lyapunov function. Such constructions have been seen in analyzing ISS of switched system [126] and certain interconnections [130, 132]. However, when dealing with state-dependent switched systems, the *patching* of the Lyapunov functions may make the resulting common Lyapunov function non-differentiable, but locally Lipschitz in most cases. This element is seen in the analysis of asymptotic stability using piecewise differentiable functions [77, 11], and to some extent for establishing ISS as well [64],[63].

In this chapter we propose sufficient conditions for ISS using locally Lipschitz Lyapunov functions for the class of differential inclusions in (5.2). The concept of set-valued derivatives for locally Lipschitz functions, introduced in Section 2.4, is crucial to properly define the notion of derivatives along the system's trajectories. In particular, considering the subclass of locally Lipschitz functions called *non-pathological functions* (as introduced in Definition 2.19), the notion of Lie derivative leads to less conservative stability conditions. The Lie derivative concept has been recently used in [78] to identify and remove infeasible directions of a differential inclusion of the form (5.2), and for stability analysis using an invariance principle

for state-dependent switched systems [79], based on the ideas already introduced in [109].

The technical novelty of this chapter starts with the use of Lie derivative for establishing ISS property of switched systems (5.1), or more generally of the differential inclusion (5.2). As an example, we will apply our results using candidate Lyapunov functions that are obtained by “gluing together” a finite number of pre-constructed  $\mathcal{C}^1$ -functions. The structure of these piecewise functions, formally introduced in [110], fits particularly well the *piecewise-smooth structure* of the switched system (5.1), as already noted in [63]. The recent paper [87] presents conditions for ISS of state-dependent switched systems when the unforced system is globally asymptotically stable, using a trajectory-based analysis rather than Lyapunov functions.

After developing this fundamental result on ISS of a single differential inclusion using the Lie derivative notion, we extend our approach to study ISS of an interconnection of two such differential inclusions. In this context, we extend the *small-gain theorem* presented in [75] by constructing a max-separable Lyapunov function where the individual functions are only Lipschitz continuous. Also, for differential inclusions, the construction proposed in [85] allows for Lipschitz continuous functions, but the authors of [85] only use the Clarke derivative. Here instead, the Lyapunov functions that we associate with each subsystem are only required to satisfy a decrease condition based on the Lie derivative, which are weaker than those using the Clarke derivative. When the interconnection (5.3) is in the so-called *cascade* form, we adapt the construction proposed in [119], constructing a common Lyapunov function in the sum-separable form, starting from two non-pathological Lyapunov functions of the subsystems in (5.3). Earlier ISS results for interconnections of switched systems have focused only on time-dependent switched systems [130, 132], while the contribution of this manuscript revolves in studying interconnections with state-dependent switching. We finally illustrate the usefulness of our results by performing output feedback stabilization of a state-dependent switched system using an observer-based controller. The arising nonlinear conditions are shown to become computationally tractable in the switched linear case.

### 5.1.1 Chapter Organization

This chapter is organized as follows: In Section 5.2 we recall the basic definitions and the necessary preliminaries from non-smooth analysis, with particular attention to the non-pathological class of locally Lipschitz functions. The first main result on ISS of system (5.2) using locally Lipschitz Lyapunov functions is given in Section 5.3, while in Section 5.4 we apply our result to state-dependent switched systems. In Section 5.5, we study interconnected differential inclusions, proposing a Lie derivative-generalization of classical small-gain and cascade arguments. We study the application of our results for feedback stabilization of switched systems in Section 5.6. In the Appendix, we prove some technical results on piecewise  $\mathcal{C}^1$  functions used in Section 5.4 and we prove the equivalence between various ISS-Lyapunov conditions.

## 5.2 Backgrounds

In this section we introduce the required technicalities, in order to treat our ISS analysis of perturbed differential inclusions.

### 5.2.1 Basic notions for perturbed differential inclusions

We recall here the formalism of differential inclusions with inputs, and recall the basic concepts of solutions and stability of equilibrium. These definition will be similar to the ones introduced in Section 2.1, but particular care is needed, since here the right-hand side of (5.2) depends not only on the state  $x \in \mathbb{R}^n$ , but also on an external input/perturbation  $u \in \mathbb{R}^m$

Let  $F : \mathbb{R}^n \times \mathbb{R}^m \rightrightarrows \mathbb{R}^n$  be a set valued map with nonempty, compact and convex values, outer semicontinuous in the first argument and continuous in the second, see Section 2.1 for a thorough discussion about continuity concepts for set-valued maps. Suppose that  $F(0,0) = \{0\}$ , and consider the differential inclusion

$$\dot{x} \in F(x, u), \quad (5.4)$$

where input  $u : \mathbb{R}_+ \rightarrow \mathbb{R}^m$  belongs to the set of measurable and locally essentially bounded functions

$$\mathcal{U} := \{u : \mathbb{R}_+ \rightarrow \mathbb{R}^m \mid u \text{ measurable, } \operatorname{ess\,sup}_{0 \leq \tau \leq T} |u(\tau)| < \infty, \forall T > 0\}.$$

For the unperturbed differential inclusion  $\dot{x} \in F(x, 0)$ , the hypotheses that  $F(\cdot, 0) : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$  has closed, convex and non-empty values together with outer semicontinuity are sufficient for the existence of solutions, and are sometimes referred as *basic assumptions* in the literature, and, as showed in Proposition 2.2 are enough to ensure local existence of solutions. On the other hand, the hypothesis that  $F : \mathbb{R}^n \times \mathbb{R}^m \rightrightarrows \mathbb{R}^n$  is continuous in the second argument is introduced to handle a large class of inputs like  $\mathcal{U}$ .

We introduce here the concepts of solutions: Given a vector  $x_0 \in \mathbb{R}^n$  and an input  $u \in \mathcal{U}$ ,  $x : [0, T) \rightarrow \mathbb{R}^n$  is a (*Carathéodory*) *solution* of system (5.4) starting at  $x_0$  if  $x : [0, T) \rightarrow \mathbb{R}^n$  is absolutely continuous,  $x(0) = x_0$ , and  $\dot{x}(t) \in F(x(t), u(t))$ , for almost every  $t \in [0, T)$ . Under the stated assumptions on the map  $F : \mathbb{R}^n \times \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ , we may prove the following existence result.

#### Proposition 5.1: Local existence

Let  $F : \mathbb{R}^n \times \mathbb{R}^m \rightrightarrows \mathbb{R}^n$  be a mapping with nonempty, compact and convex values, outer semicontinuous in the first argument and continuous in the second argument. Given an input  $u \in \mathcal{U}$ , system (5.4) has solutions from any initial point  $x_0 \in \mathbb{R}^n$ , i.e. there exists (at least) a Caratheodory solution  $x : [0, T) \rightarrow \mathbb{R}^n$  of system (5.4) starting at  $x_0$ , for some  $T > 0$ .

*Proof* : Considering any input  $u \in \mathcal{U}$ , we define  $\tilde{F} : \mathbb{R}_+ \times \mathbb{R}^n \rightrightarrows \mathbb{R}^n$  by

$$\tilde{F}(t, x) := F(u(t), x).$$

We now prove the existence of solutions of the non-autonomous differential inclusion

$$\dot{x}(t) \in \tilde{F}(t, x(t)).$$

By hypothesis,  $\tilde{F}$  is upper semicontinuous with respect to the  $x$ -component. Moreover, by continuity of  $F$  with respect to the second component, for every  $x \in \mathbb{R}^n$ , we can extract a continuous selection, i.e.

$$f(x, \cdot) \in \mathcal{C}(\mathbb{R}^m, \mathbb{R}^n), \quad f(x, u) \in F(x, u), \text{ for every } u \in \mathbb{R}^m,$$

see for example [105, Example 5.57] or [45, Lemma 2.1]. Thus  $f(x, u(\cdot)) : \mathbb{R}_+ \rightarrow \mathbb{R}^n$  is a measurable selection of  $\tilde{F}(\cdot, x)$  since

$$f(x, u(t)) \in F(x, u(t)) = \tilde{F}(t, x).$$

Now consider any  $T > 0$ , and  $r := \text{ess sup}_{0 \leq \tau \leq T} |u(\tau)|$ . Define the compact ball  $B := \bar{\mathbb{B}}(0, r)$ , and consider any compact set  $K \subset \mathbb{R}^n$ ; since  $F : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$  is upper semicontinuous with compact values, by [55, Proposition 2.3]  $F(K \times B)$  is compact, and thus  $\exists M > 0$  such that

$$\sup_{f \in F(K \times B)} |f| =: |F(K \times B)| \leq M.$$

By definition of  $\tilde{F}$ , this implies  $|\tilde{F}(t, x)| \leq M$  for almost every  $(t, x) \in [0, T] \times K$ , i.e.  $\tilde{F}$  is locally essentially bounded. This trivially implies that  $\tilde{F}$  is locally bounded by integrable functions. We can now apply [45, Corollary 5.2]) to conclude local existence of solutions.  $\diamond$

Next, we recall the input-to-state stability (ISS) concept, firstly introduced in [115].

### Definition 5.2

System (5.4) is *input-to-state stable (ISS)* with respect to  $u$  if there exist a class  $\mathcal{KL}$  function  $\beta$ , and a class  $\mathcal{K}$  function  $\chi$  such that, for any  $x_0 \in \mathbb{R}^n$  and for any input  $u \in \mathcal{U}$ , all the solutions starting at  $x_0$  satisfy

$$|x(t)| \leq \beta(|x_0|, t) + \chi\left(\text{ess sup}_{0 \leq \tau \leq t} |u(\tau)|\right), \quad \forall t \geq 0. \quad (5.5)$$

Recalling the definition of  $\mathcal{U}$ , bound (5.5) ensures that the solutions  $x$  be uniformly bounded, and thus *complete*, i.e.  $\text{dom}(x(\cdot)) = [0, +\infty)$ . It is clear that ISS of (5.4) implies global asymptotic stability (GAS) in the unperturbed case  $u \equiv 0$ .

## 5.2.2 Generalized derivatives

Our aim is to prove ISS of system (5.4) via non-smooth Lyapunov functions, and thus in the following we recall various notions of generalized derivatives. For a general introduction of non-smooth analysis tool, we refer to Section 2.4. In particular, given a locally Lipschitz function  $V : \mathbb{R}^n \rightarrow \mathbb{R}$  and a set valued map  $F : \mathbb{R}^n \times \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ , it is easy to adapt the concept of Clarke and Lie derivative of  $V$  along  $F$ , as we present in the following statement.

**Definition 5.3: Set-valued directional derivatives, Perturbed Case**

Given a locally Lipschitz function  $V : \mathbb{R}^n \rightarrow \mathbb{R}$ , and a set-valued map  $F : \mathbb{R}^n \times \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ , given any  $(x, u) \in \mathbb{R}^n \times \mathbb{R}^m$  we define the Clarke and Lie derivatives of  $V$  along  $F$ , respectively, as follows:

$$\begin{aligned}\dot{V}_F(x, u) &:= \{\langle p, f \rangle \mid p \in \partial V(x), f \in F(x, u)\}, \\ \dot{\bar{V}}_F(x, u) &:= \{a \in \mathbb{R} \mid \exists f \in F(x, u) : \langle p, f \rangle = a, \forall p \in \partial V(x)\}.\end{aligned}\tag{5.6}$$

For each  $(x, u) \in \mathbb{R}^n \times \mathbb{R}^m$  the sets  $\dot{V}_F(x, u)$  and  $\dot{\bar{V}}_F(x, u)$  are closed and bounded intervals, possibly empty, see [30]. In particular

$$\dot{\bar{V}}_F(x, u) \subset \dot{V}_F(x, u).\tag{5.7}$$

Moreover, if  $V$  is continuously differentiable at  $x$ , one has  $\partial V(x) = \{\nabla V(x)\}$  and thus

$$\dot{\bar{V}}_F(x, u) = \dot{V}_F(x, u) = \{\langle \nabla V(x), f \rangle \mid f \in F(x, u)\}.$$

In this chapter, we study ISS of differential inclusions with inputs, considering the class of non-pathological candidate Lyapunov functions and investigating how the non-pathological property could be exploited in the context of interconnected differential inclusions. Since it will be used in what follows, we recall here the crucial property of *non-pathological functions*, as introduced in Definition 2.19.

Given a non-pathological function  $V : \mathbb{R}^n \rightarrow \mathbb{R}$ , for any  $u \in \mathcal{U}$ , any initial condition  $x_0 \in \mathbb{R}^n$  and any solution  $x : \text{dom}(x(\cdot)) \rightarrow \mathbb{R}^n$  of (5.4), we have that

$$\frac{d}{dt}V(x(t)) \in \dot{\bar{V}}_F(x(t), u(t))\tag{5.8}$$

for almost every  $t \in \text{dom}(x(\cdot))$ . The proof of this result is obtained following the same reasoning of Lemma 2.23. In Section 5.4 we will define a family of locally Lipschitz functions and we will prove that it is a subset of the non-pathological functions.

### 5.3 ISS Lyapunov result

In this section, we provide sufficient conditions for ISS of system (5.4). In what follows, due to the fact that the set  $\dot{\bar{V}}_F(x, u)$  is possibly empty, we adopt the convention  $\max \emptyset = -\infty$ . The following theorem (and its proof) is an adaptation, in the setting of non-pathological functions, of the reasoning already proposed many times in literature, for example [118] and [82]. The novelty of this result lies in the fact that we require *Lie generalized derivative* of the Lyapunov function to be negative definite. Recalling the inclusion (5.7), this statement can be seen as a generalization of the existing results on ISS of differential inclusions relying on the notion of Clarke derivative, in particular [85].

**Theorem 5.4: Non-Pathological ISS-Lyapunov Functions**

Let  $V : \mathbb{R}^n \rightarrow \mathbb{R}$  be a locally Lipschitz and non-pathological function such that there exist  $\underline{\alpha}, \bar{\alpha} \in \mathcal{K}_\infty$ ,  $\rho \in \mathcal{PD}$  and  $\gamma \in \mathcal{K}$  such that

$$\underline{\alpha}(|x|) \leq V(x) \leq \bar{\alpha}(|x|), \quad (5.9)$$

$$V(x) > \gamma(|u|) \Rightarrow \max \dot{\bar{V}}_F(x, u) \leq -\rho(|x|), \quad (5.10)$$

then system (5.4) is ISS w.r.t.  $u$ , and  $V$  is called a *non-pathological ISS-Lyapunov function* for system (5.4).

*Proof :* Given any input  $u \in \mathcal{U}$ , and any  $x_0 \in \mathbb{R}^n$ , consider any solution of system (5.4) starting at  $x_0$  and with input  $u$ , and denote it by  $x : \text{dom}(x(\cdot)) \rightarrow \mathbb{R}^n$ . The function  $V \circ x : \text{dom}(x(\cdot)) \rightarrow \mathbb{R}$  is absolutely continuous because it is the composition of a locally Lipschitz continuous function and an absolutely continuous function. Then  $\frac{d}{dt}V(x(t))$  exists almost everywhere.

Consider now an input  $u \in \mathcal{U}$  and any  $T \geq 0$ . Define  $c = \gamma(\text{ess sup}_{0 \leq \tau \leq T} |u(\tau)|)$  and the set  $D = \{x \in \mathbb{R}^n \mid V(x) \leq c\}$ .

*Step 1.* Consider first an  $x_0 \in D$ , we want to prove that any solution  $x$  starting at  $x_0$  remains in  $D$ , at least until  $T$ , i.e.  $x(t) \in D$ , for all  $0 \leq t \leq T$ . Consider thus a solution  $x : \text{dom}(x(\cdot)) \rightarrow \mathbb{R}^n$  of system (5.4) starting at  $x_0$  and with input  $u$ . Firstly, we note that for almost every time  $t \in [0, T]$ , such that  $x(t) \notin D$ , we have

$$V(x(t)) > c \geq \gamma(|u(t)|).$$

Thus by (5.10), we have

$$\max \dot{\bar{V}}_F(x(t), u(t)) \leq -\rho(|x(t)|), \quad (5.11)$$

for almost all  $t \in \text{dom}(x(\cdot))$  such that  $x(t) \notin D$ . Let us now suppose, ad absurdum, that the solution leaves  $D$  before  $T > 0$ , that is, there exists a  $0 < \bar{t} \leq T$  such that  $x(\bar{t}) \notin D$  implying  $V(x(\bar{t})) \geq c + \varepsilon$ , for an  $\varepsilon > 0$ . Defining  $\underline{t} := \sup\{0 \leq t < \bar{t} \mid V(x(t)) \leq c\}$ , by continuity of  $V \circ x$ , we have that  $V(x(\underline{t})) = c$ . Recalling that  $V$  is a non-pathological function, by (5.8), we have that  $\frac{d}{dt}V(x(t)) \in \dot{\bar{V}}_F(x(t), u(t))$  for almost every  $t \in [\underline{t}, \bar{t}]$ . From equation (5.11), we conclude

$$\frac{d}{dt}V(x(t)) \leq -\rho(|x(t)|), \quad \text{a.e. in } [\underline{t}, \bar{t}],$$

which implies

$$\varepsilon \leq V(x(\bar{t})) - V(x(\underline{t})) = \int_{\underline{t}}^{\bar{t}} \frac{d}{dt}V(x(\tau))d\tau \leq - \int_{\underline{t}}^{\bar{t}} \rho(|x(\tau)|)d\tau \leq 0,$$

and thus we have a contradiction, proving that, for any solution  $x$  starting in  $D$ ,  $x(t) \in D$ , for any  $0 \leq t \leq T$ .

*Step 2.* Now consider the case  $x_0 \notin D$  and consider any solution  $x : \text{dom}(x(\cdot)) \rightarrow \mathbb{R}^n$  of system (5.4) starting at  $x_0$  and with input  $u$ . We define  $t_1 := \inf\{0 \leq t \leq$

$T \mid x(t) \in D\}$ , if the set  $\{0 \leq t \leq T \mid x(t) \in D\}$  is empty, we set  $t_1 = +\infty$ . By *Step 1.* and by (5.9) we have

$$\underline{\alpha}(|x(t)|) \leq V(x(t)) \leq c$$

for all  $t \geq t_1$ ,  $t \in [0, T]$ , and thus, writing  $\chi := \underline{\alpha}^{-1} \circ \gamma$  and recalling the definition of  $c$ , we obtain

$$|x(t)| \leq \chi\left(\operatorname{ess\,sup}_{0 \leq \tau \leq T} |u(\tau)|\right), \quad \forall t \geq t_1. \quad (5.12)$$

By definition of  $t_1$ , for  $t < t_1$ ,  $t \in [0, T]$ , we have  $x(t) \notin D$ . Reasoning as in *Step 1.* we have

$$\frac{d}{dt}V(x(t)) \leq -\rho(|x(t)|), \quad \text{a.e. in } [0, T] \cap [0, t_1],$$

and thus from [86, Lemma 4.4], there exists a  $\mathcal{KL}$  function  $\tilde{\beta}$  (depending only on  $\rho$ ) such that  $V(x(t)) \leq \tilde{\beta}(V(x_0), t)$ , for all  $t \in [0, T] \cap [0, t_1]$ . Hence, defining  $\beta(\cdot, \cdot) := \underline{\alpha}^{-1} \circ \tilde{\beta}(\underline{\alpha}(\cdot), \cdot)$ , it holds that

$$|x(t)| \leq \beta(|x_0|, t), \quad \forall t \leq t_1. \quad (5.13)$$

Combining (5.12) and (5.13), we finally have

$$|x(t)| \leq \beta(|x_0|, t) + \chi\left(\operatorname{ess\,sup}_{0 \leq \tau \leq T} |u(\tau)|\right). \quad (5.14)$$

Collecting *Steps 1.* and *2.* we have proved that, for any,  $x_0 \in \mathbb{R}^n$  any solution  $x : \operatorname{dom}(x(\cdot)) \rightarrow \mathbb{R}^n$  of system (5.4) starting at  $x_0$  and with input  $u$  satisfies the inequality (5.14). Note that the functions  $\beta$  and  $\chi$  *do not depend* on  $u \in \mathcal{U}$ , on  $T \geq 0$  and on  $x_0 \in \mathbb{R}^n$ , or on the particular chosen solution. By arbitrariness of  $u \in \mathcal{U}$  and  $T > 0$  we can conclude.  $\diamond$

### Remark 5.5

As already observed in the literature, for example in [27], the existence of  $\rho \in \mathcal{PD}$  and  $\gamma \in \mathcal{K}$  such that condition (5.10) holds is implied by the existence of two functions  $\hat{\rho} \in \mathcal{K}_\infty$  and  $\hat{\gamma} \in \mathcal{K}$  such that

$$\max \dot{\bar{V}}_F(x, u) \leq -\hat{\rho}(|x|) + \hat{\gamma}(|u|) \quad \forall (x, u) \in \mathbb{R}^n \times \mathbb{R}^m. \quad (5.15)$$

Indeed, implication (5.15)  $\Rightarrow$  (5.10) holds by choosing  $\rho := \frac{1}{2}\hat{\rho}$  and  $\gamma = \hat{\rho}^{-1} \circ 2\hat{\gamma}$ . The converse implication (5.10)  $\Rightarrow$  (5.15) holds if  $F(0, 0) = \{0\}$  and  $\rho \in \mathcal{K}_\infty$ , and it is proven in Lemma B.2 in the Appendix B.

Moreover, using (5.9), another equivalent formulation of condition (5.15) corresponds to asking that there exist two functions  $\tilde{\rho} \in \mathcal{K}_\infty$  and  $\tilde{\gamma} \in \mathcal{K}$  such that

$$\max \dot{\bar{V}}_F(x, u) \leq -\tilde{\rho}(V(x)) + \tilde{\gamma}(|u|), \quad \forall (x, u) \in \mathbb{R}^n \times \mathbb{R}^m. \quad (5.16)$$

The advantage of (5.16) is that in this formulation the function  $\tilde{\rho}$  represents the decay rate of  $V$  along the solutions.  $\triangle$

## 5.4 State Dependent Switched Systems

In this section, we apply our ISS result to a specific differential inclusion with inputs arising from a suitable regularization of state-dependent switched systems. We recall the concept of a “well-behaved” partition of the state space, of the associated switched system, and of a family of functions related to this partition, and finally we provide the specialization of Theorem 5.4 in this setting.

Since these concepts will be used intensively, we found useful to partially recall the notions/definitions arising in the state-dependent switching setting, as already presented in Section 2.2.

### Definition 5.6: Proper State-Space Partition

Given a finite set of indexes  $\mathcal{I} := \{1, \dots, K\}$ , let us consider closed sets  $X_1, \dots, X_K \subset \mathbb{R}^n$  and open sets  $\mathcal{O}_1, \dots, \mathcal{O}_K \subset \mathbb{R}^n$  such that

- a)  $\bigcup_{i=1}^K X_i = \mathbb{R}^n$ ,
- b)  $X_i \subset \mathcal{O}_i$ , for all  $i \in \mathcal{I}$ ,
- c)  $\overline{\text{int}(X_i)} = X_i$ , for all  $i \in \mathcal{I}$ ,
- d) For every  $i \in \mathcal{I}$ ,  $\text{bd}(X_i)$  has zero Lebesgue measure,
- e)  $X_i \cap X_j = \text{bd}(X_i) \cap \text{bd}(X_j)$ , for all  $i, j \in \mathcal{I}$ ,  $i \neq j$ .

In this situation, we say that  $\mathcal{X} := \{X_i, \mathcal{O}_i\}_{i \in \mathcal{I}}$  is a *proper partition* of  $\mathbb{R}^n$ . We define  $\partial X := \bigcup_{i \in \mathcal{I}} \text{bd}(X_i)$ .

Given a proper partition  $\mathcal{X}$  of  $\mathbb{R}^n$ , the “index indicator map” is the set valued map  $\mathcal{I}_{\mathcal{X}} : \mathbb{R}^n \rightrightarrows \mathcal{I}$  defined as

$$\mathcal{I}_{\mathcal{X}}(x) := \{i \in \mathcal{I} \mid x \in X_i\}. \quad (5.17)$$

We underline that  $\mathcal{I}_{\mathcal{X}}$  is almost everywhere single valued. In fact, by Definition 5.6, item e), if  $x \in \text{int}(X_\ell)$  for some  $\ell \in \mathcal{I}$  then  $\mathcal{I}_{\mathcal{X}}(x) = \{\ell\}$ .

### Definition 5.7: (Perturbed) State-Dependent Switched System

Given  $\mathcal{X} = \{X_i, \mathcal{O}_i\}_{i \in \mathcal{I}}$  a proper partition of  $\mathbb{R}^n$ , consider  $f_i \in \mathcal{C}^1(\mathcal{O}_i \times \mathbb{R}^m, \mathbb{R}^n)$ ,  $i \in \mathcal{I}$ . The *state-dependent switched system* associated to  $\{X_i, \mathcal{O}_i, f_i\}_{i \in \mathcal{I}}$  is, by definition, the differential inclusion

$$\dot{x} \in F^{\text{sw}}(x, u) = \text{co}\{f_i(x, u) \mid i \in \mathcal{I}_{\mathcal{X}}(x)\} \quad (5.18)$$

considering again signals  $u \in \mathcal{U}$ .

We recall the differential inclusion (5.18) is obtained as the Filippov regularization of the discontinuous differential equation defined by  $f_\sigma : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$ ,  $x \mapsto f_{\sigma(x)}(x, u)$ , considering a switching signal  $\sigma : \mathbb{R}^n \rightarrow \mathcal{I}$  such that  $\sigma(x) = i$ , if  $x \in \text{int}(X_i)$ , for any  $i \in \mathcal{I}$ . see Section 2.2 for the proof. Note that the map



$F^{\text{sw}} : \mathbb{R}^n \times \mathbb{R}^m \rightrightarrows \mathbb{R}^n$  satisfies the hypotheses of Section 5.2:  $F^{\text{sw}}(x, u)$  is a compact, convex and non empty set for every  $(x, u) \in \mathbb{R}^n \times \mathbb{R}^m$ , the map  $x \mapsto F^{\text{sw}}(x, u)$  is outer semicontinuous for any  $u \in \mathbb{R}^m$  and the map  $u \mapsto F^{\text{sw}}(x, u)$  is continuous for any  $x \in \mathbb{R}^n$ . In general, in order to have an equilibrium in 0, it is sufficient to suppose that  $0 \in F^{\text{sw}}(0, 0)$ , but, from now on, we will assume the stronger property that  $f_i(0, 0) = 0$ , for all  $i \in \mathcal{I}$  such that  $0 \in \mathcal{O}_i$ , and thus  $F^{\text{sw}}(0, 0) = \{0\}$ .

We introduce here a family of locally Lipschitz functions that we propose as candidate Lyapunov functions for (5.18).

### Definition 5.8: Piecewise $\mathcal{C}^1$ Functions Associated to Proper Partitions

Consider  $\mathcal{J} = \{1, \dots, N\}$  and  $\mathcal{Y} = \{Y_j, \mathcal{V}_j\}_{j \in \mathcal{J}}$ , a proper partition of  $\mathbb{R}^n$ . A function  $V : \mathbb{R}^n \rightarrow \mathbb{R}$  is called a *piecewise  $\mathcal{C}^1$  function* with respect to the proper partition  $\mathcal{Y}$  (and we write  $V \in \mathcal{P}(\mathcal{Y})$ ) if there exist real-valued functions  $V_1, \dots, V_N$  such that

1.  $V_j \in \mathcal{C}^1(\mathcal{V}_j, \mathbb{R})$  for each  $j \in \mathcal{J}$ ,
2.  $V(x) = V_j(x)$ , if  $x \in Y_j$ .

Piecewise  $\mathcal{C}^1$  functions with respect to a proper partition are a particular kind of “*piecewise  $\mathcal{C}^1$  functions*” as introduced in Definition 3.5. More specifically, conditions in Definition 5.8 are equivalent to the ones given in Definition 3.9, where we introduced the concept of “*proper piecewise  $\mathcal{C}^1$  functions*”. Since here we are interested in application to state-dependent switching systems, we found useful to reintroduce this Definition, relying on the concept of proper state-space partition, simplifying the presentation. For a general introduction of piecewise smooth functions, the interested reader is referred to [31],[98] and [110]. We recall here some important properties of piecewise  $\mathcal{C}^1$  w.r.t. proper partitions.

### Proposition 5.9

Consider  $V \in \mathcal{P}(\mathcal{Y})$ , with respect to a proper partition  $\mathcal{Y} = \{Y_j, \mathcal{V}_j\}_{j \in \mathcal{J}}$ , in the sense of Definition 5.8. Then the following hold:

1.  $V$  is locally Lipschitz, non-pathological and

$$\partial V(x) = \text{co} \{ \nabla V_j(x) \mid j \in \mathcal{I}_{\mathcal{Y}}(x) \}. \quad (5.19)$$

2. Given  $F^{\text{sw}} : \mathbb{R}^n \times \mathbb{R}^m \rightrightarrows \mathbb{R}^n$  defined as in (5.18), we have

$$\dot{V}_{F^{\text{sw}}}(x, u) = \left\{ a \in \mathbb{R} \mid \exists f \in F^{\text{sw}}(x, u) : \langle \nabla V_j(x), f \rangle = a, \forall j \in \mathcal{I}_{\mathcal{Y}}(x) \right\}. \quad (5.20)$$

The equivalence in (5.19) is proven in Lemma 3.7 and thus (5.20) trivially holds. We postpone to the Appendix B the proof of non-pathology of functions in  $\mathcal{P}(\mathcal{Y})$ . We can now specialize the results stated in Section 5.3 in this setting. First, we consider a candidate Lyapunov function in the class of *piecewise  $\mathcal{C}^1$  functions*  $\mathcal{P}(\mathcal{Y})$ ,

where the proper partition  $\mathcal{Y}$  does not necessarily coincide with  $\mathcal{X}$ , the proper partition associated with the considered switched system. Then we present specifically the case  $\mathcal{X} = \mathcal{Y}$ , see the subsequent Remark 5.12 for further discussion.

**Corollary 5.10: ISS for state dependent switching**

Consider a proper partition  $\mathcal{X} = \{X_i, \mathcal{O}_i\}_{i \in \mathcal{I}}$  and an associated switched system (5.18). Let us consider another proper partition  $\mathcal{Y} = \{Y_j, \mathcal{V}_j\}_{j \in \mathcal{J}}$  and let  $V \in \mathcal{P}(\mathcal{Y})$ . Suppose that there exist  $\underline{\alpha}, \bar{\alpha} \in \mathcal{K}_\infty$ ,  $\rho \in \mathcal{PD}$  and  $\gamma \in \mathcal{K}$  such that:

**A)** for each  $j \in \mathcal{J}$ , for each  $x \in Y_j$ ,

$$\underline{\alpha}(|x|) \leq V_j(x) \leq \bar{\alpha}(|x|);$$

**B)** for each  $j \in \mathcal{J}$ , for each  $x \in \text{int}(Y_j) \setminus \partial X$  and for each  $u \in \mathbb{R}^m$ ,

$$V(x) > \gamma(|u|) \Rightarrow \langle \nabla V_j(x), f_{\sigma(x)}(x, u) \rangle \leq -\rho(|x|);$$

**C)** for each  $(x, u) \in \partial X \times \mathbb{R}^m$ ,

$$V(x) > \gamma(|u|) \Rightarrow \max \dot{\bar{V}}_{F^{\text{sw}}}(x, u) \leq -\rho(|x|);$$

then system (5.18) is ISS.

*Proof :* By the non-pathological property of  $V$  established in Proposition 5.9, we can apply Theorem 5.4. Getting inequality (5.9) from **A)** is straightforward. We check inequality (5.10) decomposing  $\mathbb{R}^n$  as follows:

$$\mathbb{R}^n = \left( \bigcup_{j \in \mathcal{J}} \text{int}(Y_j) \setminus \partial X \right) \cup \partial X \cup (\partial Y \setminus \partial X).$$

Consider first a point  $x \in \text{int}(Y_j) \setminus \partial X$  for some  $j \in \mathcal{J}$ . Function  $V \in \mathcal{P}(\mathcal{Y})$  is  $\mathcal{C}^1$  at  $x$ , and  $F^{\text{sw}}(x, u) = \{f_{\sigma(x)}(x, u)\}$  is single-valued. Thus, for any  $u \in \mathbb{R}^m$ , we have

$$\dot{\bar{V}}_{F^{\text{sw}}}(x, u) = \{\langle \nabla V_j(x), f_{\sigma(x)}(x, u) \rangle\},$$

and by **B)**, the implication in (5.10) holds.

If  $x \in \partial X$ , the assertion follows directly by **C)**.

As the last step, consider a point  $x \in \partial Y \setminus \partial X$ , and thus  $x \in \text{int}(X_i)$  for some  $i \in \mathcal{I}$ . In particular at  $x$ , we have  $F^{\text{sw}}(x, u) = \{f_i(x, u)\}$ . Consider  $u \in \mathbb{R}^m$  and suppose that  $V(x) > \gamma(|u|)$ . Recalling (5.19) in Lemma 5.9 and by Definition 5.6, for each  $j \in \mathcal{I}_Y(x)$ , there exists a sequence  $x_k^j \rightarrow x$  such that  $x_k^j \in \text{int}(Y_j) \cap \text{int}(X_i)$  for all  $k \in \mathbb{N}$ . By continuity of  $V$  and  $\gamma$ , we can suppose  $V(x_k^j) > \gamma(|u|)$ . At these points, from **B)**, we get  $\langle \nabla V_j(x_k^j), f_i(x_k^j, u) \rangle \leq -\rho(|x_k^j|)$ . By continuity of  $\nabla V_j$ ,  $f_i$  and  $\rho$  we have

$$\langle \nabla V_j(x), f_i(x, u) \rangle \leq -\rho(|x|),$$

for each  $j \in \mathcal{I}_Y(x)$ .

We have proved (5.10) for all  $x \in \mathbb{R}^n$ , concluding the proof.  $\diamond$

We underline that we are not explicitly checking (5.10) on the zero Lebesgue measure set  $\partial Y \setminus \partial X$ : this is possible thanks to the continuity of  $F^{\text{sw}}$  when restricted to  $\text{int}(X_i)$  for some  $i \in \mathcal{I}$ .

As a special case of Corollary 5.10, we present the situation  $\mathcal{X} = \mathcal{Y}$ .

### Corollary 5.11

Consider a proper partition  $\mathcal{X} = \{X_i, \mathcal{O}_i\}_{i \in \mathcal{I}}$  and the associated switched system (5.18). Consider  $V \in \mathcal{P}(\mathcal{X})$  such that there exist  $\underline{\alpha}, \bar{\alpha} \in \mathcal{K}_\infty$ ,  $\rho \in \mathcal{PD}$  and  $\gamma \in \mathcal{K}$  satisfying

**A)'** for each  $i \in \mathcal{I}$ , for each  $x \in X_i$

$$\underline{\alpha}(|x|) \leq V_i(x) \leq \bar{\alpha}(|x|);$$

**B)'** for each  $i \in \mathcal{I}$ , for each  $(x, u) \in \text{int}(X_i) \times \mathbb{R}^m$ ,

$$V(x) > \gamma(|u|) \Rightarrow \langle \nabla V_i(x), f_i(x, u) \rangle \leq -\rho(|x|);$$

**C)'** for each  $(x, u) \in \partial X \times \mathbb{R}^m$ ,

$$V(x) > \gamma(|u|) \Rightarrow \max \dot{\bar{V}}_{F^{\text{sw}}}(x, u) \leq -\rho(|x|);$$

then system (5.18) is ISS.

### Remark 5.12: Comparison between Cor. 5.10 and Cor. 5.11

To check the conditions of Corollary 5.11 for each  $i \in \mathcal{I}$ , we must find a smooth Lyapunov function  $V_i$  for the system  $f_i$  in the set  $\mathcal{O}_i \supset X_i$ . Then we must construct a continuous function  $V$  by “gluing together” the  $V_i$ ’s on  $\partial X$  and finally check the Lie-based condition **C)'** on the switching surface  $\partial X$ . On the other hand, in some situations, it can be difficult to construct a smooth Lyapunov function even for a single subsystem in its region of activation if the subsystem is unstable. For this reason, in Corollary 5.10 we allow the candidate Lyapunov functions to be possibly non-differentiable in the interior of the  $X_i$ , and we do not need to check the conditions at the point of non-differentiability of  $V$ , as long as  $F(\cdot, u)$  is continuous in a neighborhood of  $\partial Y \setminus \partial X$ .  $\triangle$

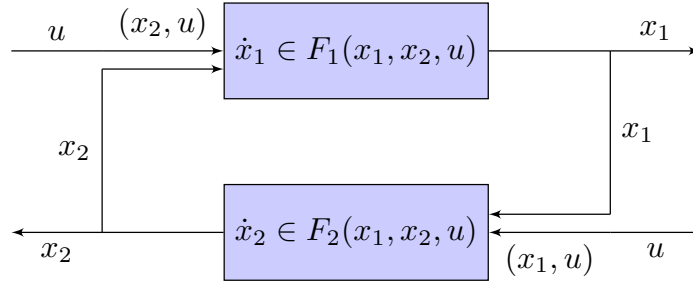


Figure 5.1: The interconnected system in (5.22).

## 5.5 Interconnected Differential Inclusions

In this section, we use Theorem 5.4 to study stability of feedback and cascade interconnections of two systems modeled by differential inclusions.

### 5.5.1 Feedback Interconnection and Small Gain Theorem

For the system shown in Figure 5.1, we establish ISS of the interconnected system by constructing a Lyapunov function from two (nonsmooth) ISS-Lyapunov functions associated with the two subsystems. Our proof techniques are inspired from [76, 75], and here our objective is to generalize those techniques to the case of non-pathological functions, using Lie derivative.

Consider  $F_1 : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \mathbb{R}^m \rightrightarrows \mathbb{R}^{n_1}$  and  $F_2 : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \mathbb{R}^m \rightrightarrows \mathbb{R}^{n_2}$ , and suppose that they have non empty, compact and convex values and are outer semicontinuous in the first two arguments and continuous in the third one. Consider the interconnection

$$\dot{x}_1 \in F_1(x_1, x_2, u), \quad (5.21a)$$

$$\dot{x}_2 \in F_2(x_1, x_2, u). \quad (5.21b)$$

We introduce the notation  $x = (x_1, x_2) := (x_1^\top, x_2^\top)^\top \in \mathbb{R}^n = \mathbb{R}^{n_1+n_2}$  and the augmented differential inclusion

$$\dot{x} \in F(x, u) := \begin{pmatrix} F_1(x_1, x_2, u) \\ F_2(x_1, x_2, u) \end{pmatrix}. \quad (5.22)$$

We suppose that the two-subsystems are ISS with respect to their inputs, and they admit non-pathological ISS-Lyapunov functions, as we state precisely in the following statement.

**Assumption 5.1** *Suppose that there exist non-pathological functions  $V_1 : \mathbb{R}^{n_1} \rightarrow \mathbb{R}$  and  $V_2 : \mathbb{R}^{n_2} \rightarrow \mathbb{R}$  such that*

1a) *There exist  $\underline{\alpha}_1, \bar{\alpha}_1 \in \mathcal{K}_\infty$  satisfying*

$$\underline{\alpha}_1(|x_1|) \leq V_1(x_1) \leq \bar{\alpha}_1(|x_1|), \quad \forall x_1 \in \mathbb{R}^{n_1}.$$

1b) There exist  $\underline{\alpha}_2, \bar{\alpha}_2 \in \mathcal{K}_\infty$  satisfying

$$\underline{\alpha}_2(|x_2|) \leq V_2(x_2) \leq \bar{\alpha}_2(|x_2|), \quad \forall x_2 \in \mathbb{R}^{n_2}.$$

2a) There exist  $\rho_1 \in \mathcal{PD}$ , and  $\chi_1, \gamma_1 \in \mathcal{K}$  satisfying

$$\begin{aligned} V_1(x_1) &> \max\{\chi_1(V_2(x_2)), \gamma_1(|u|)\} \\ &\Downarrow \\ \max \dot{V}_{1,F_1}(x_1, x_2, u) &\leq -\rho_1(V_1(x_1)) \end{aligned}$$

2b) There exist  $\rho_2 \in \mathcal{PD}$ , and  $\chi_2, \gamma_2 \in \mathcal{K}$  satisfying

$$\begin{aligned} V_2(x_2) &> \max\{\chi_2(V_1(x_1)), \gamma_2(|u|)\} \\ &\Downarrow \\ \max \dot{V}_{2,F_2}(x_1, x_2, u) &\leq -\rho_2(V_2(x_2)) \end{aligned}$$

Since we want to combine the functions  $V_1$  and  $V_2$  to obtain a non-pathological ISS function  $W : \mathbb{R}^n \rightarrow \mathbb{R}$  for the interconnected system (5.22), we need to recall the following results from non-smooth analysis.

**Fact 5.1** [34, Theorem 2.6.6] Consider a locally Lipschitz function  $V : \mathbb{R}^k \rightarrow \mathbb{R}$  and  $\sigma \in \mathcal{C}^1(\mathbb{R}, \mathbb{R})$ , and define  $U := \sigma \circ V$ . We have

$$\partial U(x) = \sigma'(V(x))\partial V(x), \quad \forall x \in \mathbb{R}^k,$$

where  $\sigma'(s)$  denotes the derivative of  $\sigma$  at  $s \in \mathbb{R}$ .

**Fact 5.2** [34, Proposition 2.3.12] Given two locally Lipschitz functions  $V_1 : \mathbb{R}^k \rightarrow \mathbb{R}$  and  $V_2 : \mathbb{R}^k \rightarrow \mathbb{R}$  consider the function  $V(x) := \max\{V_1(x), V_2(x)\}$ . Given any  $z \in \mathbb{R}^k$  such that  $V_1(z) = V_2(z)$ , it holds that

$$\partial V(z) \subset \text{co}\{\partial V_1(z), \partial V_2(z)\}.$$

Moreover, we need this well-known comparison result, for the proof, see [75, Theorem 3.1].

**Fact 5.3** Given  $\chi_1, \chi_2 \in \mathcal{K}$  such that

$$\chi_1 \circ \chi_2(r) < r, \quad \forall r > 0.$$

There exists  $\sigma \in \mathcal{K}_\infty \cap \mathcal{C}^1(\mathbb{R}_+, \mathbb{R}_+)$  with  $\sigma'(s) > 0$  for all  $s \in [0, \infty)$  such that

$$\chi_2(r) < \sigma(r), \quad \text{and} \quad \chi_1(\sigma(r)) < r, \quad \forall r > 0. \quad (5.23)$$

The geometrical intuition behind (5.23) is that the graph of the function  $\sigma$  lies between the graphs of  $\chi_2$  and  $\chi_1^{-1}$ , see for example Fig.1 in [75]. Finally, the following lemma will be used in the proof.

**Lemma 5.13**

Suppose  $V_1 : \mathbb{R}^{n_1} \rightarrow \mathbb{R}$  and  $V_2 : \mathbb{R}^{n_2} \rightarrow \mathbb{R}$  are two non-pathological functions satisfying Assumption 5.1. Consider  $\sigma \in \mathcal{C}^1(\mathbb{R}_+, \mathbb{R}_+)$  such that  $\sigma'(s) > 0$  for all  $s > 0$  and the composite function  $U_1 := \sigma \circ V_1$ . Let  $W(x_1, x_2) := \max\{U_1(x_1), V_2(x_2)\}$ , and consider a point  $z = (z_1, z_2) \neq (0, 0)$ ,  $z_1 \in \mathbb{R}^{n_1}$ ,  $z_2 \in \mathbb{R}^{n_2}$  such that  $U_1(z_1) = V_2(z_2)$ . It holds that

$$\partial W(z_1, z_2) = \text{co} \left\{ \partial \widehat{U}_1(z_1), \partial \widehat{V}_2(z_2) \right\} \quad (5.24)$$

where

$$\begin{aligned} \partial \widehat{U}_1(z_1) &:= \left\{ \begin{pmatrix} v_1 \\ 0 \end{pmatrix} \mid v_1 \in \partial U_1(z_1) = \sigma'(V_1(z_1)) \partial V_1(z_1) \right\} \quad \text{and} \\ \partial \widehat{V}_2(z_2) &:= \left\{ \begin{pmatrix} 0 \\ v_2 \end{pmatrix} \mid v_2 \in \partial V_2(z_2) \right\}. \end{aligned}$$

*Proof :* Consider a point  $z = (z_1, z_2) \neq (0, 0)$  such that  $U_1(z_1) = V_2(z_2)$ , the inclusion  $\partial W(z) \subset \text{co}\{\partial \widehat{U}_1(z_1), \partial \widehat{V}_2(z_2)\}$  is obtained by Fact 5.2.

For the converse inclusion, due to convexity of  $\partial W(z_1, z_2)$ , it suffices to show that  $\partial \widehat{U}_1(z_1) \subset \partial W(z_1, z_2)$  and  $\partial \widehat{V}_2(z_2) \subset \partial W(z_1, z_2)$ . We only prove the first inclusion, as the other one can be proved with a similar reasoning. Consider any  $\hat{v} = \begin{pmatrix} v_1 \\ 0 \end{pmatrix} \in$

$\partial \widehat{U}_1(z_1)$ , where  $v_1 \in \partial U_1(z_1)$ . Recalling Definition 2.13 of the generalized gradient, there exists a sequence  $x_1^k \in \mathbb{R}^{n_1}$  such that  $x_1^k \rightarrow z_1$  and such that  $\nabla U_1(x_1^k)$  is defined and  $\nabla U_1(x_1^k) \rightarrow v_1$ . Moreover, from 1b) and 2b) of Assumption 5.1, the function  $V_2$  has no local minima other than 0 because  $V_2$  is a Lyapunov function for the unperturbed system  $\dot{x}_2 \in F_2(0, x_2, 0)$ . Thus, considering any point  $x_2 \neq 0$ ,  $V_2$  is decreasing along the solutions starting at  $(0, x_2)$  with zero input. By local existence of solutions from any initial point, we have that  $x_2$  cannot be a local minimum of  $V_2$ . Thus  $z_2$  is not a local minima for  $V_2$ , and we can consider a sequence  $x_2^\ell \rightarrow z_2$  such that  $V_2(z_2) > V_2(x_2^\ell)$ , for all  $\ell \in \mathbb{N}$ . Now, by continuity of  $U_1$  and  $V_2$ , for each  $\ell \in \mathbb{N}$ , there exists  $k_\ell \in \mathbb{N}$  such that

$$U_1(x_1^{k_\ell}) > V_2(x_2^\ell). \quad (5.25)$$

Consider the sequence  $\bar{x}_\ell := \begin{pmatrix} x_1^{k_\ell} \\ x_2^\ell \end{pmatrix} \in \mathbb{R}^n$ . We have  $\bar{x}_\ell \rightarrow z = (z_1, z_2)$ , and from equation (5.25)

$$W(\bar{x}_\ell) = \max\{U_1(x_1^{k_\ell}), V_2(x_2^\ell)\} = U_1(x_1^{k_\ell}), \quad \forall \ell \in \mathbb{N}.$$

Thus,  $W$  is differentiable at all  $\bar{x}_\ell \in \mathbb{R}^n$  and

$$\lim_{\ell \rightarrow \infty} \nabla W(\bar{x}_\ell) = \begin{pmatrix} \lim_{\ell \rightarrow \infty} \nabla U_1(x_1^{k_\ell}) \\ 0 \end{pmatrix} = \begin{pmatrix} v_1 \\ 0 \end{pmatrix} = \hat{v}.$$

By definition of  $\hat{v}$  and the generalized gradient, it follows that  $\hat{v} \in \partial W(z_1, z_2)$  and hence  $\partial \hat{U}_1(z_1) \subset \partial W(z_1, z_2)$ . Similarly, one can prove that  $\partial \hat{V}_2(z_2) \subset \partial W(z_1, z_2)$ , and thus the equality (5.24) holds.  $\diamond$

We have now all the necessary tools to present a small gain theorem involving non-pathological ISS functions, adapting the idea firstly proposed in [75].

### Theorem 5.14: Generalized Small Gain Theorem

Consider the non-pathological functions  $V_1, V_2$  satisfying Assumption 5.1 and suppose that

$$\chi_1 \circ \chi_2(r) < r, \quad \forall r > 0. \quad (5.26)$$

Considering a function  $\sigma \in \mathcal{K}_\infty \cap \mathcal{C}^1(\mathbb{R}_+, \mathbb{R}_+)$  satisfying property (5.23) in Fact 5.3, define  $W : \mathbb{R}^n \rightarrow \mathbb{R}$  as

$$W(x_1, x_2) := \max\{\sigma(V_1(x_1)), V_2(x_2)\}. \quad (5.27)$$

Then  $W$  is a non-pathological ISS function and thus system (5.22) is ISS w.r.t.  $u$ .

*Proof :* We want to show that  $W : \mathbb{R}^n \rightarrow \mathbb{R}$  satisfies all the conditions of Theorem 5.4. To this end, it is enough to show that

- A)  $\sigma \circ V_1 : \mathbb{R}^{n_1} \rightarrow \mathbb{R}$  is non-pathological, and  $W : \mathbb{R}^n \rightarrow \mathbb{R}$  is non-pathological.
- B) There exist  $\rho \in \mathcal{PD}$  and  $\gamma \in \mathcal{K}$  such that

$$W(x) > \gamma(|u|) \Rightarrow \max \bar{W}_F(x, u) \leq -\rho(|x|). \quad (5.28)$$

*Proof of A):* We recall that  $V_1 : \mathbb{R}^{n_1} \rightarrow \mathbb{R}$  is non-pathological and  $\sigma \in \mathcal{C}^1(\mathbb{R}, \mathbb{R})$  and  $\sigma'(r) > 0$  for all  $r > 0$ . Defining  $U_1 := \sigma \circ V_1$ , by Fact 5.1 we have  $\partial U_1(x) = \sigma'(V_1(x))\partial V_1(x)$  for all  $x \in \mathbb{R}^{n_1}$ . Moreover for any absolutely continuous function  $\varphi : [0, T) \rightarrow \mathbb{R}^{n_1}$ , we have that  $\partial V_1(\varphi(t))$  is a subset of an affine subspace orthogonal to  $\dot{\varphi}(t)$ , for almost every  $t \in [0, T)$ , and the same holds for  $\partial U_1(\varphi(t)) = \sigma'(V_1(\varphi(t)))\partial V_1(\varphi(t))$ . Thus  $U_1 : \mathbb{R}^{n_1} \rightarrow \mathbb{R}$  is non-pathological. The non-pathology of  $W : \mathbb{R}^n \rightarrow \mathbb{R}$  follows from the fact that pointwise maximum of non-pathological functions is non-pathological, as stated in Lemma 2.20.

*Proof of B):* We proceed by considering three cases. Let us define the sets

$$\begin{aligned} \mathcal{O}_1 &:= \{(x_1, x_2) \in \mathbb{R}^n \mid V_2(x_2) < \sigma(V_1(x_1))\}, \\ \mathcal{O}_2 &:= \{(x_1, x_2) \in \mathbb{R}^n \mid V_2(x_2) > \sigma(V_1(x_1))\}, \\ \Gamma &:= \{(x_1, x_2) \in \mathbb{R}^n \mid V_2(x_2) = \sigma(V_1(x_1))\}. \end{aligned}$$

For  $z = (z_1, z_2) \in \mathcal{O}_1$ , by continuity there exists a neighborhood  $\mathcal{U}$  of  $z$  where  $W(x) = \sigma(V_1(x_1))$ , for all  $x = (x_1, x_2) \in \mathcal{U}$ . By Fact 5.1, we have  $\partial W(z) = \sigma'(V_1(z_1))\partial V_1(z_1) \times 0$  and in particular,

$$\dot{\bar{W}}_F(z, u) = \sigma'(V_1(z_1))\dot{\bar{V}}_{1, F_1}(z_1, z_2, u). \quad (5.29)$$

Recalling that  $z \in \mathcal{O}_1$  and equation (5.23), we have  $\chi_1(V_2(z_2)) < \chi_1(\sigma(V_1(z_1))) < V_1(z_1)$ . Thus, by condition 2a) of Assumption 5.1, we have from (5.29) that

$$W(z) > \hat{\gamma}_1(|u|) \Rightarrow \max \dot{\bar{W}}_F(z, u) \leq -\hat{\rho}_1(W(z)), \quad \forall z \in \mathcal{O}_1, \quad (5.30)$$

where  $\hat{\rho}_1(s) := \sigma'(\sigma^{-1}(s))\rho_1(\sigma^{-1}(s))$  is a positive definite function and  $\hat{\gamma}_1(s) := \sigma(\gamma_1(s))$  is class  $\mathcal{K}$ .

For  $z = (z_1, z_2) \in \mathcal{O}_2$ , following the same reasoning (but without the complications introduced by  $\sigma$ ), one has that

$$W(z) > \gamma_2(|u|) \Rightarrow \max \dot{\bar{W}}_F(z, u) \leq -\rho_2(W(z)), \quad \forall z \in \mathcal{O}_2. \quad (5.31)$$

Before addressing  $z = (z_1, z_2) \in \Gamma$ , using an idea proposed in [78], we introduce the following notation motivated by definition (5.6): Given  $F : \mathbb{R}^n \times \mathbb{R}^m \rightrightarrows \mathbb{R}^n$  and a locally Lipschitz function  $V : \mathbb{R}^n \rightarrow \mathbb{R}$  we define

$$F^V(z, u) := \{f \in F(z, u) \mid \exists a \in \mathbb{R} : \langle v, f \rangle = a, \forall v \in \partial V(z)\}.$$

By Definition 5.3, it is clear that

$$\dot{\bar{V}}_F(z, u) = \{\langle v, f \rangle \mid v \in \partial V(x), f \in F^V(z, u)\}. \quad (5.32)$$

We continue by using the following set inclusion whose proof is postponed a few lines to avoid breaking the flow of the exposition:

$$F^W(z, u) \subset F_1^{V_1}(z_1, z_2, u) \times F_2^{V_2}(z_1, z_2, u). \quad (5.33)$$

Finally consider  $z = (z_1, z_2)^\top \in \Gamma$  and take any  $w \in \partial W(z)$ , by Lemma 5.13, there exist  $v_1 \in \partial V_1(z_1)$ ,  $v_2 \in \partial V_2(z_2)$  and  $\lambda \in [0, 1]$  such that

$$w = \begin{pmatrix} \lambda \sigma'(V_1(z_1))v_1 \\ (1 - \lambda)v_2 \end{pmatrix}.$$

Consider  $f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \in F^W(z, u)$ , so that, from (5.33),  $f_1 \in F_1^{V_1}(z_1, z_2, u)$  and  $f_2 \in F_2^{V_2}(z_1, z_2, u)$ . Using (5.32), we may proceed as in (5.30) and (5.31) and use continuity of  $W$  to get, for all  $z \in \Gamma = \text{bd}(\mathcal{O}_1) \cap \text{bd}(\mathcal{O}_2)$

$$\begin{aligned} W(z) > \hat{\gamma}_1(|u|) &\Rightarrow \max_{\substack{f_1 \in F_1^{V_1}(z, u) \\ v_1 \in \partial V_1(z_1)}} \sigma'(V_1(z_1))\langle v_1, f_1 \rangle \leq -\hat{\rho}_1(W(z)) \\ W(z) > \gamma_2(|u|) &\Rightarrow \max_{\substack{f_2 \in F_2^{V_2}(z, u) \\ v_2 \in \partial V_2(z_2)}} \langle v_2, f_2 \rangle \leq -\rho_2(W(z)). \end{aligned} \quad (5.34)$$



Using (5.34) we finally get that  $W(z) > \max\{\hat{\gamma}_1(|u|), \gamma_2(|u|)\}$  implies

$$\begin{aligned} \langle w, f \rangle &= \lambda \sigma'(V_1(z_1)) \langle v_1, f_1 \rangle + (1 - \lambda) \langle v_2, f_2 \rangle \\ &\leq -\lambda \hat{\rho}_1(W(z)) - (1 - \lambda) \rho_2(W(z)) \\ &\leq -\min\{\hat{\rho}_1(W(z)), \rho_2(W(z))\}. \end{aligned}$$

Thus, letting  $\gamma(s) := \max\{\hat{\gamma}_1(s), \gamma_2(s)\}$  and  $\rho(s) := \min\{\hat{\rho}_1(s), \rho_2(s)\}$ , we have

$$W(z) > \gamma(|u|) \Rightarrow \max \bar{W}_F(z, u) \leq -\rho(W(z)), \quad \forall z \in \Gamma. \quad (5.35)$$

Collecting (5.30), (5.31) and (5.35) we can conclude (5.28), and prove item B).

We complete the proof by proving (5.33). To this, take any  $f \in F^W(z, u)$ . By definition of  $F$  in (5.22), we have that  $f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$  for some  $f_1 \in F_1(z_1, z_2, u)$  and  $f_2 \in F_2(z_1, z_2, u)$ . By Lemma 5.13 and Fact 5.1, for any  $v_1 \in \partial V_1(z_1)$ , the vector  $w = \begin{pmatrix} \sigma'(V_1(z_1))v_1 \\ 0 \end{pmatrix} \in \partial W(z)$  and thus

$$\langle w, f \rangle = \sigma'(V_1(z_1)) \langle v_1, f_1 \rangle.$$

By varying  $v_1$  in  $\partial V_1(z_1)$  and recalling that  $f \in F^W(z, u)$  (and thus  $\langle w, f \rangle$  is constant for all  $w \in \partial W(z)$ ), we obtain that  $f_1 \in F_1^{V_1}(z_1, z_2, u)$ , that is  $\langle v_1, f_1 \rangle$  constant w.r.t.  $v_1 \in \partial V_1(z_1)$ . The same reasoning applies to  $f_2$ , considering a vector  $w = \begin{pmatrix} 0 \\ v_2 \end{pmatrix} \in \partial W(z)$ , with  $v_2 \in \partial V_2(z_2)$ , concluding the proof of the claim.  $\diamond$

### Remark 5.15

The idea of analyzing the derivative of the composite function  $W$  in the three sets  $\mathcal{O}_1, \mathcal{O}_2, \Gamma$ , appeared firstly in [75], and is the common idea of many results on small-gain theorems for interconnected systems, see for example [85] or [72]. The analysis in  $\mathcal{O}_1$  and  $\mathcal{O}_2$  was straightforward, but because of non-differentiability of  $V_1$  and  $V_2$ , the analysis in the set  $\Gamma$  is different from [75]. In particular the additional tools from nonsmooth analysis have been used to study the Lie-derivative of  $W$  along  $F$  on the set  $\Gamma$ .

## 5.5.2 Cascade System

We now apply Theorem 5.4 to *cascade interconnections* as in Figure 5.2. More precisely, given two maps  $F_1 : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \mathbb{R}^m \rightrightarrows \mathbb{R}^{n_1}$ , and  $F_2 : \mathbb{R}^{n_2} \times \mathbb{R}^m \rightrightarrows \mathbb{R}^{n_2}$  we consider the *cascade* system defined as follow:

$$\dot{x}_1 \in F_1(x_1, x_2, u), \quad (5.36a)$$

$$\dot{x}_2 \in F_2(x_2, u). \quad (5.36b)$$

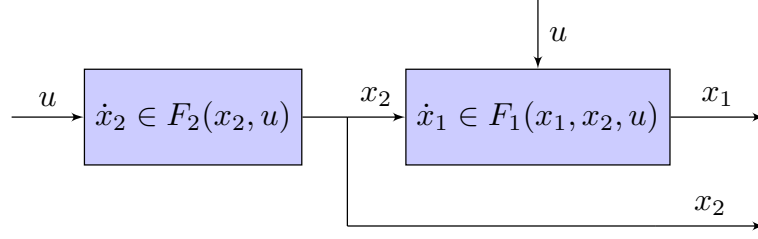


Figure 5.2: The cascade system in (5.36).

Defining again  $n := n_1 + n_2$  we will write  $F : \mathbb{R}^n \times \mathbb{R}^m \rightrightarrows \mathbb{R}^n$  defined by

$$F(x_1, x_2, u) := \begin{pmatrix} F_1(x_1, x_2, u) \\ F_2(x_2, u) \end{pmatrix}.$$

The cascade system (5.36) can be seen as a system of the form (5.22) where  $F_2$  does not depend on  $x_1$ , see also Figure 5.2. Therefore Theorem 5.14 can be applied with  $\chi_1 \equiv 0$  and condition (5.26) holds for any  $\chi_2 \in \mathcal{K}$ . On the other hand, the cascade structure allows us to construct a different ISS-Lyapunov function, based on two non-smooth ISS-Lyapunov functions associated with each subsystem. The function that we construct is in the so-called *sum-separable* form, that has some clear advantages with respect to the *max-separable* form (5.27) in Theorem 5.14, see [73] and references therein for a thorough discussion. In particular, the sum-separable architecture preserves regularity, and in our setting also leads to a more direct proof of ISS of the cascade interconnection.

As a starting point of our construction, we assume that both subsystems admit a non-pathological ISS Lyapunov function, using the formulation (5.15) in Remark 5.5.

**Assumption 5.2** *The following conditions hold for system (5.36):*

**(A.1)** ISS of (5.36b). *There exist a non-pathological function  $V_2 : \mathbb{R}^{n_2} \rightarrow \mathbb{R}$  and  $\underline{\alpha}_2, \bar{\alpha}_2, \rho_2 \in \mathcal{K}_\infty$  and  $\gamma_2 \in \mathcal{K}$  satisfying*

$$\begin{aligned} \underline{\alpha}_2(|x_2|) &\leq V_2(x_2) \leq \bar{\alpha}_2(|x_2|), \quad \forall x_2 \in \mathbb{R}^{n_2}, \\ \max \dot{V}_{2,F_2}(x_2, u) &\leq -\rho_2(V_2(x_2)) + \gamma_2(|u|), \end{aligned}$$

*for all  $x_2 \in \mathbb{R}^{n_2}$  and for all  $u \in \mathbb{R}^m$ .*

**(A.2)** ISS of (5.36a). *There exist a non-pathological function  $V_1 : \mathbb{R}^{n_1} \rightarrow \mathbb{R}$  and  $\underline{\alpha}_1, \bar{\alpha}_1, \rho_1, \gamma_1 \in \mathcal{K}_\infty$  satisfying*

$$\begin{aligned} \underline{\alpha}_1(|x_1|) &\leq V_1(x_1) \leq \bar{\alpha}_1(|x_1|), \quad \forall x_1 \in \mathbb{R}^{n_1}, \\ \max \dot{V}_{1,F_1}(x_1, x_2, u) &\leq -\rho_1(V_1(x_1)) \\ &\quad + \gamma_1(V_2(x_2)) + \gamma_2(|u|), \end{aligned}$$

*for all  $x_1 \in \mathbb{R}^{n_1}$ ,  $x_2 \in \mathbb{R}^{n_2}$  and all  $u \in \mathbb{R}^m$ .*

(A.3) Defining  $\bar{\nu}(s) := \gamma_1(s)/\rho_2(s)$ , there exists a scalar  $M > 0$  such that

$$\lim_{s \rightarrow 0^+} \bar{\nu}(s) = \lim_{s \rightarrow 0^+} \frac{\gamma_1(s)}{\rho_2(s)} \leq M.$$

In other words,  $\gamma_1(s) \in O(\rho_2(s))$  as  $s \rightarrow 0^+$ .

### Remark 5.16: Tightness of Assumption 5.2

Condition (A.3), which is used in the construction of  $W$  in the proof of the following Proposition 5.17, is not restrictive: if it does not hold it is possible to modify the function  $V_1$  in such a way that it holds, following the same idea proposed in [117]. Due to this fact, Proposition 5.17 establishes that when system (5.21) is in the cascade-form presented in equation (5.36), it suffices to have ISS-Lyapunov functions (satisfying the Lie-derivative conditions presented in (A.1) and (A.2)) for each subsystem, to conclude ISS of the interconnected system. In this context, the *small gain condition* required in the general construction of Theorem 5.14 is somehow trivially satisfied.  $\triangle$

Using Assumption 5.2, we can construct a non-pathological Lyapunov function for the cascade system (5.36), by adapting a Lyapunov design developed in [117] and [132].

### Proposition 5.17

Consider the cascade system (5.36), and suppose that Assumption 5.2 holds. There exists a continuous and nondecreasing function  $\nu : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  satisfying  $\nu(s) \geq 4\bar{\nu}(s)$ , for all  $s \in \mathbb{R}_+$ . Moreover, the function

$$W(x_1, x_2) := \int_0^{V_2(x_2)} \nu(s) ds + V_1(x_1) \quad (5.37)$$

is a non-pathological ISS functions for system (5.36); that is there exist  $\underline{\alpha}, \bar{\alpha} \in \mathcal{K}_\infty$  such that

$$\underline{\alpha}(|(x_1, x_2)|) \leq W(x_1, x_2) \leq \bar{\alpha}(|(x_1, x_2)|) \quad (5.38)$$

for all  $(x_1, x_2) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ , and there exist  $\rho \in \mathcal{K}_\infty$  and  $\gamma \in \mathcal{K}$  such that

$$\max \dot{\bar{W}}_F(x_1, x_2, u) \leq -\rho(W(x_1, x_2)) + \gamma(|u|), \quad (5.39)$$

for all  $(x_1, x_2) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$  and for all  $u \in \mathbb{R}^m$ .

*Proof:* The existence of function  $\nu : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  under (A.3) of Assumption 5.2 is established in [117, Lemmas 1 and 2]. Introduce the function  $\ell : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  defined by

$$\ell(s) := \int_0^s \nu(r) dr, \quad \forall s \in \mathbb{R}_+.$$

Since  $\nu(s) \geq 4\bar{\nu}(s) > 0, \forall s > 0$ , then  $\ell$  is a class  $\mathcal{K}_\infty$  function. Moreover  $\ell \in \mathcal{C}^1(\mathbb{R}_+, \mathbb{R}_+)$ . We can thus rewrite

$$W(x_1, x_2) = \ell \circ V_2(x_2) + V_1(x_1).$$

Non-pathology of  $W$  follows from Proposition 5.9 and Fact 5.1 since  $\ell$  is  $\mathcal{C}^1$  by construction. Moreover, the functions  $\underline{\alpha}$  and  $\bar{\alpha}$  of equation (5.38) are easily constructed as  $\underline{\alpha}(s) := \int_0^{\underline{\alpha}_2(s)} \nu(r)dr + \underline{\alpha}_1(s)$  and  $\bar{\alpha}(s) := \int_0^{\bar{\alpha}_2(s)} \nu(r)dr + \bar{\alpha}_1(s)$ .

Let us now define  $U_2 := \ell \circ V_2$ ; noting that  $\ell'(s) = \nu(s)$  and using Fact 5.1, we have that  $\dot{\bar{U}}_{2,F_2}(x_2, u) = \nu(V_2(x_2))\dot{\bar{V}}_{2,F_2}(x_2, u)$ . Recalling **(A.1)**, we can write

$$\begin{aligned} \max \dot{\bar{U}}_{2,F_2}(x_2, u) &\leq -\nu(V_2(x_2))\rho_2(V_2(x_2)) \\ &\quad + \nu(V_2(x_2))\gamma_2(|u|), \end{aligned} \quad (5.40)$$

$\forall x_2 \in \mathbb{R}^{n_2}, \forall u \in \mathbb{R}^m$ . Defining  $\theta(s) := \rho_2^{-1}(2\gamma_2(s))$ , we prove the following inequality

$$\begin{aligned} \max \dot{\bar{U}}_{2,F_2}(x_2, u) &\leq -\frac{1}{2}\nu(V_2(x_2))\rho_2(V_2(x_2)) \\ &\quad + \nu(\theta(|u|))\gamma_2(|u|). \end{aligned} \quad (5.41)$$

Indeed, by (5.40), if  $\gamma_2(|u|) \leq \frac{1}{2}\rho_2(V_2(x_2))$ , (5.41) trivially holds. Otherwise, we see that

$$\gamma_2(|u|) \geq \frac{1}{2}\rho_2(V_2(x_2)) \Leftrightarrow V_2(x_2) \leq \rho_2^{-1}(2\gamma_2(|u|)) = \theta(|u|),$$

and by the nondecreasing property of  $\nu$ , inequality (5.41) holds. Before proceeding to proving (5.39) we observe the following equality

$$\dot{\bar{W}}_F(x_1, x_2, u) = \dot{\bar{U}}_{2,F_2}(x_2, u) + \dot{\bar{V}}_{1,F_1}(x_1, x_2). \quad (5.42)$$

To show (5.42), we recall that any locally Lipschitz function  $G : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \rightarrow \mathbb{R}$  defined by  $G(x_1, x_2) = G_1(x_1) + G_2(x_2)$ , satisfies

$$\partial G(x_1, x_2) = \left\{ \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \mid v_1 \in \partial G_1(x_1), v_2 \in \partial G_2(x_2) \right\} \quad (5.43)$$

and thus, using definition (5.6), we obtain (5.42). From **(A.2)**, (5.41) and (5.42), we have

$$\begin{aligned} \max \dot{\bar{W}}_F(x_1, x_2, u) &\leq -\rho_1(V_1(x_1)) + \gamma_1(V_2(x_2)) + \gamma_2(|u|) \\ &\quad - \frac{1}{2}\nu(V_2(x_2))\rho_2(V_2(x_2)) + \nu(\theta(|u|))\gamma_2(|u|). \end{aligned}$$

From the assumption  $\nu(s) \geq 4\frac{\gamma_1(s)}{\rho_2(s)}$  for all  $s \in \mathbb{R}_+$ , and following [80, Lemma 10], we finally have

$$\begin{aligned} \max \dot{\bar{W}}_F(x_1, x_2) &\leq -\gamma_1(V_2(x_2)) - \rho_1(V_1(x_1)) \\ &\quad + \nu(\theta(|u|))\gamma_2(|u|) + \gamma_2(|u|) \\ &\leq -\rho(W(x_1, x_2)) + \gamma(|u|), \end{aligned}$$

where we have defined

$$\begin{aligned} \gamma(s) &:= (\nu(\theta(s)) + 1) \gamma_2(s), \\ \rho(s) &:= \min \left\{ \rho_1\left(\frac{1}{2}s\right), \gamma_1\left(\frac{1}{2}\ell^{-1}(s)\right) \right\}. \end{aligned}$$

◇

## 5.6 Feedback Stabilization of a 2-mode Switched System

We now use the tools developed in the previous section to solve an output feedback stabilization problem for a class of switched systems with two modes. In particular, we consider the state dependent switched system defined as

$$\mathcal{S} : \begin{cases} \dot{x} = \begin{cases} f_1(x) + g(x)u & \text{if } x \in X_1 := \{q(x) \geq 0\}, \\ f_2(x) + g(x)u & \text{if } x \in X_2 := \{q(x) \leq 0\}, \end{cases} \\ y = h(x), \end{cases} \quad (5.44)$$

where  $x \in \mathbb{R}^n$  and  $u \in \mathbb{R}^m$ . The basic assumptions we impose on the system (5.44) are the following:

**Assumption 5.3** *The data in (5.44) is such that:*

- $f_1, f_2 \in \mathcal{C}^1(\mathbb{R}^n, \mathbb{R}^n)$ ;
- $g \in \mathcal{C}^1(\mathbb{R}^n, \mathbb{R}^{n \times m})$ ;
- $h \in \mathcal{C}^1(\mathbb{R}^n, \mathbb{R}^p)$ ;
- $q \in \mathcal{C}^1(\mathbb{R}^n, \mathbb{R})$  and  $\mathcal{X} = \{X_i, \mathbb{R}^n\}_{i \in \{1,2\}}$  form a proper partition of  $\mathbb{R}^n$  (recall Definition 5.6);
- $q(0) \geq 0 \Rightarrow f_1(0) = 0$  and  $q(0) \leq 0 \Rightarrow f_2(0) = 0$ ;
- There exists  $\beta_g \in \mathcal{C}(\mathbb{R}_+, \mathbb{R})$ ,  $\beta_g(s) \geq 0$  for all  $s \geq 0$ , such<sup>1</sup> that  $\|g(x)\| \leq \beta_g(|x|)$ , for all  $x \in \mathbb{R}^n$ ,
- There exists  $\beta_f \in \mathcal{K}_\infty$  such that  $|f_1(x) - f_2(x)| \leq \beta_f(|x|)$ , for all  $x \in \mathbb{R}^n$ .

**Example 5.1 (Switched Linear Case)** *As a simple paradigm, one can think of a state-dependent switched linear system, such as*

$$f_i(x) = A_i x, \quad g(x) \equiv B, \quad h(x) = Cx,$$

where  $A_i \in \mathbb{R}^{n \times n}$  for  $i \in \{1, 2\}$ ,  $B \in \mathbb{R}^{n \times m}$  and  $C \in \mathbb{R}^{p \times n}$ . Regarding the function  $q \in \mathcal{C}^1(\mathbb{R}^n, \mathbb{R})$ , the simplest non-trivial cases are the halfspace partitions or the symmetric conic partitions, described respectively by the functions

$$q_v(x) := \langle v, x \rangle \quad \text{or} \quad q_Q(x) := x^\top Qx,$$

for some  $v \in \mathbb{R}^n$ , or  $Q \in \text{Sym}(\mathbb{R}^n)$ ,  $Q$  is neither negative, nor positive semi-definite. These cases satisfy Assumption 5.3, by selecting

$$\beta_g(s) := \|B\|, \quad \text{and} \quad \beta_f(s) := \|A_1 - A_2\|s. \quad \triangle$$

<sup>1</sup>Here and in what follows we consider the 2-induced matrix norm, i.e., given  $A \in \mathbb{R}^{n \times m}$  we define

$$\|A\| := \sup_{|x|=1} |Ax|$$

Under Assumption 5.3, we design next an observer-based controller for system (5.44) of the form

$$\mathcal{C} : \begin{cases} \dot{z} = \begin{cases} f_1(z) + g(z)u + \ell_1(y - h(z)) & \text{if } z \in X_1, \\ f_2(z) + g(z)u + \ell_2(y - h(z)) & \text{if } z \in X_2, \end{cases} \\ u = k(z), \end{cases} \quad (5.45)$$

where  $\ell_1, \ell_2 \in \mathcal{C}^1(\mathbb{R}^p, \mathbb{R}^n)$ , and the globally Lipschitz map  $k : \mathbb{R}^n \rightarrow \mathbb{R}^m$  are design parameters.

We consider the interconnected system (5.44)-(5.45), and in particular its Filippov regularization, which can be written as follows

$$\dot{x} \in \text{co} \{f_i(x) \mid i \in \mathcal{I}_{\mathcal{X}}(x)\} + g(x)k(z) =: \tilde{F}_x(x, z), \quad (5.46a)$$

$$\begin{aligned} \dot{z} \in \text{co} \{f_i(z) + \ell_i(h(x) - h(z)) \mid i \in \mathcal{I}_{\mathcal{X}}(z)\} \\ + g(z)k(z) =: \tilde{F}_z(x, z), \end{aligned} \quad (5.46b)$$

where the function  $\mathcal{I}_{\mathcal{X}}$  is defined as in (5.17).

The maps  $\tilde{F}_x, \tilde{F}_z : \mathbb{R}^n \times \mathbb{R}^n \rightrightarrows \mathbb{R}^n$  satisfy the conditions of Section 5.2: they have non-empty, compact and convex values and are outer semicontinuous with respect to the states ( $x$  and  $z$  respectively) and continuous with respect to the inputs ( $z$  and  $x$  respectively). We can thus conclude local existence of solutions for the systems (5.46a), (5.46b) using Proposition 5.1.

To design (5.45), we first characterize stability of the interconnection (5.46). To this end, we perform the change of coordinates  $(x, z) \mapsto (x, e) := (x, x - z)$  and we construct the Filippov regularization of the corresponding dynamics, resulting in

$$\dot{x} \in F_x(x, e) := \tilde{F}_x(x, x - e) \quad (5.47a)$$

$$\dot{e} \in F_e(x, e), \quad (5.47b)$$

where the map  $F_e : \mathbb{R}^n \times \mathbb{R}^n \rightrightarrows \mathbb{R}^n$  is defined as the Filippov regularization of the discontinuous map  $f_e(x, e) :=$

$$\begin{aligned} & f_1(x) - f_1(z) - \ell_1(h(x) - h(z)) + \tilde{g}(x, z) \text{ if } q(x) \geq 0, q(z) \geq 0, \\ & f_2(x) - f_1(z) - \ell_1(h(x) - h(z)) + \tilde{g}(x, z) \text{ if } q(x) \leq 0, q(z) \geq 0, \\ & f_1(x) - f_2(z) - \ell_2(h(x) - h(z)) + \tilde{g}(x, z) \text{ if } q(x) \geq 0, q(z) \leq 0, \\ & f_2(x) - f_2(z) - \ell_2(h(x) - h(z)) + \tilde{g}(x, z) \text{ if } q(x) \leq 0, q(z) \leq 0, \end{aligned} \quad (5.48)$$

with  $\tilde{g}(x, z) := (g(x) - g(z))k(z)$ .

In our construction, we first use Theorem 5.14 to ensure ISS of (5.47a) based on two functions  $V_1, V_2$ , each of them associated to a mode.

**Property 5.1** *There exist  $V_1, V_2 \in \mathcal{C}^1(\mathbb{R}^n, \mathbb{R})$ , and  $\underline{\psi}_x, \bar{\psi}_x, \rho_x, \alpha_x \in \mathcal{K}_\infty$ , such that, for each  $x \in \mathbb{R}^n$ ,  $(-1)^{i-1}q(x) > 0$  implies*

$$\underline{\psi}_x(|x|) \leq V_i(x) \leq \bar{\psi}_x(|x|), \quad (5.49a)$$

$$|\nabla V_i(x)| \leq \rho_x(|x|), \quad (5.49b)$$

$$\langle \nabla V_i(x), f_i(x) + g(x)k(x) \rangle \leq -\alpha_x(|x|). \quad (5.49c)$$

Moreover, there exists a function  $\theta_x \in \mathcal{K}_\infty$  such that

$$\theta_x(s) \leq \frac{\alpha_x(s)}{\beta_g(s)\rho_x(s)}, \quad \forall s \in \mathbb{R}_+. \quad (5.50)$$

Finally, defining

$$V_x(x) := V_i(x), \quad \text{if } x \in X_i \quad i = 1, 2, \quad (5.51)$$

we suppose that  $V_x$  is continuous, that is,

$$(q(x) = 0) \Rightarrow V_1(x) = V_2(x), \quad (5.52)$$

and there exist functions  $\gamma_x^q \in \mathcal{K}$  and  $\alpha_x^q \in \mathcal{PD}$  such that for all  $x \in \mathbb{R}^n$  satisfying  $q(x) = 0$ , it holds that

$$(|x| \geq \gamma_x^q(|e|)) \Rightarrow \max \dot{\bar{V}}_{x, F_x}(x, e) \leq -\alpha_x^q(|x|). \quad (5.53)$$

Based on Property 5.1, we may prove the next result.

### Proposition 5.18

Under Property 5.1, there exists  $\hat{\alpha}_x \in \mathcal{PD}$  and  $\hat{\gamma}_x \in \mathcal{K}$  such that

$$(|x| \geq \hat{\gamma}_x(|e|)) \Rightarrow \max \dot{\bar{V}}_{x, F_x}(x, e) \leq -\hat{\alpha}_x(|x|)$$

and thus system (5.47a) is ISS w.r.t.  $e$ .

*Proof* : First of all we rewrite system (5.47a) as

$$\dot{x} \in \text{co} \{f_i(x) \mid i \in \mathcal{I}_{\mathcal{X}}(x)\} + g(x)k(x) - g(x)[k(x) - k(z)].$$

Let us note that equation (5.49a) assures continuity of the function  $V_x$ , and thus  $V_x$  is a piecewise  $\mathcal{C}^1$  function with respect to  $\mathcal{X}$ , in the sense of Definition 5.8. Consider first a point  $x \in \text{int}(X_i)$  for some  $i \in \{1, 2\}$ , that is an  $x \in \mathbb{R}^n$  such that  $(-1)^{i-1}q(x) > 0$ . We have

$$F_x(x, e) = \{f_i(x) + g(x)k(x) - g(x)[k(x) - k(z)]\} =: \{\tilde{f}_i\},$$

and thus by equation (5.49c) it follows that, for  $x \in X_i$ ,

$$\begin{aligned} \langle \nabla V_x(x), \tilde{f}_i \rangle &= \langle \nabla V_i(x), \tilde{f}_i \rangle \\ &\leq -\alpha_x(|x|) + |\nabla V_i(x)| \|g(x)\| |k(x) - k(z)| \\ &\leq -\alpha_x(|x|) + \rho_x(|x|)\beta_g(|x|)L_k|x - z|, \end{aligned}$$

where  $\beta_g$  comes from Assumption 5.3,  $L_k > 0$  is the Lipschitz constant of the map  $k : \mathbb{R}^n \rightarrow \mathbb{R}$  and  $\rho_x$  is given by equation (5.49b). Choosing  $0 < \varepsilon < 1$ , we have

$$\begin{aligned} \langle \nabla V_x(x), \tilde{f}_i \rangle &\leq -(1 - \varepsilon)\alpha_x(|x|) \\ &\quad - \varepsilon\alpha_x(|x|) + \rho_x(|x|)\beta_g(|x|)L_k|x - z| \end{aligned}$$

and thus, for  $i \in \{1, 2\}$ , and each  $x \in X_i$ ,

$$\langle \nabla V_x(x), \tilde{f}_i \rangle \leq -(1 - \varepsilon)\alpha_x(|x|), \text{ if } |e| \leq \frac{\varepsilon\alpha_x(|x|)}{L_k\rho_x(|x|)\beta_g(|x|)}.$$

Thanks to (5.50), the function  $\hat{\theta}_x(s) := \frac{\varepsilon\theta_x(s)}{L_k} \leq \frac{\varepsilon\alpha_x(s)}{L_k\rho_x(s)\beta_g(s)}$  is of class  $\mathcal{K}_\infty$ . Defining  $\alpha_x^c := (1 - \varepsilon)\alpha_x$  and  $\gamma_x^c := \hat{\theta}_x^{-1}$ , by arbitrariness of  $i \in \{1, 2\}$ , the previous inequality implies that, for any  $x \in \text{int}(X_1) \cup \text{int}(X_2)$ ,

$$(|x| \geq \gamma_x^c(|e|)) \Rightarrow \max \dot{\bar{V}}_{x, F_x}(x, e) \leq -\alpha_x^c(|x|). \quad (5.54)$$

Consider now a point  $x \in \partial X_1 \cup \partial X_2$ . By definition of the proper partition  $\{X_1, X_2\}$ , we have  $q(x) = 0$ , and thus implication (5.53) holds. Collecting (5.53) and (5.54) we obtain that, for all  $x, e \in \mathbb{R}^n$ ,

$$(|x| \geq \hat{\gamma}_x(|e|)) \Rightarrow \max \dot{\bar{V}}_{x, F_x}(x, e) \leq -\hat{\alpha}_x(|x|),$$

where  $\hat{\gamma}_x(s) := \max\{\gamma_x^c(s), \gamma_x^q(s)\}$  and  $\hat{\alpha}_x(s) := \min\{\alpha_x^c(s), \alpha_x^q(s)\}$ , concluding the proof.  $\diamond$

#### Remark 5.19: Clarke derivative based condition

It is possible to obtain a corollary of Proposition 5.18, based on the Clarke derivative, as in Definition 5.3. To this end, it is sufficient to replace implication (5.53) in Property 5.1, with the following:

(CL.1) For all  $x \in \mathbb{R}^n$  such that  $q(x) = 0$ , for each  $i \in \{1, 2\}$ ,

$$\langle \nabla V_i(x), f_{3-i}(x) + g(x)k(x) \rangle \leq -\alpha_x(|x|),$$

where  $\alpha_x \in \mathcal{K}_\infty$  satisfies also (5.49c).

The proof carries over straightforwardly, recalling the inclusion (5.7).  $\triangle$

**Example 5.1 (Continued)** *In the switched linear case of Example 5.1, Property 5.1 can be guaranteed with quadratic functions  $V_i(x) := x^\top P_i x$ ,  $i \in \{1, 2\}$ . Indeed, since the partitions given by  $q_v$  (or  $q_Q$ ) are conic, i.e.  $X_i$  is a cone for each  $i \in \{1, 2\}$ , we can look for Lyapunov functions homogeneous of degree 2, see [106] and the extension [123]. From now on we focus on the case  $q_Q(x) = x^\top Q x$ . The half-space partition case (i.e. considering  $q_v(x) = \langle v, x \rangle$ ) can be developed analogously to [64].*

*Considering  $q_Q(x)$ , it suffices to find  $K \in \mathbb{R}^{m \times n}$ , positive definite matrices  $P_1, P_2 \in \mathbb{R}^{n \times n}$ ,  $\mu_{12}, \mu_{21}, \mu_Q \in \mathbb{R}$ ,  $a_x > 0$  and  $\mu_1, \mu_2 \geq 0$  such that*

$$P_1 - P_2 = \mu_Q Q; \quad (5.55a)$$

$$\mu_1 Q + P_1(A_1 + BK) + (A_1 + BK)^\top P_1 + a_x I < 0; \quad (5.55b)$$

$$-\mu_2 Q + P_2(A_2 + BK) + (A_2 + BK)^\top P_2 + a_x I < 0; \quad (5.55c)$$

$$\mu_{12} Q + P_1(A_2 + BK) + (A_2 + BK)^\top P_1 + a_x I < 0; \quad (5.55d)$$

$$\mu_{21} Q + P_2(A_1 + BK) + (A_1 + BK)^\top P_2 + a_x I < 0; \quad (5.55e)$$



for each  $i \in \{1, 2\}$ . Then all the conditions of Property 5.1 hold with  $V_x(x) := x^\top P_i x$ , if  $x \in X_i$ . Indeed, first we note that (5.55a) implies (5.52) of Property 5.1. Moreover, we can define

$$\underline{\lambda}_x := \min_{i \in \{1, 2\}} \{\lambda_{\min}(P_i)\}, \quad \overline{\lambda}_x := \max_{i \in \{1, 2\}} \{\lambda_{\max}(P_i)\},$$

where  $\lambda_{\max}(P)$ ,  $\lambda_{\min}(P)$  represent respectively the largest and the smallest eigenvalues of a positive definite matrix  $P > 0$ . The bound functions in (5.49a) and (5.49b) of Property 5.1 are thus obtained by defining

$$\underline{\psi}_x(s) := \underline{\lambda}_x s^2, \quad \overline{\psi}_x(s) := \overline{\lambda}_x s^2, \quad \rho_x(s) := 2\overline{\lambda}_x s.$$

Via the S-Procedure (see for example [20, Section 2.6.3]), equation (5.55b) implies

$$x^\top (P_1(A_1 + BK) + (A_1 + BK)^\top P_1)x < -a_x |x|^2,$$

if  $x^\top Qx \geq 0$  and equation (5.55c) implies

$$x^\top (P_2(A_2 + BK) + (A_2 + BK)^\top P_2)x < -a_x |x|^2,$$

if  $x^\top Qx \leq 0$ . We have thus proved (5.49c) of Property 5.1 with  $\alpha_x(s) := a_x s^2$ . Similarly, using Finsler's Lemma (see [43], equations (5.55d) and (5.55e) imply item (CL.1) in Remark 5.19, again with  $\alpha_x(s) := a_x s^2$ . The function  $\theta_x \in \mathcal{K}_\infty$  in (5.50) can be defined as  $\theta_x(s) := \frac{a_x}{2\|B\|\overline{\lambda}_x} s$ .  $\triangle$

Let us now consider the error dynamics (5.47b) and characterize ISS from  $x$ , using a  $\mathcal{C}^1$ -Lyapunov function, satisfying the next property.

**Property 5.2** Suppose that there exist  $V_e \in \mathcal{C}^1(\mathbb{R}^n, \mathbb{R})$ , and  $\underline{\psi}_e, \overline{\psi}_e, \alpha_e, \rho_e \in \mathcal{K}_\infty$  such that

$$\underline{\psi}_e(|e|) \leq V_e(e) \leq \overline{\psi}_e(|e|), \quad \forall e \in \mathbb{R}^n \quad (5.56a)$$

$$\left| \frac{\partial V_e}{\partial e}(e) \right| \leq \rho_e(|e|), \quad \forall e \in \mathbb{R}^n. \quad (5.56b)$$

Moreover, for all  $x \in \mathbb{R}^n$ , for all  $z \in \mathbb{R}^n$  and for each  $i \in \{1, 2\}$ ,

$$\left\langle \frac{\partial V_e}{\partial e}(e), f_i(x) - f_i(z) - \ell_i(h(x) - h(z)) + \tilde{g}(x, z) \right\rangle \leq -\alpha_e(|e|), \quad (5.57)$$

with  $e = x - z$ . Finally there exists  $\theta_e \in \mathcal{K}_\infty$  such that

$$\theta_e(s) \leq \frac{\alpha_e(s)}{\rho_e(s)} \quad \forall s \in \mathbb{R}_+. \quad (5.58)$$

Based on Property 5.2 we can prove the next result.

### Proposition 5.20

Under Property 5.2, there exist  $\hat{\gamma}_e \in \mathcal{K}$  and  $\hat{\alpha}_e \in \mathcal{K}_\infty$  such that

$$(|e| \geq \hat{\gamma}_e(|x|)) \Rightarrow \left\langle \frac{\partial V_e}{\partial e}(e), f_e \right\rangle \leq -\hat{\alpha}_e(|e|),$$

for all  $f_e \in F_e(x, e)$ , and thus the system (5.47b) is ISS w.r.t.  $x$ .

*Proof* : It is easy to see that the second and third expression in (5.48) can be rewritten respectively as

$$\begin{aligned} f_1(x) - f_1(z) - \ell_1(h(x) - h(z)) + \tilde{g}(x, z) + (f_2(x) - f_1(x)) \\ f_2(x) - f_2(z) - \ell_2(h(x) - h(z)) + \tilde{g}(x, z) + (f_1(x) - f_2(x)) \end{aligned}$$

and thus we can rewrite  $f_e : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$  as

$$f_e(x, z) := \begin{cases} f_1(x) - f_1(z) - \ell_1(h(x) - h(z)) \\ \quad + \tilde{g}(x, z) + \mathcal{I}_-(q(x))\tilde{f}(x) & \text{if } q(z) \geq 0, \\ f_2(x) - f_2(z) - \ell_2(h(x) - h(z)) \\ \quad + \tilde{g}(x, z) - \mathcal{I}_+(q(x))\tilde{f}(x) & \text{if } q(z) \leq 0. \end{cases}$$

where we defined  $\tilde{f}(x) := f_2(x) - f_1(x)$  and  $\mathcal{I}_+, \mathcal{I}_-$  are the indicator functions of the positive and negative real numbers respectively. We note that, by Assumption 5.3,

$$\max\{|\mathcal{I}_-(q(x))\tilde{f}(x)|, |\mathcal{I}_+(q(x))\tilde{f}(x)|\} \leq \beta_f(|x|)$$

for all  $x \in \mathbb{R}^n$ . Thus, we can now apply the same reasoning used in proof of Proposition 5.18, concluding that

$$(|e| \geq \hat{\gamma}_e(|x|)) \Rightarrow \left\langle \frac{\partial V}{\partial e}(e), f_e \right\rangle \leq -\hat{\alpha}_e(|e|), \quad \forall f_e \in F_e(x, e),$$

where  $\hat{\alpha}_e := (1 - \varepsilon)\alpha_e$  and  $\hat{\gamma}_e := (\varepsilon\theta_e)^{-1} \circ \beta_f$ , for some  $0 < \varepsilon < 1$ . Note that condition (5.58) ensures that  $\hat{\gamma}_e \in \mathcal{K}$ . The ISS property follows again from Theorem 5.4.  $\diamond$

To clarify our construction, the idea behind Property 5.2 and Proposition 5.20 is to search for a common  $\mathcal{C}^1$  Lyapunov function for the two vector fields  $f_i(x) - f_i(z) - \ell_i(h(x) - h(z)) + \tilde{g}(x, z)$ ,  $i \in \{1, 2\}$ . If  $x$  and the estimated state  $z$  are not in the same region  $\mathcal{X}_i$ , then the  $(x - z)$ -dynamics is perturbed by a factor  $\pm(f_1(x) - f_2(x))$ , which is treated as an external disturbance. The injection gains  $\ell_i$  induce ISS with respect to these disturbances.

**Example 5.1 (Continued)** For the switched linear case presented in Example 5.1, Property 5.2 can be ensured using a quadratic function  $V_e(e) := e^\top P_e e$ , with  $P_e > 0$ , by finding matrices  $L_1, L_2 \in \mathbb{R}^{n \times p}$  and  $a_e > 0$  such that

$$P_e(A_i - L_i C) + (A_i - L_i C)^\top P_e + a_e I < 0, \quad (5.59)$$

for each  $i \in \{1, 2\}$ . Indeed, defining

$$\begin{aligned} \underline{\lambda}_e &:= \lambda_{\min}(P_e), \quad \overline{\lambda}_e := \lambda_{\max}(P_e), \quad \underline{\psi}_e(s) := \underline{\lambda}_e s^2, \\ \overline{\psi}_e(s) &:= \overline{\lambda}_e s^2, \quad \rho_e(s) := 2\overline{\lambda}_e s, \quad \hat{\alpha}_e(s) := a_e s^2, \end{aligned}$$

equations (5.56a), (5.56b) and (5.57) are satisfied, and the function  $\theta_e \in \mathcal{K}_\infty$  in (5.58) is defined as  $\theta_e(s) = \frac{a_e}{2\overline{\lambda}_e} s$ .  $\triangle$

We are finally ready to state our stability conditions, based on Theorem 5.14, for the interconnection in (5.46).

### Corollary 5.21

Assume that Properties 5.1 and 5.2 hold, and define  $\eta_1 := \bar{\psi}_x \circ \hat{\gamma}_e \circ \underline{\psi}_e^{-1}$  and  $\eta_2 := \bar{\psi}_e \circ \hat{\gamma}_x \circ \underline{\psi}_x^{-1}$ . If

$$\eta_1 \circ \eta_2(s) < s, \quad \forall s > 0, \quad (5.60)$$

then system (5.46) is globally asymptotically stable.

*Proof :* Since by Propositions 5.18 and 5.20 we can construct non-pathological ISS-Lyapunov functions  $V_x$  and  $V_e$  as in Assumption 5.1, it remains to check that condition (5.60) implies the small gain condition (5.26) in Theorem 5.14. First we note that, by (5.49a) and (5.56a),

$$\begin{aligned} V_x(x) \geq \eta_1(V_e(e)) &= \bar{\psi}_x \circ \hat{\gamma}_e \circ \underline{\psi}_e^{-1}(V_e(e)) \\ &\Leftrightarrow \bar{\psi}_x^{-1}(V_x(x)) \geq \hat{\gamma}_e \circ \underline{\psi}_e^{-1}(V_e(e)) \\ &\Rightarrow |x| \geq \hat{\gamma}_e(|e|), \end{aligned}$$

since  $V_x(x) \leq \bar{\psi}_x(|x|)$ , and  $V_e(e) \geq \underline{\psi}_e(|e|)$ . By Proposition 5.18, this implies that

$$(V_x(x) \geq \eta_1(V_e(e))) \Rightarrow \max \dot{\bar{V}}_{x,F_x}(x, e) \leq -\hat{\alpha}_x(|x|).$$

Following the same path for  $\eta_2$ , we obtain the implication

$$(V_e(e) \geq \eta_2(V_x(x))) \Rightarrow \max \dot{\bar{V}}_{e,F_e}(x, e) \leq -\hat{\alpha}_e(|e|),$$

proving that (5.60) implies (5.26).  $\diamond$

### Remark 5.22: Concluding Example 5.1

Considering again the switched system presented in Example 5.1, we can check the small-gain condition (5.60) as follows. Recalling the definitions of  $\hat{\gamma}_x$  and  $\hat{\gamma}_e$  in the proofs of Propositions 5.18 and 5.20 we can write

$$\hat{\gamma}_x(s) = \frac{2\|B\|\|K\|\bar{\lambda}_x}{\varepsilon a_x} s \quad \text{and} \quad \hat{\gamma}_e(s) = \frac{2\|A_1 - A_2\|\bar{\lambda}_e}{\varepsilon a_e} s.$$

Thus, by arbitrariness of  $0 < \varepsilon < 1$ , condition (5.60) holds if

$$\frac{16\|B\|^2\|K\|^2\|A_1 - A_2\|^2\bar{\lambda}_x^3\bar{\lambda}_e^3}{\lambda_x\lambda_e a_x^2 a_e^2} < 1.$$

## 5.7 Summary

In this chapter, we focused on ISS of differential inclusions using locally Lipschitz Lyapunov functions. We provided sufficient conditions based on the notion of Lie

derivative of the candidate Lyapunov function, which generalize previous results relying on the study of the Clarke derivative. We applied our results to state-dependent switched systems and proposed a new formulation of the well-known small gain theorem in the context of interconnected differential inclusions. We finally studied the design of an observer-based output feedback controller for a bimodal switched system. As possible further research, we may investigate convex LMI-based algorithms, based on using Lipschitz non quadratic functions and Lie derivative.



# 6

## Conclusions and Perspectives

### 6.1 General Conclusions

This thesis has considered the stability problem for differential inclusions, with two main application areas: hybrid systems and state-dependent switched systems. From a control point of view, Lyapunov function design is a fruitful tool for proposing stability certificates for a large class of dynamical systems. In this work, we focused on proposing Lyapunov results relying on locally Lipschitz functions, analyzing some specific class, and underlining the subtleties and the limitations with respect to the considered problems. We found it useful, in the first part of this thesis (Chapters 1 and 2), to propose an “almost complete” formal introduction of the considered frameworks, namely (constrained) differential inclusions, hybrid systems and switching systems. Moreover, in Chapter 2, we proposed an overview of the main results built upon locally Lipschitz Lyapunov functions, with particular care in defining and developing the differential concepts needed, together with an historical discussion and a comparison with the recent literature. Even if the content of these first two chapters is fairly standard, we hope this brief summary provides a clear and brief introduction for a reader who would like to familiarize and discover this interesting subject. The work was then organized in three core chapters, whose technical contributions are listed in what follows.

- In **Chapter 3** we studied the problem of stability for a class of inner semicontinuous constrained differential inclusions, with a particular interest in applications to hybrid dynamical systems. For a particular class of piecewise-defined locally Lipschitz functions we underlined how the Lyapunov decrease inequality could be verified only on a dense subset of the given domain of interest  $\mathcal{C}$ . This was in particular interesting since the proposed result turns out to be less conservative than the “classical” conditions relying on the computation of the Clarke gradient, and, moreover, it is tailored also for cases where the domain  $\mathcal{C}$  is closed, and possibly with empty interior. This “geometrical elasticity” of the proposed conditions turns out to be quite important when the results are applied to hybrid systems, since it is fairly common to have closed flows sets in this setting. Several examples are provided to show these subtleties, and the limitations of our approach. We studied the relations between our proposed class of locally Lipschitz functions and the piecewise  $\mathcal{C}^1$  functions recently proposed in the literature. We applied our result to the Clegg integrator model, a classical example from the reset systems literature.

- In **Chapter 4** we introduced and studied a family of locally Lipschitz functions obtained by max-min combinations. Based on two notions of generalized directional derivatives, we proposed sufficient conditions for global asymptotic stability of differential inclusions, with particular attention to the state-dependent switched systems case. This allowed us to clarify the relations between the notion of Lie derivative, and the sliding motion phenomenon, which often arise in switching systems, thus complicating the analysis. For a class of systems with conic switching regions and linear dynamics within each of these regions, we studied some conditions under which a max-min certificate can be obtained by solving a set of matrix inequalities. These results crucially depend on the combinatorial structure (given by max-min combinations) of the chosen class of functions, and motivated our choice of studying this particular case.
- In **Chapter 5**, considering differential inclusions perturbed by external signals, we focused on proposing ISS certificates using locally Lipschitz functions. Relying on the notion of Lie derivatives of non-pathological functions, we provided sufficient conditions that generalize previous results relying on the study of the Clarke generalized derivative. Again, we studied the application of our main results to state-dependent switched systems. We proposed a new formulation of the well-known small-gain theorem in the context of interconnected differential inclusions, and we analyzed deeply the case of interconnections in the so-called cascade form. We finally studied the design of an observer-based output feedback controller for a bimodal switched system.

Concluding, in this thesis we mainly concentrated in proposing various kinds of relaxed Lyapunov conditions considering locally Lipschitz functions. The possible routes of research are detailed in the following section.

## 6.2 Perspectives

In this concluding section, we introduce possible paths for future research, suggested by the work presented in this thesis.

### 6.2.1 Algorithmic Verification

The results presented in this thesis showed how, for a large class of systems, one can obtain relaxed Lyapunov conditions when looking at locally Lipschitz functions, choosing the appropriate notions of “*derivative of a scalar function along solutions of dynamical systems*”. The main drawback of this approach is that the conditions we obtained, even if appealing from a theoretical point of view, turn out to be computational hard to implement and verify, also for relatively simple systems (for example switched systems composed by linear subsystems). During this work, we started exploring the possibility of proposing algorithmic procedures, in order to verify our proposed conditions, as for example was sketched in Section 3.6 for hybrid systems, or in Section 4.5 for state-dependent switched systems. Nevertheless,

it would be interesting to investigate and propose computationally tractable conditions, at least for the case of hybrid and state-dependent switched systems with linear sub-dynamics, and subjected to specific geometric properties of flow/jump sets and state-space partition. More specifically, in this thesis we focused on operating regions/flow sets defined as *symmetric cones*, that is sets  $\mathcal{X} \subset \mathbb{R}^n$  defined by  $\mathcal{X} := \{x \in \mathbb{R}^n \mid x^\top Q x \geq 0\}$ , considering  $Q \in \text{Sym}(\mathbb{R}^{n \times n})$ . A similar and promising setting would arise considering *polyhedral* partitions of the state space, or polyhedral flows and jump sets, i.e. sets defined as intersection of a finite number of closed half-spaces. Interesting recent results in this context can be found in [11], [70] and [71].

### 6.2.2 Smooth Approximation

During this PhD project, the following interesting question arose, and unfortunately we were able to provide only a partial answer. As we have already underlined, if the origin of differential inclusion

$$\dot{x} \in F(x), \quad (6.1)$$

(where  $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$  is outer semicontinuous with non-empty, compact and convex values) is globally asymptotically stable (GAS), then there exists a *smooth* Lyapunov function, see [122] or [36] for the formal statement and [81] for a thorough review. Since we proposed various locally Lipschitz Lyapunov constructions, we faced the following question:

*Given a locally Lipschitz Lyapunov function  $V : \mathbb{R}^n \rightarrow \mathbb{R}$  for system (6.1), is it possible to approximate  $V$  in order to find a smooth Lyapunov function?*

More precisely, we proposed the following conjecture:

#### Conjecture 6.1: Smooth Approximation

Let us suppose  $V : \mathbb{R}^n \rightarrow \mathbb{R}$  is a Clarke or Lie locally Lipschitz Lyapunov function for system (6.1) (recall Theorems 2.18 and 2.24). Then, for any compact set  $\mathcal{K} \subset \mathbb{R}^n$  and any  $\varepsilon > 0$  there exists a function  $\tilde{V} \in \mathcal{C}^1(\mathbb{R}^n, \mathbb{R})$  such that

- $\|V(x) - \tilde{V}(x)\| \leq \varepsilon$ , for every  $x \in \mathcal{K}$ ,
- $\tilde{V}$  is a *smooth* Lyapunov function for (6.1), as defined in Theorem 1.3

Intuitively, this conjecture postulates that, given a locally Lipschitz Lyapunov function for (6.1) (satisfying certain conditions), there always exists a smooth Lyapunov function arbitrarily close to it (in the uniform norm sense), at least on compact sets. This conjecture is true when considering Clarke locally Lipschitz Lyapunov functions, and we analyzed this case in the conference paper [50]. Not surprising, the result follows from a “regularization-via-convolution” technique, a common tool used in various converse Lyapunov theorems (as the ones in [86],[122], [36]), and formally introduced for example in [24, Proposition 4.21]. Unfortunately, this tool can not be applied “mutatis mutandis” to the Lie-locally Lipschitz Lyapunov



functions case. In the paper [50] some preliminary and partial results are given, exploring and underlining the subtleties. This investigation was not satisfactory, and for that reason Conjecture 6.1 is still open, and it will be the subject of further research. Despite its theoretical appeal, an important motivation for constructing smooth approximations is that they provide more information about the robustness of the system. For example in [50, Proposition 15], we showed a connection between the existence of smooth Lyapunov functions and the ISS property with respect to additive inputs.

### 6.2.3 Time-Dependent Switched Systems: Minimum Dwell Time Computation

Another potentially interesting path of research would be to explore the possibility of using locally Lipschitz functions in the study of stability of *time-dependent* switched systems. As we have briefly sketched in Section 1.3, in this context it is quite common to find in literature stability results relying on *multiple Lyapunov functions*, i.e. constructions obtained combining/gluing together a finite number of functions. Thus, it seems promising to look at generalizations of the constructions proposed here for state-dependent switched systems, but it will need some care. Indeed, while the Lyapunov functions proposed in this thesis have always been obtained “patching” together a finite number of functions in the *state space* variable  $x$ , it seems reasonable that, in a time-dependent switching context, this patching should be done in the time variable, thus obtaining time-varying or signal-dependent Lyapunov functions. For this reason a serious investigation is needed, in order to develop the appropriate notions of derivative/decrease conditions. More specifically, an interesting (and still open) problem was presented almost twenty years ago in [66], and concern the finiteness and behavior of the  $\mathcal{L}_2$ -norm of switched linear systems defined by

$$\begin{cases} \dot{x}(t) = A_{\sigma(t)}x(t) + B_{\sigma(t)}u(t), \\ y(t) = C_{\sigma(t)}x(t) + D_{\sigma(t)}u(t), \end{cases} \quad (6.2)$$

given a finite set of subsystems  $\mathcal{M} = \{(A_i, B_i, C_i, D_i) \mid i \in \{1, \dots, K\}\}$  and under a dwell-time constraint on the time-dependent switching signal  $\sigma : \mathbb{R}_+ \rightarrow \{1, \dots, K\}$ . Recently, in [33], it was proved that, under certain conditions, the  $\mathcal{L}_2$  stability of (6.2) is equivalent to the GAS of system  $\dot{x}(t) = A_{\sigma(t)}x(t)$ . Since the main characterizations of stability for switched systems under dwell-time assumptions (for example [127], [32]) are given by using multiple Lyapunov constructions involving non-smooth Lyapunov functions, we believe that our approach could give interesting developments in this direction. Preliminary results in this direction are given in the conference paper [48].

### 6.2.4 Control Design/State-Partition Control

We derived in this thesis various Lyapunov conditions ensuring (asymptotic) stability for hybrid systems and state dependent switched systems. One step forward in our analysis would be the use of similar conditions in constructing a feedback

control law, when the system admits an input variable. A particular example of application of our ISS result in this context was discussed in Section 5.6. We recall that the study and use of non-smooth functions as control Lyapunov functions has been a rich and active field of research in the past years, since it is well-known that a system can be globally asymptotically controllable (or equivalently, feedback stabilizable when considering feedback solutions, see [35]), even if a *smooth* control Lyapunov function does not exist, see for example [116], [103] and references therein.

More specifically, we believe that our results could be applied to the following specific problem: considering a discrete set of vector fields  $\{f_i\}_{i \in \mathcal{I}} \subset Lip_{loc}(\mathbb{R}^n, \mathbb{R}^n)$ , we would like to *construct* a state-space partition  $\mathcal{X} = \{X_i\}_{i \in \mathcal{I}}$  (as introduced in Section 2.2) for which the resulting state-dependent switched system is globally asymptotically stable. Roughly speaking, the state-dependent switching rule in this case is seen as an input parameter that the designer can choose, instead of a given data of the considered system. Preliminary discussions and results in this context are provided in [84, Section 3.4].



# A

## Appendix to Chapter 4

This appendix is dedicated to clarify and develop some concepts introduced in Chapter 4. In particular, in Section A.1 we provide the proof of a technical result, while in Section A.2 we study, as a further insight, the family of max-min functions obtained from 3 quadratics, and show how the computational complexity of the proposed conditions depends on the max-min structure.

### A.1 A Technical Result

#### Lemma A.1

Consider  $Q \in \text{Sym}(\mathbb{R}^n)$  invertible and any max-min function  $V \in \mathbf{Mm}(x^\top P_1 x, \dots, x^\top P_K x)$ , such that  $P_1, \dots, P_K > 0$  satisfy (4.65). Consider a point  $x \in \mathbb{R}^n$  such that  $x^\top Q x = 0$  and  $\alpha_V(x) = \{\ell_1, \dots, \ell_p\}$  ( $p > 1$ ). If  $\forall \ell', \ell'' \in \alpha_V(x)$ ,  $\ell' \neq \ell''$ ,  $x^\top (P_{\ell'} - P_{\ell''}) \neq \tau x^\top Q$ , for all  $\tau \in \mathbb{R} \setminus \{0\}$ , then there exists a sequence  $x_k \rightarrow x$  such that  $x_k^\top Q x_k = 0$  and  $\alpha_V(x_k) = \{\bar{\ell}\}$ , for all  $k \in \mathbb{N}$ , for an  $\bar{\ell} \in \alpha_V(x)$ .

*Proof :* Since  $Q$  is invertible, for all  $x \in \mathbb{R}^n \setminus \{0\}$  such that  $x \in \mathcal{Q} = \{x \in \mathbb{R}^n \mid x^\top Q x = 0\}$ , we can define the tangent space of  $\mathcal{Q}$  at  $x$  as  $T_x(\mathcal{Q}) := \{w \in \mathbb{R}^n \mid x^\top Q w = 0\}$ , see for example [13, Page 23]. Consider  $v \in \mathbb{R}^n$ ,  $v \neq 0$ , such that  $x^\top Q v = 0$  and  $x^\top (P_{\ell'} - P_{\ell''}) v \neq 0$ , for all  $\ell', \ell'' \in \alpha_V(x)$ ,  $\ell' \neq \ell''$ . Such a  $v \in \mathbb{R}^n$  exists, since, by (4.65),  $Q$  and  $P_{\ell'} - P_{\ell''}$  are invertible and  $x^\top Q$  and  $x^\top (P_{\ell'} - P_{\ell''})$  are linearly independent, for all  $\ell', \ell'' \in \alpha_V(x)$ ,  $\ell' \neq \ell''$ . By definition of  $T_x(\mathcal{Q})$ , given  $\beta > 0$ , there exists continuously differentiable function  $\psi : (-\beta, \beta) \rightarrow \mathbb{R}^n$  such that  $\psi(0) = x$ ,  $\dot{\psi}(0) = v$ , and  $\psi(\tau)^\top Q \psi(\tau) = 0$ , (i.e.  $\psi(\tau) \in \mathcal{Q}$ ),  $\forall \tau \in (-\beta, \beta)$ . For all  $\ell', \ell'' \in \alpha_V(x)$ ,  $\ell' \neq \ell''$ , define  $\Psi_{\ell', \ell''} : (-\beta, \beta) \rightarrow \mathbb{R}$  as

$$\Psi_{\ell', \ell''}(\tau) := \psi(\tau)^\top (P_{\ell'} - P_{\ell''}) \psi(\tau).$$

Since  $\ell', \ell'' \in \alpha_V(x)$  we have  $\Psi_{\ell', \ell''}(0) = 0$ ; moreover  $\Psi_{\ell', \ell''}$  is continuously differentiable at 0 and by the chain rule  $\dot{\Psi}_{\ell', \ell''}(0) = x^\top (P_{\ell'} - P_{\ell''}) v \neq 0$ . This means that there exists a  $\beta' < \beta$ ,  $\beta' \neq 0$  such that

$$\Psi_{\ell', \ell''}(\tau) = \psi(\tau)^\top P_{\ell'} \psi(\tau) - \psi(\tau)^\top P_{\ell''} \psi(\tau) \neq 0, \quad (\text{A.1})$$

for all  $\tau \in (0, \beta')$ , for all  $\ell', \ell'' \in \alpha_V(x)$ ,  $\ell' \neq \ell''$ . We now consider a sequence  $\tau_k \rightarrow 0$  such that  $\tau_k \in (0, \beta')$ ,  $\forall k \in \mathbb{N}$ , and define  $x_k := \psi(\tau_k)$ . Without loss of generality we

can suppose  $x_k \in \mathcal{U}$ , for all  $k \in \mathbb{N}$ , where  $\mathcal{U}$  is the open neighborhood of  $x$  defined in Lemma 4.6. By (A.1) we have

$$x_k^\top P_{\ell'} x_k \neq x_k^\top P_{\ell''} x_k, \quad \forall \ell', \ell'' \in \alpha_V(x), \ell' \neq \ell''.$$

By Lemma 4.6 this implies that,  $\forall k \in \mathbb{N}$ , there exists  $\ell_k \in \alpha_V(x)$  such that  $\alpha_V(x_k) = \{\ell_k\}$ . By finiteness of  $\alpha_V(x)$ , possibly considering a subsequence, we can suppose  $\alpha_V(x_k) = \{\bar{\ell}\} \forall k \in \mathbb{N}$ , with  $\bar{\ell} \in \alpha_V(x)$ . Since, by definition of  $\psi$ ,  $x_k^\top Q x_k = 0, \forall k \in \mathbb{N}$ , we also have  $I(x_k) = \{1, 2\}, \forall k \in \mathbb{N}$ .  $\diamond$

## A.2 Max-Min Functions with Three quadratics

In this section, we analyze some max-min functions of 3 quadratics defined by positive-definite and symmetric matrices  $P_1, P_2$  and  $P_3$ . It can be taken as a simple useful model to underline some remarks and how the number of inequalities in (4.53), resulting from the S-procedure depends on the choice of the max-min composition. With an abuse of notation, we will write  $\min\{P_i, P_j\}$  instead of  $\min\{x^\top P_i x, x^\top P_j x\}$ . The set  $\mathbf{Mm}\{P_1, P_2, P_3\}$  has the following elements:

- Common Lyapunov function:  $V = \max\{\min\{P_i\}\}$ ;
- Min of 2 quadratics:  $V = \max\{\min\{P_i, P_j\}\}$ ;
- Max of 2 quadratics:  $V = \max\{\min\{P_i\}, \min\{P_j\}\}$ ;
- Min of 3 quadratics:  $V = \max\{\min\{P_1, P_2, P_3\}\}$ ;
- Max of 3 quadratics:

$$V = \max\{\min\{P_1\}, \min\{P_2\}, \min\{P_3\}\};$$

- Quasi-max functions:

$$V = \max\{\min\{P_1\}, \min\{P_2, P_3\}\};$$

- Quasi-min functions:

$$V = \max\{\min\{P_1, P_3\}, \min\{P_2, P_3\}\};$$

- Mid-of-quadratics function:

$$V = \max\{\min\{P_1, P_2\}, \min\{P_2, P_3\}, \min\{P_1, P_3\}\}.$$

Our interest particularly lies in the last three cases because the remaining cases can be obtained more simply by considering maximum or minimum of (3 or less) quadratic functions. Moreover the cases of quasi-max and quasi-min are in some sense dual as we observe that  $\max\{\min\{P_i, P_k\}, \min\{P_j, P_k\}\} = \min\{P_k, \max\{P_i, P_j\}\}$ .

### A.2.1 Comparison of Max Construction with Other Results

Let us consider the max function  $V = \max\{P_1, P_2, P_3\}$ . Without loss of generality, we write down only the inequalities corresponding to the regions where  $x^\top P_3 x$  has the maximum value. We want to show that the two inequalities, corresponding to a fixed  $i \in \{1, \dots, m\}$ , can be reduced to a single inequality, and hence the total computational burden can be reduced from  $6m$  to  $3m$  inequalities.

#### Lemma A.2

Denote  $A := A_i$  for a fixed  $i \in \{1, \dots, m\}$ . Consider the following statements:

1.  $\exists \tau_{21}, \tau_{32} \geq 0$  such that  $A^\top P_3 + P_3 A + \tau_{21}(P_2 - P_1) + \tau_{32}(P_3 - P_2) < 0$ .
2.  $\exists \tau_{12}, \tau_{31} \geq 0$  such that  $A^\top P_3 + P_3 A + \tau_{12}(P_1 - P_2) + \tau_{31}(P_3 - P_1) < 0$ .
3.  $\exists \lambda_1, \lambda_2 \geq 0$  such that  $A^\top P_3 + P_3 A + \lambda_1(P_3 - P_1) + \lambda_2(P_3 - P_2) < 0$ .

Then, it holds that  $1 \wedge 2 \iff 3$ .

*Proof* :  $1 \wedge 2 \Rightarrow 3$ . If  $\tau_{21} = 0$  then 3 holds with  $\lambda_1 = 0$  and  $\lambda_2 = \tau_{32}$ . The case  $\tau_{12} = 0$  is analogous. If  $\tau_{21} \neq 0, \tau_{12} \neq 0$  it suffices to multiply the inequality in item 1 by  $\frac{1}{\tau_{21}}$ , then add it to the inequality given in 2 multiplied by  $\frac{1}{\tau_{12}}$  to arrive at 3.

$3 \Rightarrow 1 \wedge 2$ : Let us take  $\lambda_1$  and  $\lambda_2$  such that  $A^\top P_3 + P_3 A + \lambda_1(P_3 - P_1) + \lambda_2(P_3 - P_2) < 0$ . We have

$$\begin{aligned} A^\top P_3 + P_3 A + \lambda_1(P_3 - P_1) + \lambda_2(P_3 - P_2) \pm \lambda_2 P_1 &= \\ A^\top P_3 + P_3 A + (\lambda_1 + \lambda_2)(P_3 - P_1) + \lambda_2(P_1 - P_2) &< 0, \end{aligned}$$

that is precisely the inequality in 2. The inequality in 1 can be derived with the same argument.  $\diamond$

With this Lemma we have recovered the sufficient conditions for computing Lyapunov function via the max of quadratics, given in [59, Corollary 4.4], while using the more general framework of max-min functions.

### A.2.2 Mid of 3 Quadratics

Let us consider the mid of quadratics described by

$$V = \max \{ \min\{P_1, P_2\}, \min\{P_2, P_3\}, \min\{P_3, P_1\} \}.$$

We have called this function *mid of quadratics* because, for every  $x \in \mathbb{R}^n$ , it takes the value  $x^\top P_\ell x$  such that  $x^\top P_j x \leq x^\top P_\ell x \leq x^\top P_k x$ , where  $i, k, \ell$  are different. Let us suppose, just to explain the ideas, to have a “trivial switching” systems, that is  $M = 1$   $Q_1 = I$  and  $A_1 \in \mathbb{R}^{n \times n}$ , that is the linear system  $\dot{x} = A_1 x$ . The general case then could be reduced using the same ideas, mutadis mutandis. In this case the condition (4.53) becomes:  $\exists \tau_{12}, \tau_{13}, \tau_{21}, \tau_{23}, \tau_{31}, \tau_{32}, \tilde{\tau}_{12}, \tilde{\tau}_{13}, \tilde{\tau}_{21}, \tilde{\tau}_{23}, \tilde{\tau}_{31}, \tilde{\tau}_{32} \geq 0$  such that

$$(123) \quad A_i^\top P_2 + P_2 A_i + \tau_{21}(P_2 - P_1) + \tau_{32}(P_3 - P_2) < 0,$$

$$(132) \quad A_i^\top P_3 + P_3 A_i + \tau_{31}(P_3 - P_1) + \tau_{23}(P_2 - P_3) < 0,$$

$$(213) \quad A_i^\top P_1 + P_1 A_i + \tau_{12}(P_1 - P_2) + \tilde{\tau}_{31}(P_3 - P_1) < 0,$$

$$(231) \quad A_i^\top P_3 + P_3 A_i + \tilde{\tau}_{32}(P_3 - P_2) + \tau_{13}(P_1 - P_3) < 0,$$

$$(312) \quad A_i^\top P_1 + P_1 A_i + \tilde{\tau}_{31}(P_1 - P_3) + \tilde{\tau}_{21}(P_2 - P_1) < 0,$$

$$(321) \quad A_i^\top P_2 + P_2 A_i + \tilde{\tau}_{23}(P_2 - P_3) + \tilde{\tau}_{12}(P_1 - P_2) < 0.$$

We have enumerated the inequalities using the triplets  $(j_1 j_2 j_3)$ , which correspond to the cone where  $x^\top P_{j_1} x \leq x^\top P_{j_2} x \leq x^\top P_{j_3} x$ . This is the worst case: we can not regroup any inequalities, and  $6m$  inequalities involving  $12m$  non-negative scalars must be solved.

### A.2.3 Quasi-Max Function

In this case, we consider the function described as

$$V = \max \{ \min\{P_1\}, \min\{P_2, P_3\} \}.$$

For the trivial system defined by  $M = 1$ ,  $Q_1 = I$ ,  $A_1 \in \mathbb{R}^{n \times n}$ , conditions given in (4.53), are in this case:  $\exists \tau_{12}, \tau_{13}, \tau_{21}, \tau_{23}, \tau_{31}, \tau_{32}, \tilde{\tau}_{12}, \tilde{\tau}_{13}, \tilde{\tau}_{21}, \tilde{\tau}_{23}, \tilde{\tau}_{31}, \tilde{\tau}_{32} \geq 0$  such that

$$(123) \quad A^\top P_2 + P_2 A + \tau_{21}(P_2 - P_1) + \tau_{32}(P_3 - P_2) < 0,$$

$$(132) \quad A^\top P_3 + P_3 A + \tau_{31}(P_3 - P_1) + \tau_{23}(P_2 - P_3) < 0,$$

$$(213) \quad A^\top P_1 + P_1 A + \tau_{12}(P_1 - P_2) + \tilde{\tau}_{31}(P_3 - P_1) < 0,$$

$$(231) \quad A^\top P_1 + P_1 A + \tilde{\tau}_{32}(P_3 - P_2) + \tau_{13}(P_1 - P_3) < 0,$$

$$(312) \quad A^\top P_1 + P_1 A + \tilde{\tau}_{31}(P_1 - P_3) + \tilde{\tau}_{21}(P_2 - P_1) < 0,$$

$$(321) \quad A^\top P_1 + P_1 A + \tilde{\tau}_{23}(P_2 - P_3) + \tilde{\tau}_{12}(P_1 - P_2) < 0.$$

Reasoning as in Lemma A.2 it easy to note that inequalities (231), (321), (213) are equivalent to the single inequality

$$\exists \tilde{\lambda} \geq 0 \text{ s.t. } A^\top P_1 + P_1 A + \tilde{\lambda}(P_1 - P_2) < 0.$$

This way, we can rewrite the sufficient conditions for the quasi-max Lyapunov function as:  $\exists \tau_{21}, \tau_{23}, \tau_{31}, \tau_{32}, \tilde{\tau}_{21}, \tilde{\tau}_{31}, \tilde{\lambda} \geq 0$  such that

$$(123) \quad A^\top P_2 + P_2 A + \tau_{21}(P_2 - P_1) + \tau_{32}(P_3 - P_2) < 0,$$

$$(132) \quad A^\top P_3 + P_3 A + \tau_{31}(P_3 - P_1) + \tau_{23}(P_2 - P_3) < 0,$$

$$(312) \quad A^\top P_1 + P_1 A + \tilde{\tau}_{31}(P_1 - P_3) + \tilde{\tau}_{21}(P_2 - P_1) < 0,$$

$$(4) \quad A^\top P_1 + P_1 A + \tilde{\lambda}(P_1 - P_2) < 0.$$

Note that, for every  $i \in \{1, \dots, m\}$ , we have just one more inequality (involving just one more non-negative scalar) as compared to the max of quadratics case.

# B

## Appendix to Chapter 5

This appendix presents some results concerning Chapter 5 and the corresponding proofs. In particular, in Section B.1 we prove that, given a proper partition  $\mathcal{X} = \{X_i, \mathcal{O}_i\}_{i \in \mathcal{I}}$  of  $\mathbb{R}^n$ , piecewise  $\mathcal{C}^1$  functions w.r.t.  $\mathcal{X}$  (Definition 5.8) are non-pathological (Definition 2.19) In Section B.2 we prove the equivalence between two different ISS-Lyapunov sufficient conditions, as discussed in Remark 5.5.

### B.1 Non-Pathology of Piecewise $\mathcal{C}^1$ functions

In this Appendix we conclude the proof of Proposition 5.9, showing that piecewise  $\mathcal{C}^1$  functions are non-pathological. We first recall that, considering  $\mathcal{X} = \{X_i, \mathcal{O}_i\}_{i \in \mathcal{I}}$ , a proper partition of  $\mathbb{R}^n$ , if  $V \in \mathcal{P}(\mathcal{X})$  then  $V$  is locally Lipschitz and

$$\partial V(x) = \text{co} \{ \nabla V_\ell(x) \mid \ell \in \mathcal{I}_\mathcal{X}(x) \}. \quad (\text{B.1})$$

#### Lemma B.1

Consider  $\mathcal{X} = \{X_i, \mathcal{O}_i\}_{i \in \mathcal{I}}$ , a *proper partition* of  $\mathbb{R}^n$ . If  $V \in \mathcal{P}(\mathcal{X})$ , then  $V$  is non-pathological.

*Proof:* Recalling Definition 2.19 we must show that, given any  $\varphi \in AC(\mathbb{R}_+, \mathbb{R}^n)$ ,  $\partial V(\varphi(t))$  is a subset of an affine subspace orthogonal to  $\dot{\varphi}(t)$ , for almost all  $t \in \mathbb{R}_+$ , namely that  $\exists a_t \in \mathbb{R}$  such that

$$\langle v, \dot{\varphi}(t) \rangle = a_t, \quad \forall v \in \partial V(\varphi(t)). \quad (\text{B.2})$$

Since  $\varphi : \mathbb{R}_+ \rightarrow \mathbb{R}^n$  is absolutely continuous and  $V : \mathbb{R}^n \rightarrow \mathbb{R}$  is locally Lipschitz we have that  $\varphi$  and  $V(\varphi(t))$  are differentiable almost everywhere, i.e. there exists a set of measure zero  $\mathcal{N} \subset \mathbb{R}_+$  such that  $\dot{\varphi}(t)$  and  $\frac{d}{dt}V(\varphi(\cdot))(t)$  both exist for every  $t \in \mathbb{R}_+ \setminus \mathcal{N}$ . Using (B.1), to ensure (B.2) it is enough to show that, for almost all  $t \in \mathbb{R}_+ \setminus \mathcal{N}$ , there exists  $a_t \in \mathbb{R}$  such that

$$\langle \nabla V_\ell(\varphi(t)), \dot{\varphi}(t) \rangle = a_t, \quad \forall \ell \in \mathcal{I}_\mathcal{X}(\varphi(t)). \quad (\text{B.3})$$

Fix any  $t \in \mathbb{R}_+ \setminus \mathcal{N}$ . Either (B.3) holds for that  $t$ , or there exist  $\ell_1, \ell_2 \in \mathcal{I}_\mathcal{X}(\varphi(t))$  such that  $\ell_1 \neq \ell_2$  and

$$\langle \nabla V_{\ell_1}(\varphi(t)), \dot{\varphi}(t) \rangle \neq \langle \nabla V_{\ell_2}(\varphi(t)), \dot{\varphi}(t) \rangle.$$



In this second case we have

$$\begin{aligned} \frac{d}{dt} (V_{\ell_1}(\varphi(t)) - V_{\ell_2}(\varphi(t))) &= \\ &= \langle \nabla V_{\ell_1}(\varphi(t)), \dot{\varphi}(t) \rangle - \langle \nabla V_{\ell_2}(\varphi(t)), \dot{\varphi}(t) \rangle \neq 0. \end{aligned}$$

Thus, by continuity, there exists  $\varepsilon > 0$  small enough such that  $V_{\ell_1}(\varphi(\tilde{t})) \neq V_{\ell_2}(\varphi(\tilde{t}))$ , for all  $\tilde{t} \in (t - \varepsilon, t + \varepsilon) \setminus \{t\}$ , which implies that either  $\ell_1 \notin \mathcal{I}_X(\varphi(\tilde{t}))$  or  $\ell_2 \notin \mathcal{I}_X(\varphi(\tilde{t}))$  (or both), for all such  $\tilde{t}$ , because, by Definition 5.8,

$$\ell_1, \ell_2 \in \mathcal{I}_X(x) \Rightarrow V(x) = V_{\ell_1}(x) = V_{\ell_2}(x).$$

This shows that for any point  $t$  where two or more scalar products  $\langle \nabla V_{\ell_j}(\varphi(t)), \dot{\varphi}(t) \rangle$  “disagree” in (B.3),  $t$  is isolated. We conclude by recalling that a set of isolated point is countable [67, Page 180] and thus has measure zero, as to be proven.  $\diamond$

## B.2 Proof of ISS-Equivalent Conditions

In this section we want to prove the following result

### Lemma B.2

Consider  $V : \mathbb{R}^n \rightarrow \mathbb{R}$  and  $F : \mathbb{R}^n \times \mathbb{R}^m \rightrightarrows \mathbb{R}^n$  a set valued map with nonempty, compact and convex values, and outer semicontinuous in the first and continuous in the second component such that  $F(0, 0) = \{0\}$ . Then:

There exists  $\rho \in \mathcal{K}_\infty$  and  $\gamma \in \mathcal{K}$  such that

$$\max \dot{\bar{V}}_F(x, u) \leq -\rho(|x|), \quad \forall |x| \geq \gamma(|u|), \quad (\text{B.4})$$

implies

There exists  $\hat{\rho}, \hat{\gamma} \in \mathcal{K}_\infty$  such that

$$\max \dot{\bar{V}}_F(x, u) \leq -\hat{\rho}(|x|) + \hat{\gamma}(|u|) \quad \forall (x, u) \in \mathbb{R}^n \times \mathbb{R}^m. \quad (\text{B.5})$$

*Proof :* The main idea of this proof was firstly introduced in [27, Proposition 2.6]. Let us define  $\Omega(r) := \mathbb{B}^x(0, \gamma(r)) \times \mathbb{B}^u(0, r) \subset \mathbb{R}^n \times \mathbb{R}^m$ , for all  $r \in \mathbb{R}_+$ . Define

$$\lambda(r) := \max_{(x, u) \in \Omega(r)} \{ \max \dot{\bar{V}}_F(x, u) + \rho(|x|) \},$$

$$\beta(r) := \max\{0, \lambda(r)\} \quad (\text{and } \beta(r) = 0 \text{ if } \lambda(r) \text{ not defined}).$$

Clearly  $\beta(r) \geq 0$  for all  $r \in \mathbb{R}_+$ , it is nondecreasing and  $\beta(0) = 0$ . We want to prove that  $\lim_{s \rightarrow 0^+} \beta(s) = 0$ . Since  $\rho(|x|) \rightarrow 0$  if  $|x| \rightarrow 0$ , defining

$$\theta(r) := \max_{(x, u) \in \Omega(r)} \{ \max \dot{\bar{V}}_F(x, u) \},$$

we have to prove that

$$\limsup_{s \rightarrow 0^+} \theta(s) \leq 0.$$

In other words we have to prove that  $\forall \varepsilon < 0$  there exists an  $\delta > 0$  such that

$$\theta(\delta') < \varepsilon \quad \forall \delta' < \delta.$$

We suppose by contradiction that this is not the case, thus  $\exists \varepsilon > 0$  such that  $\forall \delta > 0$  there exists  $\delta' \leq \delta$  such that

$$\theta(\delta') \geq \varepsilon.$$

Let fix this “special”  $\varepsilon > 0$ . Developing it means that for all  $\delta > 0$ , there exists a  $(x, u) \in \Omega(\delta)$  such that

$$\max \dot{V}_F(x, u) \geq \varepsilon.$$

That means that for all  $\delta$ , there exists a  $(x, u) \in \Omega(\delta)$ , there exists an  $f \in F(x, u)$  such that

$$\text{cost} \equiv \langle v, f \rangle \geq \varepsilon, \quad \forall v \in \partial V(x). \quad (\text{B.6})$$

We have the following fact

- $\partial V$  is locally bounded around 0, see [34, Prop. 2.1.2(a)]
- Since  $F(0, 0) = \{0\}$  and  $F$  outer semicontinuous in  $(x, u)$ , for every  $\phi > 0$ , there exists a  $m(\phi) > 0$  such that  $F(x, u) \subset \mathbb{B}^x(0, \phi)$ , for all  $(x, u) \in \Omega(m(\phi))$ , see [105, Theorem 5.19].

By the local boundedness of  $\partial V$  there exist  $a_1 > 0$  and  $M > 0$  such that  $|v| \leq M$ , for all  $v \in \partial V(x)$ , for all  $x \in \mathbb{B}^x(0, \gamma(a_1))$ . Let us chose  $a_2 < \frac{\varepsilon}{M}$ , thus there exist  $m(a_2) > 0$  such that  $F(x, u) \subset \mathbb{B}^x(0, a_2)$ , for all  $(x, u) \in \Omega(m(a_2))$ . Thus take  $a < \min\{a_1, a_2\}$ , by (B.6) we have that there exists a  $(x, u) \in \Omega(a)$  and a  $f \in F(x, u)$  such that  $\varepsilon \leq \langle v, f \rangle$ , but this would imply

$$\varepsilon \leq \langle v, f \rangle \leq |v||f| < M \frac{\varepsilon}{M} = \varepsilon,$$

leading us to a contradiction. Thus there exists an  $\hat{\gamma} \in \mathcal{K}_\infty$  such that  $\beta(r) \leq \hat{\gamma}(r)$ , for all  $r \in \mathbb{R}_+$ . Now if  $|x| \geq \gamma(|u|)$ , (B.5) follows directly from (B.4), if  $|x| \leq \gamma(|u|)$  we have

$$\begin{aligned} \max \dot{V}_F(x, u) + \rho(|x|) &\leq \max_{(x,u) \in \Omega(|u|)} \{\max \dot{V}_F(x, u) + \rho(|x|)\} \\ &\leq \hat{\gamma}(|u|). \end{aligned}$$

◇



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