Lyapunov Functions and Ensemble Approximations for Constrained Systems using Semidefinite Programming

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THÈSE

En vue de l'obtention du

DOCTORAT DE L'UNIVERSITÉ DE TOULOUSE

Délivré par :
l'Institut National des Sciences Appliquées de Toulouse (INSA de Toulouse)

Présentée et soutenue le *15/10/2021* **par :**

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Remerciements

Tout d'abord, je tiens à remercier mes directeurs de thèse, Aneel et Didier. Sans eux, le travail n'aurait pas pu être complet. Je les remercie pour leurs soutiens, leurs disponibilités et leurs judicieux conseils durant ces trois années de thèse et pour toutes les heures qu'ils m'ont consacrées pour diriger les travaux de recherches de cette thèse en répondant à toutes mes questions. Leurs connaissances approfondies m'ont permis de mener à bien ce travail. De plus, je les remercie pour leur bonne humeur durant chaque réunion, pour savoir si bien rendre agréable le cadre de travail et pour avoir été présents à mes côtés pour m'orienter et m'écouter. Il m'ont aidé à approfondir au maximum mes travaux afin de pouvoir être fière aujourd'hui du travail réalisé. Je leur en suis profondément reconnaissante.

J'exprime mes sincères remerciements à Vincent ACARY et Vincent ANDRIEU pour avoir accepté d'être rapporteurs et de consacrer du temps pour examiner le manuscrit. Il m'ont fait l'honneur de bien vouloir étudier avec attention le travail. Je suis très honorée de remercier aussi Kanat CAMLIBEL et Mirjam DÜR pour avoir accepté l'invitation d'être parmi le jury de ma thèse.

Je remercie toutes les personnes de l'équipe MAC pour leur acceuil chaleureux, leur soutien, leurs encouragements, leurs savoirs scientifiques et leurs échanges amicaux pendant ces trois années. Ils ont été toujours présents pour n'importe quel sujet. Je remercie tous les doctorants de l'équipe MAC pour toutes les discussions qu'on a eu ensemble. J'étais ravi de leur faire connaissance. J'exprime mes remerciements à mes collègues du bureau (Matteo, Mathias, Corbinian, Swann) pour leur bonne humeur et gentillesse, et je leur souhaite une très bonne continuation dans leurs futurs travaux. Je souhaiterais remercier toutes les personnes du laboratoire que j'ai pu fréquenter au cours de ma thèse.

Je voudrais également remercier Liviu NICU, directeur actuel du LAAS, de m'y avoir accueilli avec les meilleures conditions de travail et pour ses efforts de rester à notre côté dans toute la période du 'Covid' et le confinement, en nous envoyant chaque jour des emails d'encouragement.

Mes remerciements et mes pensées vont aussi à tous mes amis qui m'ont soutenu lors de mon séjour dans cette belle ville de Toulouse. Je remercie en particulier Dany, Dima et Chaymaa qui m'ont accompagné et encouragé pendant tous les moments au LAAS et qui m'ont apporté leur support moral tout au long des trois années. Je les remercie pour tous les bons moments passés ensemble au LAAS (déjeuners, pauses,...) et aussi les moments en dehors du LAAS. Je leur dis que nos souvenirs sont gravés dans ma mémoire et je n'oublierai jamais chaque jour passé ensemble. On restera amis pour toujours. Je n'oublie encore pas d'ajouter de remercier Dima pour tous les moments de travail passés ensemble (toutes les soirées passées au LAAS pour la rédaction). Je souhaiterais bien d'être aussi présente à sa soutenance de thèse. Je présente mes remerciements à tous mes autres amis à Toulouse (Elena, Michella, Perla, Joe, Kamil, Ralph,...) et ceux qui sont au Liban (Sara, Joyce, Miriam,...) et dans d'autres régions en France (Cynthia, Grace, Hiba,...)

Je ne terminerai pas mes remerciements sans remercier ma famille sans lesquelle mon travail n'aurait pas vu le jour. Un très GRAND MERCI à tous les membres de ma famille au Liban qui m'ont soutenu et été présents dans tous les moments que j'ai vécus depuis mon arrivée en France. Je remercie en particulier mon frère Marc pour son encouragement qui est pour moi le pilier fondateur de ce que je suis et de ce que je fais. Je remercie ma mère et mon père, confiants en mes capacités, qui m'ont toujours épaulée dans mes études et mes décisions. Je ne pourrais jamais assez les remercier !

DU FOND DU COEUR, MERCI A TOUS ! !

Abstract

This thesis deals with analysis of constrained dynamical systems, supported by some numerical methods. The systems that we consider can be broadly seen as a class of nonsmooth systems, where the state trajectory is constrained to evolve within a prespecified (and possibly time-varying) set. The possible discontinuities in these systems arise due to sudden change in the vector field at the boundary of the constraint set. The general framework that we adopt has been linked to different classes of nonsmooth systems in the literature, and it can be described by an interconnection of an ordinary differential equation with a static relation (such as variational inequality, or a normal cone inclusion, or complementarity relations). Such systems have found applications in modeling of several engineering and physical systems, and the results of this dissertation make some contributions to the analysis and numerical methods being developed for such system class.

The first problem that we consider is related to the stability of an equilibrium point for the aforementioned class of nonsmooth systems. We provide appropriate definitions for stability of an equilibrium, and the Lyapunov functions, which take into consideration the presence of constraints in the system. In the presence of conic constraints, it seems natural to work with cone-copositive Lyapunov functions. To confirm this intuition, and as the first main result, we prove that, for a certain class of cone-constrained systems with an exponentially stable equilibrium, there always exists a smooth cone-copositive Lyapunov function. Putting some more structure on the system vector field, such as homogeneity, we can show that the aforementioned functions can be approximated by a rational function of cone-copositive homogeneous polynomials.

This later class of functions is seen to be particularly amenable for numerical computation as we provide two types of algorithms precisely for that purpose. These algorithms consist of a hierarchy of either linear or semidefinite optimization problems for computing the desired cone-copositive Lyapunov function. For conic constraints, we provide a discretization algorithm based on simplicial partitioning of a simplex, so that the search of the desired function is addressed by constructing a hierarchy (associated with the diameter of the cells in the partition) of linear programs. Our second algorithm is tailored to semi-algebraic sets, where a hierarchy of semidefinite programs is constructed to compute Lyapunov functions as a sum of squares of polynomials. Some examples are given to illustrate our approach.

Continuing with our study of state-constrained systems, we next consider the time evolution of a probability measure which describes the distribution of the state over a set. In contrast with smooth ordinary differential equations, where the evolution of this probability measure is described by the Liouville equation, the flow map associated with the nonsmooth differential inclusion is not necessarily invertible and one cannot directly derive a continuity equation to describe the evolution of the distribution of states. Instead, we consider Lipschitz approximation of our original nonsmooth system and construct a sequence of measures obtained from Liouville equations corresponding to these approximations. This sequence of measures converges in weak-star topology to the measure describing the evolution of the distribution of states for the original nonsmooth system. This allows us to approximate numerically the evolution of moments (up to some finite order) for our original nonsmooth system, using a hierarchy of semidefinite programs. Using similar methodology, we study the approximation of the support of the solution (described by a measure at each time) using polynomial approximations.

Keywords: Constrained systems; complementarity systems; converse Lyapunov theorem; moment-sums-of-squares optimization; ensemble approximations.

Résumé

Cette thèse traite de l'analyse de systèmes dynamiques avec contraintes, avec certaines méthodes numériques. Les systèmes que nous considérons peuvent être considérés comme une classe de systèmes non lisses, où la trajectoire d'état est contrainte d'évoluer dans un ensemble prédéfini (et éventuellement variable en temps). Les discontinuités possibles dans ces systèmes surviennent en raison d'un changement soudain du champ vectoriel à la frontière de l'ensemble de contraintes. Le cadre général que nous adoptons est relié à différentes classes de systèmes non lisses dans la littérature, et peut être décrit par une interconnexion d'une équation différentielle ordinaire avec une relation statique (telle qu'une inégalité variationnelle, ou une inclusion dans le cône normal, ou des relations de complémentarité). De tels systèmes ont trouvé des applications dans la modélisation de systèmes d'ingénierie et physiques, et les résultats de cette thèse apportent des contributions à l'analyse et aux méthodes numériques développées pour une telle classe de systèmes.

Le premier problème que nous considérons est lié à la stabilité d'un point d'équilibre pour la classe susmentionnée de systèmes non lisses. Nous proposons des définitions appropriées pour la stabilité d'un équilibre et les fonctions de Lyapunov, qui prennent en considération la présence de contraintes dans le système. En présence de contraintes coniques, il semble naturel de travailler avec des fonctions de Lyapunov cône-copositives. Pour confirmer cette intuition, et comme premier résultat principal, nous prouvons que, pour une certaine classe de systèmes avec contraintes coniques avec un équilibre exponentiellement stable, il existe toujours une fonction de Lyapunov lisse cône-copositive. En mettant un peu plus de structure sur le champ de vecteurs du système, comme l'homogénéité, nous pouvons montrer que les fonctions susmentionnées peuvent être approchées par une fonction rationnelle de polynômes homogènes cône-copositifs.

Cette dernière classe de fonctions est particulièrement adaptée au calcul numérique et nous fournissons deux types d'algorithmes dans ce but. Ces algorithmes consistent en une hiérarchie de problèmes d'optimisation linéaires ou semi-définis pour le calcul de la fonction de Lyapunov cône-copositive. Pour les contraintes coniques, nous proposons un algorithme de discrétisation basé sur le partitionnement simplicial d'un simplexe, de sorte que la recherche de la fonction souhaitée est abordée en construisant une hiérarchie (associée au diamètre des cellules de la partition) de programmes linéaires. Notre deuxième algorithme est adapté aux ensembles semi-algébriques, où une hiérarchie de programmes semi-définis est construite pour calculer les fonctions de Lyapunov sous la forme de polynômes sommes de carrés. Quelques exemples sont donnés pour illustrer notre approche.

Poursuivant notre étude des systèmes à état contraint, nous considérons ensuite l'évolution temporelle d'une mesure de probabilité qui décrit la distribution de l'état sur un ensemble. Contrairement aux équations différentielles ordinaires lisses, où l'évolution de cette mesure de probabilité est décrite par l'équation de Liouville, le flot associé à l'inclusion différentielle non lisse n'est pas nécessairement inversible et on ne peut pas directement dériver une équation de continuité pour décrire l'évolution de la distribution des états. Au lieu de cela, nous considérons l'approximation de Lipschitz pour notre système original non lisse et construisons une séquence de mesures obtenue à partir des équations de Liouville correspondant à ces approximations. Cette séquence de mesures converge en topologie faible étoile vers la mesure décrivant l'évolution de la distribution des états pour le système original non lisse. Cela nous permet d'approximer numériquement l'évolution des moments (jusqu'à un certain ordre fini) pour notre système original non lisse, en utilisant une hiérarchie de programmes semi-définis. En utilisant une méthodologie similaire, nous étudions l'approximation du support de la solution (décrite par une mesure à chaque instant) à l'aide d'approximations polynomiales.

Mots clés : Systèmes avec contraintes ; systèmes de complémentarité ; converse du théorème de Lyapunov ; optimisation moments - sommes de carrés ; approximations d'ensemble.

Contents

Notation

This section provides the notations used all along the thesis.

- $\mathbb{N}$: The set of positive integers.
- $\mathbb{N}^n$: The set of multi-indices $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$ of non-negative integers.
- $\mathbb{N}^n_m$: The set of vectors $\alpha \in \mathbb{N}^n$ such that $\sum_{i=1}^n \alpha_i \leqslant m$.
- $\mathbb{R}$: The set of real numbers.
- $\mathbb{R}^n$: Real n-dimensional space.
- $\mathbb{R}^n_+$: Nonnegative orthant of $\mathbb{R}^n$.
- $\|.\|$: Euclidean norm on $\mathbb{R}^n$.
- $x^\top$: Transpose of a vector x.
- $\langle x, y\rangle = x^\top y$: Standard inner product of vectors in $\mathbb{R}^n$.
- $x \perp y$: $x^\top y = 0$.
- L^∞_{loc}: Space of functions $f \in L^\infty(K)$ for every compact set $K \subset \mathbb{R}^n$.
- $\operatorname{im} F$: Image of a set-valued mapping F.
- $\mathbb{S}^n$: Space of symmetric matrices in $\mathbb{R}^{n\times n}$.
- $\mathbb{R}[x]$: The ring of polynomials.
- $\mathbb{R}[x]_d$: The vector space of polynomials of total degree at most d.
- $\deg p$: Total degree of a polynomial p.
- $\dim S$: Dimension of the set S.
- $\operatorname{bd} S$: Boundary of the set S.
- $\operatorname{cl} S$: Closure of the set S.
- $\operatorname{int} S$: Interior of the set S.
- $\operatorname{rint} S$: Relative interior of the set S.

- ψ_S: Indicator function of the set S.
- $\mathcal{N}_S(x)$: Normal cone to the set S at point x.
- $\mathcal{T}_S(x)$: Tangent cone to the set S at point x.
- S°: Polar cone of the set S.
- $S^\star$: Dual cone of the set S.
- $d(x,S)$: Euclidean distance between vector x and set S.
- $d_H(X,Y)$: Hausdorff distance between sets X and Y.
- $\partial\varphi$: Subdifferential of the function φ.
- $\mathbb{B}(x,r)$: Closed Euclidean ball of radius r centered at x.
- $\mathscr{C}(X)$: Space of all continuous functions on X.
- $\mathscr{C}_+(X)$: Cone of all nonnegative continuous functions on X.
- $\mathscr{C}(X;Y)$: Space of all continuous functions from X to Y.
- $\mathscr{M}(X)$: Space of all signed Borel measures on X.
- $\mathscr{M}_+(X)$: Cone of all nonnegative Borel measures on X.
- $\mathcal{P}(X)$: Set of probability measures on X.
- ∇f: Gradient of f. If f is a function of (t,x), then $\nabla f = \frac{\partial f}{\partial x}$.
- $\operatorname{dom} f$: Domain of the function f.
- $\inf f$: Infimum of the function f.
- $\sup f$: Supremum of the function f.
- $\min f$: Minimum of the function f.
- $\max f$: Maximum of the function f.

Acronyms

This section provides the acronyms used all along the thesis.

- CP: Complementarity problem.
- LCP/LCS: Linear complementarity problem/system.
- LCCP/LCCS: Linear cone complementarity problem/system.
- SOS: Sum-of-squares of polynomials.
- LMI: Linear matrix inequality.
- LP: Linear program, linear programming.
- SDP: Semidefinite program, semidefinite programming.

List of Publications

1. M. Souaiby, A. Tanwani and D. Henrion. Cone-copositive Lyapunov functions for complementarity systems: Converse result and polynomial approximation. IEEE Transactions on Automatic Control, 2021. DOI: 10.1109/TAC.2021.3061557.

2. M. Souaiby, A. Tanwani and D. Henrion. Computation of Lyapunov Functions under State Constraints using Semidefinite Programming Hierarchies. IFAC World Congress - Germany and IFAC-PapersOnline, 53(2): 6281-6286, 2020.

3. M. Souaiby, A. Tanwani and D. Henrion. Ensemble approximations for constrained dynamical systems using Liouville equation. Submitted for publication, 2021.

1 Introduction

1.1 Overview

Constrained dynamical systems, where the evolution of state trajectories is confined to a predefined set, arise in different applications. Examples include electrical circuits [1] where the voltages and currents have to respect some algebraic relations in addition to the differential equations arising from active elements. Mathematically, such systems can be modeled using different approaches, but in this thesis, we adopt an approach which models constrained systems as a particular class of nonsmooth dynamical systems using the framework of differential inclusions. More precisely, given a closed convex set $\mathcal{S} \subset \mathbb{R}^n$, and a locally Lipschitz continuous function $f : \mathbb{R}^n \to \mathbb{R}^n$, we consider the evolution of state trajectories described as:

$$\dot{x} \in f(x) - \mathcal{N}_{\mathcal{S}}(x), \tag{1.1}$$

where x is the state of the system and $\mathcal{N}_{\mathcal{S}}(x) \in \mathbb{R}^n$ denotes the outward normal cone to the set $\mathcal{S}$ at the point $x \in \mathbb{R}^n$. Using the definition of the normal cone[1], one can also write (1.1) as an evolution variational inequality, described as

$$\langle \dot{x}(t) - f(x(t)), y - x(t) \rangle \geqslant 0,$$

for all $y \in \mathcal{S}$, $x(t) \in \mathcal{S}$, $t \in [0, T]$. Such dynamical systems have been a matter of extensive study in past decades due to their relevance in engineering and physical systems and its connections to different classes of nonsmooth mathematical models. Analysis of such systems requires tools from variational analysis, nonsmooth analysis, set-valued analysis [14, 115, 139].

An absolutely continuous function $x : [0, T] \to \mathbb{R}^n$ is a solution of (1.1) if there exists a (possibly discontinuous and state-dependent) function $\eta : [0, T] \to \mathbb{R}^n$ such that $\eta(t) \in -\mathcal{N}_{\mathcal{S}}(x(t))$, for all $t \geqslant 0$ and $\dot{x}(t) = f(x(t)) + \eta(t)$

[1]The definition of a normal cone appears in Chapter 2, Definition 10.

holds for almost every t. In other words, if at a time $t \in [0, T]$, $x(t)$ is in the interior of $\mathcal{S}$, then $\eta(t)$ is essentially equal to 0. However, if $x(t)$ is on the boundary of the set $\mathcal{S}$, then the vector $\eta(t) \in -\mathcal{N}_{\mathcal{S}}(x(t))$ is chosen such that $\dot{x}(t) = f(x(t)) + \eta(t)$ points inside the set $\mathcal{S}$, which allows the motion to continue within the set $\mathcal{S}$. In other words, one can also interpret the evolution of the trajectories of system (1.1) to be constrained in such a manner that $x(t) \in \mathcal{S}$, for each $t \in [0, T]$.

One sees that the system (1.1) involves discontinuities, which depend on the position of x. Discontinuous systems, i.e. dynamical systems whose right-hand side is not a continuous vector field, have been studied in various scientific fields like applied mathematics, systems and control, mechanics, biology, and electronics. They model a whole variety of applications: dry friction, forced vibrations, electrical circuits, elasto-plasticity, oscillating systems with viscous damping [1, 5, 107]. Discontinuous systems, or *nonsmooth systems* in general, form a rather large class of systems (just like nonlinear systems) and it is important to develop tools which address peculiar features of such systems. In mathematics and optimization, there is an increasing interest in studying problems involving nonsmooth phenomenon and there is a need to study them rigorously. The utility of the tools of nonsmooth analysis [54, 56] are not confined only to situations in which nonsmoothness is present. Sometimes in order to solve difficult smooth problems, we need to recall methods from nonsmooth analysis for simplifying the problem.

In the literature, we find several references related to analysis, numerics, and control of dynamical systems of the form (1.1). A recent survey article [35] provides an overview on this topic. In this dissertation, we are primarily concerned with questions related to analysis and computational feasibility for certain problems related to system class (1.1), or close variants of such systems.

The first set of questions that we address in our work is related to stability of an equilibrium point. Stability analysis of hybrid, or nonsmooth dynamical systems, where the vector field is set-valued with possible discontinuities, is of particular relevance with respect to several applications. Naturally, Lyapunov functions for such systems provide a potent tool for studying stability related properties and the underlying theory strongly influences our understanding of the motion of dynamic systems. Several advances have been made on the theoretical side to establish existence of Lyapunov functions for various classes of dynamical systems, see e.g. [65, 96, 98, 78, 138, 29] for examples of standard expositions. An important question in stability analysis is to determine a class of Lyapunov functions whose existence is necessary and sufficient for proving stability. For constrained systems of the form (1.1), such questions have not received much attention in the literature. This dissertation addresses the existence of Lyapunov functions rather rigorously with certain mild assumptions on the system structure. In particular, we specify a function class for the Lyapunov functions which takes into consideration the constraints on the system dynamics.

Having a candidate for the Lyapunov function, the theory provides an analysis tool to ensure the stability of dynamical systems. However, it does not give a procedure for finding the Lyapunov function. Then, constructing a suitable Lyapunov function is a hard problem since there are no general methods for computing such functions. In general, the problem of finding the Lyapunov function is a challenging task, which has attracted the attention of many researchers, see for example [77] for an overview. If one looks at the literature on computing Lyapunov functions numerically using appropriate algorithms, the fundamental question behind the works in this direction boils down to checking the positivity of certain functions over the state space, which is a challenging problem numerically [120]. Modern developments in the field of real algebraic geometry [136, 141] provide certificates of positivity of (polynomial) functions with Positivstellensätze relying on *sums-of-squares* (SOS) decompositions. Since it has been observed in [134, 53, 128] that finding SOS decompositions is equivalent to semidefinite programming (SDP) or linear matrix inequalities (LMI), numerical tools based on SOS optimization have been developed extensively over the past two decades to compute Lyapunov functions, see e.g. [128, 135, 85, 52]. While checking if a function is positive everywhere is numerically hard, checking if it admits an SOS decomposition is a semidefinite program [134, 53, 128]. An overview of sum-of-squares techniques can be found in [102], and applications of semidefinite programming for solving polynomial inequalities in control systems related problems appear in [84]. When the system is modeled by switching vector fields over the whole state space, then the construction of Lyapunov functions using SOS is studied in [126, 10, 8]. Other approaches for computing Lyapunov functions for differential inclusions based on linear programming appear in [15]. However, we are concerned with a certain class of differential inclusions which is useful in modeling systems with state constraints, where the vector field exhibits discontinuous behaviour on the boundary of the constraints so that the state trajectory is forced to evolve within the prespecified set.

Stepping aside from the stability related problems, one observes that there is a considerable amount of effort being put into developing the simulation tools for system of form (1.1). In these methods, we study the evolution of state trajectories, defined as absolutely continuous functions of time, by taking the initial condition to be a given vector in the constraint set. For several applications, the initial condition is not known exactly and it is natural to model the initial condition via some probability distribution supported on the constraint set. Evolution of probability measures through the system dynamics of the form (1.1) has received very little attention in the literature, and once again our interest lies in proposing appropriate numerical methods that allow us to approximate the resulting solution (which now evolves in the space of probability measures).

By and large, our efforts have been focused on studying some analysis related problems for systems of the form (1.1), with a special attention on developing appropriate numerical routines to support our analytical observations. The interesting aspect of our work comes from the fact that the presence of constraints introduces nonsmooth vector fields which needs special care. By restricting ourselves to convex sets, one can borrow tools from the theory of convex analysis and maximal monotone operators to provide constructive statements. On the numerical side, the use of semidefinite programs (such as the ones based on SOS decomposition) remained unexplored for simulation or stability analysis of the constrained systems (1.1) before this dissertation, and in this thesis, we provide some instances of how such tools can be adapted.

1.2 Motivating Examples

To present some motivation behind the system class (1.1), let us rewrite it as follows:

$$\begin{aligned} \dot{x} &= f(x) + \eta \\ \eta &\in -\mathcal{N}_{\mathcal{S}}(x). \end{aligned} \tag{1.2}$$

One can, therefore, see (1.1) as interconnection of an ordinary differential equation $\dot{x} = f(x)$, with a static relation $\eta \in -\mathcal{N}_{\mathcal{S}}(x)$. This sort of static relation, described by normal cone inclusion, can be represented in several forms. In the particular case, when $\mathcal{S}$ is the positive orthant of $\mathbb{R}^n$, that is, $\mathcal{S} = \mathbb{R}^n_+$, then

$$\eta \in -\mathcal{N}_{\mathcal{S}}(x) \Leftrightarrow 0 \leqslant \eta \perp x \geqslant 0,$$

where the expression on the right-hand of the equivalence denotes the following three algebraic relations:

$$x \geqslant 0, \quad \eta \geqslant 0, \quad x^\top \eta = 0, \tag{1.3}$$

with $\eta, x \in \mathbb{R}^n$, and the inequality $x \geqslant 0$ is componentwise.

This section provides some examples where the relations of the form (1.3) appear. This will provide the motivation about why we are interested in such nonsmooth relations. In the next section, we will see some examples of dynamical systems which includes such static relations in their description.

1.2.1 Piecewise Linear Functions

It is possible to model certain piecewise linear relations using the framework of complementarity relations [34, 35, 106, 150]. Let us consider two such examples:

Example 1. For some $a \in \mathbb{R}$, consider the piecewise linear function:

$$x \mapsto y = f(x) = \max\{a, x\}$$

then we can write it as

$$y = x + \lambda, \quad w = x + \lambda - a, \quad 0 \leqslant \lambda \perp w \geqslant 0,$$

or

$$y = a + \lambda, \quad w = -x + a + \lambda, \quad 0 \leqslant \lambda \perp w \geqslant 0.$$

Example 2. Now, let us consider the saturation function:

$$y = f(x) = \text{sat}(x) = \begin{cases} 1, & \text{if } x \geqslant 1 \\ x, & \text{if } -1 \leqslant x \leqslant 1 \\ -1, & \text{if } x \leqslant -1 \end{cases}$$

We can write it in complementarity form as

$$y = -1 + \lambda_1 - \lambda_2,\ w_1 = -x + \lambda_1 - 1,\ w_2 = -x + \lambda_2 + 1,\ 0 \leqslant \lambda \perp w \geqslant 0.$$

1.2.2 Constrained Optimization Problem

This section is inspired by [45]. Let us consider the optimization problem

$$\begin{aligned} \min_{x} &\quad g(x) \\ \text{subject to} &\quad x \in C, \end{aligned} \tag{1.4}$$

where $g : \mathbb{R}^n \to \mathbb{R}$ is a continuously differentiable convex function, and $C \subset \mathbb{R}^n$ is a closed convex set. We can write the constrained optimization problem as an unconstrained optimization problem

$$\min_{x} \quad g(x) + \psi_C(x) \tag{1.5}$$

where ψ_C is an indicator function associated with the set C, defined by,

$$\psi_C(x) = \begin{cases} 0 & \text{if } x \in C, \\ +\infty & \text{otherwise.} \end{cases} \tag{1.6}$$

As a special case, let us consider the following quadratic program

$$\begin{aligned} \min_x \quad & \frac{1}{2} x^\top A x + b^\top x \\ \text{subject to} \quad & Cx \geqslant c, \end{aligned} \tag{1.7}$$

where $A \in \mathbb{R}^{n \times n}$, $b \in \mathbb{R}^n$, $C \in \mathbb{R}^{m \times n}$ and $c \in \mathbb{R}^m$. We assume that A is symmetric. In addition, if A is positive semidefinite, then the cost function is convex, and we have a convex quadratic program.

We can characterize the optimal solution using Karush-Kuhn-Tucker (KKT) conditions. If $x^\star$ is a locally optimal solution of the quadratic program, then there exists $\lambda^\star \in \mathbb{R}^m$ such that

$$\begin{aligned} Ax^\star + b - C^\top \lambda^\star = 0, \\ \lambda^\star \geqslant 0, \quad Cx^\star - c \geqslant 0, \quad {\lambda^\star}^\top (Cx^\star - c) = 0. \end{aligned} \tag{1.8}$$

These points are described as the stationary points of the linear complementarity system

$$\begin{aligned} \dot{x} &= -Ax - b + C^\top \lambda, \\ 0 &\leqslant \lambda \perp Cx - c \geqslant 0. \end{aligned} \tag{1.9}$$

1.2.3 Electrical Circuits with Ideal Diodes

Modeling of electrical circuits has attracted much interest over the past few decades and it has been studied in numerous articles and books [1, 3, 4]. The following framework has been proposed in [2].

Let us consider electrical circuits in which the diodes are supposed to be ideal, i.e., the characteristic between the current $i(t)$ and the voltage $v(t)$ satisfies the complementarity conditions:

$$0 \leqslant i(t) \perp v(t) \geqslant 0. \tag{1.10}$$

This set of conditions means that both the variables current $i(t)$ and voltage $v(t)$ have to remain nonnegative at all times t and they have to be orthogonal one to each other. So $i(t)$ can be positive only if $v(t) = 0$, and vice versa. The complementarity condition (1.10) between the current across the diode and its voltage represents the way to define the diode characteristic.

- We can rewrite the complementarity relations in (1.10) as:

$$i(t) \in -\partial \psi_{\mathbb{R}_+}(v(t)), \quad v(t) \in -\partial \psi_{\mathbb{R}_+}(i(t)). \tag{1.11}$$

  where $\partial \psi_{\mathbb{R}_+}$ is equal to $\mathcal{N}_{\mathbb{R}_+}$, the normal cone to $\mathbb{R}_+$.

- Both (1.10) and (1.11) are in turn equivalent to the variational formulation: for all $i(t) \geqslant 0$,

$$\langle i(t), z - v(t) \rangle \geqslant 0, \ \forall z \geqslant 0. \tag{1.12}$$

A simple electrical circuit containing an ideal diode, a current source and an inductor set up in parallel, satisfies the following dynamical system [34, Example 8]:

$$\begin{cases} \dot{x} = v \\ i = -x - u \\ 0 \leqslant -i \perp v \geqslant 0 \end{cases} \tag{1.13}$$

where we can see that the complementarity relation (1.3) appears in (1.13). The variables in system (1.13) are defined by: $x(t)$ is the inductor current, $v(t)$ is the voltage across the diode, $i(t)$ is the current across the diode, $u(t)$ is the current variable of the current source.

1.3 Some Mathematical Models of Nonsmooth Systems

In this section, we recall some mathematical models of different classes of nonsmooth set-valued dynamical systems, that constitute an active area of research over the past several years and motivated in particular by engineering applications. The Subsection 1.3.1 is devoted to present the model of Moreau's sweeping processes. The complementarity system is presented in Subsection 1.3.2 and the model of projected dynamical system appears in Subsection 1.3.3.

1.3.1 Moreau's Sweeping Processes

The sweeping, or Moreau, process was introduced and extensively studied by Jean Jacques Moreau in the seventies, in [116, 117, 118, 119], to model an elastoplastic mechanical system. Today, it remains an object of mathematical research.

Significant applications of sweeping processes have been given, specifically in electrical circuits [1, 3, 7], crowd motion modeling [113, 112], hysteresis in elastoplastic models [99], mathematical economics [62, 75], dynamic networks, nonsmooth mechanics [100] and many other. The sweeping process theory has become an important area of nonlinear and variational analysis with various mathematical achievements and some challenging open questions; see, e.g., [61, 100] and the references therein.

Sweeping processes are considered as an evolution variational inequality, or a differential variational inequality. The main concept of these processes is to describe the movement of a point belonging to a moving set, and since the set is moving with time, the point is swept by it. In general, the time-dependent moving set is given.

Mathematically, the most simple formulation of the sweeping process is the following. We consider a set-valued mapping $\mathcal{S}(t) : \mathbb{R}_+ \to \mathbb{R}^n$, such that $\mathcal{S}(t) \subseteq \mathbb{R}^n$ is a nonempty closed and convex set parametrized by the time variable $t \geqslant 0$. We also assume that the moving set $\mathcal{S}(t)$ is Lipschitz continuous with respect to the Hausdorff metric. The sweeping process, [116, 117, 118, 119], corresponds to finding a function $x : \mathbb{R}_+ \to \mathbb{R}^n$ which take the form of the following set of inclusions:

$$x(t) \in \mathcal{S}(t), \ \forall t \geqslant 0, \tag{1.14a}$$

$$\dot{x} - f(t,x) \in -\mathcal{N}_{\mathcal{S}(t)}(x) \tag{1.14b}$$

for a given time-varying vector field $f : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$. Under the assumption on the set $\mathcal{S}(t)$, the normal cone defines a maximal monotone mapping[2] for each fixed t. Because of the presence of the normal cone, for any solution $x(t)$, we see that $x(t)$ is constrained to stay in $\mathcal{S}(t)$. This means in particular that (1.14) appears as a constrained differential inclusion.

A solution of (1.14) corresponds to finding a function x, and a selection $\eta : \mathbb{R}_+ \to \mathbb{R}^n$ such that $\eta(t) \in -\mathcal{N}_{\mathcal{S}(t)}(x(t))$ and $\dot{x}(t) - f(t, x(t)) = \eta(t)$ holds for Lebesgue almost every $t \geqslant 0$. The interpretation (1.14) arises for the way how the point x(t) is swept : as long as the point $x(t)$ happens to be in the interior of $\mathcal{S}(t)$, the normal cone $\mathcal{N}_{\mathcal{S}(t)}(x(t))$ is reduced to zero, so $x(t)$ does not move. When the point $x(t)$ is at the boundary of $\mathcal{S}(t)$, and $\dot{x}(t)$ points outside $\mathcal{S}(t)$, we choose $\eta(t^+)$ that points strictly inside the set $\mathcal{S}(t)$ and rectify the vector field in such a manner that $x(.)$ satisfies the constraint $x(t) \in \mathcal{S}(t)$.

For the sweeping process with nonconvex sets $\mathcal{S}(t)$, we refer the reader to [17, 27, 48] and the references therein. And several extensions of the sweeping process as well-posedness and optimal control have been studied in the literature, e.g., [7, 6, 35].

[2] The definition of a maximal monotone mapping appears in Definition 23, Chapter 2.

1.3.2 Complementarity System

In many applications, one encounters systems that consists of a combination of differential equations and inequalities. The inequalities play an important role at the level of modeling in problems arising in mathematical programming and economics. Complementarity systems, which consist of ordinary differential equations coupled to complementarity conditions, have been used for a long time in the context of specific applications such as electrical networks with ideal diodes (see e.g.[106]), mechanical objects subject to unilateral constraints [108] or Coulomb friction, control systems with saturation or deadzones, piecewise linear and variable structure systems, relay systems and hydraulic processes with one-way valves. Complementarity systems have the potential to play a major role in developing systematic methods to overcome analysis and synthesis problems in a wide range of applications.

The idea of coupling complementarity conditions to a general input/output dynamical system has first been proposed in [148]. The theory of complementarity problems has witnessed an impressive development essentially motivated by optimization problems. Recently, it has been the object of in depth studies in the control literature. The combination of inequalities and differential equations causes the system description to be of hybrid nature as it contains both continuous and discrete dynamics. As a consequence, complementarity systems form a subclass of hybrid dynamical systems [130, 131].

A subclass of particular complementarity system is the resulting linear complementarity system [82] which is described by relations of the form

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{1.15a}$$
$$y(t) = Cx(t) + Du(t) \tag{1.15b}$$
$$0 \leqslant y(t) \perp u(t) \geqslant 0, \tag{1.15c}$$

where A, B, C, and D are linear mappings. The linear complementarity systems were introduced in [148] and studied in [81, 82, 40].

As mentioned in the beginning of Section 1.2, the complementarity relation in (1.15) can be also written in terms of an inclusion. In particular, when $D = 0$ and the matrices B and C satisfy certain conditions, we can write the system (1.15) in the form of the differential inclusion with a normal cone operator (1.2).

A nonlinear complementarity system is described as following

$$\dot{x}(t) = f(x(t), u(t)) \tag{1.16a}$$
$$y(t) = h(x(t), u(t)) \tag{1.16b}$$
$$0 \leqslant y(t) \perp u(t) \geqslant 0. \tag{1.16c}$$

In some applications it is natural to allow an external input (forcing term) in a complementarity system. The previous system is then replaced by equations of the form

$$\dot{x}(t) = f(x(t), u(t), v(t)) \tag{1.17a}$$

$$y(t) = h(x(t), u(t), v(t)) \tag{1.17b}$$

$$0 \leqslant y(t) \perp u(t) \geqslant 0, \tag{1.17c}$$

where $v(t)$ denotes the forcing term.

1.3.3 Projected Dynamical System

The class of projected dynamical systems in which the right-hand side of the ordinary differential equation is a projection operator, was introduced in [89, 90] and it has been studied in [70, 122, 13, 58, 94]. These systems are used for studying the behaviour of urban transportation networks, traffic networks, international trade, and agricultural and energy markets. Their stationary points can be identified by variational inequalities; hence one may say that projected dynamical systems present a dynamic extension of variational inequalities. One can also write projected dynamical systems as complementarity systems [83] since variational inequalities and complementarity problems are related. The general theory of stability analysis of such dynamical systems was developed in [154].

In this subsection, we recall the definition of projected dynamical systems [70, 122]. The projected dynamical systems model the trajectories confined to a given set. To present this system class, let us consider $\mathcal{S} \subset \mathbb{R}^n$ a nonempty closed and convex set, and let f be a vector field whose domain contains $\mathcal{S}$. The projected dynamics are roughly described by the equation $\dot{x}(t) = f(x(t))$ in the interior of $\mathcal{S}$, and by a suitable modification of $f(\cdot)$ on the boundary of the set $\mathcal{S}$, which involves taking the projection on the tangent space so that the solution is confined to the constraint set $\mathcal{S}$.

More formally, for a given vector $v \in \mathbb{R}^n$, let $\Pi_{\mathcal{S}}(x; v)$ denotes the directional derivative of $\mathcal{P}_{\mathcal{S}}(x)$ which is defined as

$$\Pi_{\mathcal{S}}(x; v) = \lim_{\delta \to 0} \frac{\mathcal{P}_{\mathcal{S}}(x + \delta v) - \mathcal{P}_{\mathcal{S}}(x)}{\delta} \tag{1.18}$$

where $\mathcal{P}_{\mathcal{S}}$ is the projection operator onto $\mathcal{S}$ that assigns to each vector $x \in \mathbb{R}^n$ the vector in $\mathcal{S}$ i.e. $\mathcal{P}_{\mathcal{S}}(x) := \arg\min_{z \in \mathcal{S}} \|x - z\|$, where $\|.\|$ is the Euclidean norm.

The projected dynamical system corresponding to the closed convex set $\mathcal{S}$ and the vector field f on $\mathcal{S}$ is defined by

$$\dot{x} = \Pi_{\mathcal{S}}(x; f(x)) \tag{1.19}$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$ is continuously differentiable. For a convex set $\mathcal{S}$, it is possible to rewrite the directional derivative $\Pi_{\mathcal{S}}(x; v)$ in terms of the tangent cone operator,

$$\Pi_{\mathcal{S}}(x; v) = \mathcal{P}_{\mathcal{T}_{\mathcal{S}}(x)}(v),$$

that is, $\Pi_{\mathcal{S}}(x; v)$ is the projection of vector v on to the tangent space $\mathcal{T}_{\mathcal{S}}(x)$. This last inequality, in particular, allows us to show that [35, Section 2.5] the solutions of (1.19) coincide with the slow solutions of

$$\dot{x} \in f(x) - \mathcal{N}_{\mathcal{S}}(x),$$

that is, the solutions which correspond to the element of least norm from the set on the right-hand side. This way, one sees the connection between the system class (1.1) and projected dynamical systems.

1.4 Contribution and Organization

The primary contribution of this thesis lies in studying some analysis related problems with numerical tractability for a class of constrained systems. These systems are broadly modeled by the differential inclusions (1.1), and as we saw earlier, they can be linked to different classes of nonsmooth dynamical systems which are described by an interconnection of the ordinary differential equation with a static relation. As we saw in the examples discussed in previous sections, the static relations have some particular structure, such as the subdifferential of a convex function, which provide some nice properties that are broadly associated with maximal monotone operators. We exploit those properties for the analysis, and also use some tools from the literature for numerical certificates for our results.

1.4.1 An Overview of Problem Statements

Let us now provide a quick summary of the problems that have been studied in this manuscript, along with brief comments about the originality of our work.

- *Converse result:* We first study the problem of analyzing the stability of an equilibrium point for the constrained system (1.1). Note that for system (1.1), the state trajectories evolve only on a given set and the vector field gets discontinuous on the boundaries. The resulting motion corresponds to a particular choice of vector field from the admissible set, and our goal is to establish appropriate stability conditions for such a setup. In particular, we address the question of existence of Lyapunov functions for stability of an equilibrium point within an appropriate function class. As the first main result, we provide an affirmative answer to this question by proving the existence of cone-copositive Lyapunov functions when the underlying constraint set is a convex cone. These results have been published in [144].

- *Computing Lyapunov functions:* After proving the existence of Lyapunov functions, we explore numerical methods for computing such functions. For the tools we use in our works, we have to add more structure to our system class by taking the drift term to be homogeneous. This motivates the introduction of homogeneous cone-copositive Lyapunov functions, and for computing such functions, we borrow tools from optimization and polynomial approximations. In particular, we work with two classes of algorithms. The first one is based on taking simplices in a cone, and discretizing the simplices to construct a set of inequalities whose solution corresponds to the coefficients of the polynomial Lyapunov function. The other method is based on representing the positivity constraint on Lyapunov functions as the *sum-of-squares*. Computing Lyapunov functions using such a method leads to semidefinite programs, for which we have rather efficient solvers. These developments have been published in [142, 144].

- *Ensemble approximations:* Continuing with our approach of studying analytical problems with computational methods, we next study the evolution of a probability measure for the aforementioned class of constrained systems. For conventional ODEs, this problem is solved by looking at the solution of a linear partial differential equation, the so-called *Liouville equation*. In our approach, we approximate the solution of nonsmooth system by a sequence of ODEs, and for each ODE, we consider a corresponding Liouville equation. For this single parameter family of PDEs, we develop some convergence results and present some numerical methods based on semidefinite programming for approximating the moments and support of the solution. These results are based on a manuscript under review [143].

1.4.2 Organization of the Thesis

The remainder of this thesis is organized as follows:

- In **Chapter 2**, we present an overview of some essential mathematical background required for the developments in later chapters. Some details about the system class (1.1) are included, along with some discussions about how the solutions of such systems evolve with time. We also provide an overview of the existing approaches for stability analysis, and how our approach is different from the existing works.

- In **Chapter 3**, we address the stability notions of our interest more formally, and the definition of Lyapunov functions with constrained domains for system trajectories. We then provide our first main result which states that if the origin is globally exponentially stable for system (1.1) with $\mathcal{S}$ being a convex cone, then there exists a continuously differentiable cone-copositive Lyapunov function. After that, we show the existence of homogeneous Lyapunov function for the case when the vector field is homogeneous, which is useful for the numerical computation.

- In **Chapter 4**, we prove the existence of a cone-copositive Lyapunov function which can be expressed as a rational function of homogeneous polynomials by using appropriate density results related to approximation of functions. Then we propose computationally tractable algorithms for finding the Lyapunov functions. We adopt two approaches: the first one is a *discretization method* which is based on finding an inner approximation of the cone of cone-copositive polynomials by using simplicial partitions, and the second approach is based on finding the *sum-of-squares* (SOS) representation of the unknown function. Then, we derive the corresponding algorithms for those two techniques with some generalizations. As an illustration, we study some academic examples which are solved by using Matlab toolboxes.

- In **Chapter 5**, we study the time evolution of a probability measure which describes the distribution of the state over a set. As opposed to smooth ordinary differential equations, one cannot directly derive a continuity equation to describe the evolution of the distribution of states. Instead, we consider Lipschitz approximation of our system (1.1) and construct a sequence of measures obtained from Liouville equations corresponding to these approximations. This sequence of measures is shown to converge in weak-star topology to the measure describing the evolution of the distribution of states for the nonsmooth system. This allows us to approximate numerically the evolution of moments for the nonsmooth system. An algorithm is also provided for approximating the support of the measure that describes the solution. We illustrate our approach by an academic example.

- In **Chapter 6**, we present some conclusions, along with some discussion about the possible future paths of research. We explain some possible generalizations that can be carried out for the problems studied in this manuscript, which includes deriving converse theorems for a broader class of nonsmooth systems and developing algorithms for computing Lyapunov functions with different descriptions for the constraint set. For the problems involving the evolution of a probability measure, there is room to improve results by relaxing certain restrictive hypotheses. In addition to these generalizations, one can study aforementioned problems for a class of nonsmooth systems where the underlying constraints are not necessarily convex. Potentially, such a setup would require different tools and it is expected that the results of this manuscript provide some direction into investigating such questions.

2

Mathematical Background

This chapter collects mathematical background for the subsequent developments in the manuscript by recalling some concepts from convex analysis, representation of positive polynomials, convex optimization, and some essentials about the solutions of nonsmooth dynamical systems. In Section 2.1, we provide an overview of some basic necessary concepts. In Section 2.2, we provide some details about the dynamical systems with complementarity relations, followed by some discussions in Section 2.2.2 on how to interpret or simulate the solutions of such systems.

2.1 Essential Concepts

2.1.1 Convex Analysis

Convex analysis is a special branch of mathematics combining classical analysis on the one side and geometry on the other. Convex analysis is widely acknowledged to have an important but specialized role in mathematical optimization. It has a significant role in the study of nonlinear problems in the calculus of variations and optimal control. Several books treat convex analysis in depth, e.g. [91].

Definition 1. (Convex set). A subset C of $\mathbb{R}^n$ is called convex if for each $x, y \in C$ and for each $\lambda \in [0,1]$, we have

$$\lambda x + (1-\lambda) y \in C, \tag{2.1}$$

i.e. the closed line segment $[x, y] \subset C$ whenever $x, y \in C$.

Definition 2. (Simplex). Suppose that $x_0, x_1, \dots, x_n \in \mathbb{R}^{n+1}$ are affinely independent. An n-simplex is an n-dimensional polytope which is the convex hull of its $n+1$ vertices $\{x_0, x_1, \dots, x_n\}$, namely

$$\Delta := \left\{ \theta_0 x_0 + \cdots + \theta_n x_n \,\middle|\, \sum_{i=0}^{n} \theta_i = 1 \text{ and } \theta_i \geqslant 0 \text{ for all } i \in \{0, \dots, n\} \right\}.$$

Example 3. Some examples of convex sets, and simplices are given below.

- In $\mathbb{R}$, convex sets are the intervals.
- In $\mathbb{R}^n$, an affine manifold is a convex set.
- In $\mathbb{R}^2$, the line connecting the points $[0\ 1]^\top$ and $[1\ 0]^\top$ is a simplex.
- The unit simplex in $\mathbb{R}^n$ is a convex set.

Definition 3. (Convex combination). A convex combination of the points $(x_i)_{1 \leqslant i \leqslant k} \subset \mathbb{R}^n$ is defined by

$$x = \sum_{i=1}^{k} \lambda_i x_i, \text{ with } \lambda_i \geqslant 0, \forall i = 1, 2, \ldots, k \text{ and } \sum_{i=1}^{k} \lambda_i = 1,$$

which means that x is a linear combination of $x_1, \ldots, x_k$, for scalars $\lambda_i \geqslant 0$, $i = 1, 2, \ldots, k$.

The idea of a convex combination can be generalized to include infinite sums, integrals, and, in the most general form, probability distributions.

A particular class of convex subsets in $\mathbb{R}^n$, which have a principle role in optimization and linear programming, are convex polyhedra which are defined as following.

Definition 4. (Convex polyhedron). A subset P of $\mathbb{R}^n$ is called a convex polyhedron if there is a matrix $A \in \mathbb{R}^{m \times n}$ and a vector $b \in \mathbb{R}^m$ such that

$$P := \{x \in \mathbb{R}^n : Ax \leqslant b\}, \tag{2.2}$$

that is

$$P := \{x \in \mathbb{R}^n : \langle a_i, x \rangle \leqslant b_i, \ \ i = 1, 2, \ldots, m\}.$$

Definition 5. (Projection onto a convex set). Let C be a nonempty closed convex set in $\mathbb{R}^n$ and $x \in \mathbb{R}^n$. The projection of x onto C is defined by

$$\text{proj}_C(x) := \arg\min_{y \in C} \|x - y\|.$$

Let us now recall some notions relative to convex functions. Then for the next definitions, let us introduce a convex function $\varphi : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$, that is, $\forall x, y \in \mathbb{R}^n$, $\forall \lambda \in [0, 1]$, we have

$$\varphi(\lambda x + (1 - \lambda)y) \leqslant \lambda\varphi(x) + (1 - \lambda)\varphi(y).$$

The effective domain of φ is defined by

$$\text{dom}(\varphi) := \{x \in \mathbb{R}^n : \varphi(x) < +\infty\}. \tag{2.3}$$

Definition 6. (Proper function). The function $\varphi : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is said to be proper if $\varphi(x) < +\infty$ for at least one $x \in \mathbb{R}^n$ and $\varphi(x) > -\infty$ for every $x \in \mathbb{R}^n$. It means that, a convex function is proper if its effective domain is nonempty and it never attains $-\infty$.

Convex functions that are not proper are called improper convex functions.

Definition 7. (Lower semicontinuous function). The function φ is said to be lower semicontinuous at $x_0 \in \mathbb{R}^n$ if for every $\epsilon > 0$ there exists a neighborhood U of x_0 such that $\varphi(x) \geqslant \varphi(x_0) - \epsilon$ for all x in U when $\varphi(x_0) < +\infty$, and $\varphi(x)$ tends to $+\infty$ as x tends towards x_0 when $\varphi(x_0) = +\infty$. Equivalently, this can be expressed as

$$\liminf_{x \to x_0} \varphi(x) \geqslant \varphi(x_0)$$

where $\liminf$ is the limit inferior of φ at point x_0.

The function φ is called lower semicontinuous if it is lower semicontinuous at every point of its domain.

Suppose that $\varphi : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is a proper, convex and lower semicontinuous function. The subdifferential of φ at $x \in \mathbb{R}^n$, denoted by $\partial\varphi(x)$ is defined as

$$\partial\varphi(x) := \{\lambda \in \mathbb{R}^n : \langle \lambda, z - x \rangle \leqslant \varphi(z) - \varphi(x),\ \forall z \in \mathrm{dom}(\varphi)\}. \tag{2.4}$$

We say that φ is subdifferentiable at $x \in \mathbb{R}^n$ if $\partial\varphi(x) \neq \emptyset$.

Definition 8. (Cone). A nonempty subset K of $\mathbb{R}^n$ is called a cone if for each $x \in K$ and each $\lambda \geqslant 0$, we have

$$\lambda x \in K.$$

A cone K is called a convex cone if for every $\alpha, \beta \geqslant 0$, and for all $x, y \in K$, we have

$$\alpha x + \beta y \in K.$$

The conic hull of a subset $\mathcal{S} \subset \mathbb{R}^n$, denoted $\mathrm{cone}(\mathcal{S})$, is the smallest convex cone that contains $\mathcal{S}$.

Definition 9. (Dual cone). The dual cone $K^\star$ of a nonempty subset K of $\mathbb{R}^n$ is defined by

$$K^\star := \{p \in \mathbb{R}^n : \langle p, v \rangle \geqslant 0,\ \forall v \in K\}. \tag{2.5}$$

Geometrically, the dual cone of K is the set of all nonnegative continuous linear functionals on K. Notice that the dual $K^\star$ is always a closed convex cone containing the origin. A cone K that satisfies $K^\star = K$ is called self-dual.

Remark 2.1. If K is a linear subspace of $\mathbb{R}^n$, then $K^\star$ coincides with the orthogonal subspace of K i.e.

$$K^\star = K^\perp,$$

with

$$K^\perp := \{p \in \mathbb{R}^n : \langle p, v \rangle = 0, \ \forall v \in K\}.$$

We note that K is a closed convex cone of $\mathbb{R}^n$ if and only if $K^{\star\star} = K$.

Definition 10. (Polar and normal cone). The polar cone K° of a nonempty subset K of $\mathbb{R}^n$ is defined by

$$K^\circ := \{p \in \mathbb{R}^n : \langle p, v \rangle \leqslant 0, \ \forall v \in K\} = -K^\star. \tag{2.6}$$

The normal cone to a nonempty subset K of $\mathbb{R}^n$ at a point $x \in K$ is defined by

$$\mathcal{N}_K(x) := \{\lambda \in \mathbb{R}^n : \langle \lambda, y - x \rangle \leqslant 0, \ \forall y \in K\}. \tag{2.7}$$

If x belongs to the interior of K i.e. $x \in \text{int}(K)$ then $\mathcal{N}_K(x) = 0$ and by convention, we let $\mathcal{N}_K(x) := \emptyset$ for all $x \notin K$.

Let $\psi_K(x)$ be the indicator function of a nonempty subset $K \subset \mathbb{R}^n$ as defined in (1.6). The indicator function is discontinuous on the boundary of K, but lower semicontinuous everywhere. Its subdifferential is related to the normal cone operator.

Proposition 2.2. *The subdifferential of ψ_K at x is the normal cone to K at $x \in K$ i.e. $\partial\psi_K(x) = \mathcal{N}_K(x)$.*

2.1.2 Positive Polynomials

The study of relationships between positive (nonnegative) polynomials and sum-of-squares polynomials is a classic question which goes back to work of Hilbert at the end of the nineteenth century. It is of real practical importance in view of numerous potential applications.

Sum-of-squares optimization is an active area of research at the interface of real algebraic geometry and convex optimization. Over the last decade, it has made a crucial influence on both discrete and continuous optimization, as well as several other disciplines, notably control theory. A specially exciting aspect of this research area is that it relies on classical results from real algebraic geometry. Additionally, it offers a modern, algorithmic view point on these concepts, which is amenable to computation and semidefinite programming.

We present in this section a brief exposition on basic definitions and results concerning positive polynomials and sum-of-squares polynomials.

We denote by $\mathbb{R}[x]$ the vector space of real polynomials in the variables $x = (x_1, ..., x_n) \in \mathbb{R}^n$.

Definition 11. (Sum-of-squares polynomial). A multivariate polynomial $p(x) = p(x_1, \ldots, x_n)$ is said to be a sum-of-squares, abbreviated as SOS, if it can be written in the form

$$p(x) = \sum_{k=1}^{m} q_k^2(x), \tag{2.8}$$

for some polynomials $q_k \in \mathbb{R}[x]$, $k = 1, \ldots, m$.

We denote by

$$\Sigma\mathbb{R}[x]^2 := \{p \in \mathbb{R}[x] : p \text{ is SOS}\},$$

the cone of elements in $\mathbb{R}[x]$ that can be written as SOS polynomials.

The existence of an SOS decomposition is an algebraic certificate for nonnegativity of a polynomial. It is obvious that every SOS polynomial is nonnegative on $\mathbb{R}^n$. But the converse is not always true, that is, a nonnegative polynomial is not necessarily SOS.

Theorem 2.3. *A multivariate polynomial $p(x) = p(x_1, \ldots, x_n)$ of degree $2d$ is SOS if and only if there exists a positive semidefinite matrix Q such that*

$$p(x) = z^T Q z, \tag{2.9}$$

where z is the vector of monomials of degree up to d

$$z = [1, x_1, x_2, \ldots, x_n, x_1x_2, \ldots, x_n^d].$$

Definition 12. A function $f : \mathbb{R}^n \to \mathbb{R}^n$ is said to be homogeneous of degree $d \geqslant 1$ if it satisfies

$$f(\lambda x) = \lambda^d f(x)$$

for each $x \in \mathbb{R}^n$ and $\lambda \geqslant 0$.

We say that a polynomial $p \in \mathbb{R}[x]$ of degree d is homogeneous if

$$p(\lambda x) = \lambda^d p(x)$$

for any scalar $\lambda \in \mathbb{R}$.

Dealing with positivity of a polynomial is hard but with SOS, it becomes easier as the problem boils down to semidefinite programming (SDP) or linear matrix inequalities (LMIs), a particular class of convex optimization problems for which efficient algorithms are available. For detailed accounts on SOS and positive polynomials and the algebraic concepts, we refer to [105, 102].

Now, we review some SOS representation results for positive polynomials. Before that, we need some definitions.

An algebraic set is an intersection of finitely many polynomial level sets. A semialgebraic set is a union of finitely many intersections of finitely many open polynomial superlevel sets. A closed basic semialgebraic set is an intersection of finitely many closed polynomial superlevel sets which is denoted by

$$K = \{x \in \mathbb{R}^n;\ g_k(x) \geqslant 0,\ k = 1, \dots, m\} \tag{2.10}$$

where $g_k(x) \in \mathbb{R}[x]$, $k = 1, \dots, m$.

We denote by $M(K)$ the quadratic module generated by g_k, $k = 1, \dots, m$, that is to say

$$M(K) = \Sigma\mathbb{R}[x]^2 + g_1\Sigma\mathbb{R}[x]^2 + \cdots + g_m\Sigma\mathbb{R}[x]^2$$

and $\Sigma^2\langle g_1, \dots, g_m\rangle$ the multiplicative convex cone generated by all possible products of the g_k

$$\Sigma^2\langle g_1, \dots, g_m\rangle = M(K) + g_1 g_2\Sigma\mathbb{R}[x]^2 + \cdots + g_1 g_2 \dots g_m\Sigma\mathbb{R}[x]^2. \tag{2.11}$$

We say that the quadratic module $M(K)$ is Archimedean if

$$N - \sum_{i=1}^{n} x_i^2 \in M(K), \text{ for some } N \in \mathbb{N}.$$

Pólya, Schmüdgen and Putinar theorems: In the literature, we find several important results which characterize the positivity of a polynomial in different contexts, and here we recall some of these statements which are relevant for our work. The first such statement, *Pólya's Positivstellensatz*, provides the conditions for positivity of the polynomials on positive orthants.

Theorem 2.4. *(Pólya [105], [133]). Let $p \in \mathbb{R}[x]$ be homogeneous such that $p > 0$ on $\mathbb{R}^n_+ \setminus \{0\}$. Then for all $k \in \mathbb{N}$ big enough, the polynomial $(x_1 + \cdots + x_n)^k p$ has only nonnegative coefficients.*

The next statement, due to *Schmüdgen*, provides a characterization of positive polynomials on a compact semialgebraic set K with no additional assumptions on K or on its description.

Theorem 2.5. *(Schmüdgen [141]). Let K be a compact semialgebraic set and let $p \in \mathbb{R}[x]$ such that $p(x) > 0$ for all $x \in K$. Then $p \in \Sigma^2\langle g_1, \dots, g_m\rangle$.*

Theorem 2.5 is a very powerful result. From computational viewpoint, when we wish to check if a polynomial is positive over the set K, we seek a representation of the polynomial in the form (2.11). However, note that the number of terms in Schmüdgen's representation is exponential in the number of polynomials that define the set K. In the following statement, the so-called *Putinar's Positivstellensatz*, we can see that by adding an assumption on $M(K)$, the number of terms in Putinar's representation is linear in the number of polynomials that define K, which becomes very useful for computation.

Theorem 2.6. *(Putinar's Positivstellensatz [136]). Let K be a compact semialgebraic set and let $M(K)$ be an Archimedean quadratic module. Let $p \in \mathbb{R}[x]$ such that $p(x) > 0$ for all $x \in K$. Then $p \in M(K)$.*

In some of the results developed in this manuscript, we will use some of these statements which basically describe the representation we seek for checking the positivity of a function.

2.1.3 Matrix Classes

In our work, we use several matrix classes, which are formally defined in this section.

Definition 13. (Positive (semi) definite matrix). A matrix $M \in \mathbb{R}^{n\times n}$ is positive (semi) definite if for all $x \in \mathbb{R}^n$ one has $x^T M x > 0$ ($\geqslant 0$) for all $x \neq 0$. It is denoted $M \succ 0$ ($\succcurlyeq 0$). It is not necessarily symmetric.

For the following definition and theorem, we recall that the submatrix of a matrix M is a matrix obtained by deleting some of the rows and/or columns of M. The principal submatrix is a submatrix in which the set of row indices that remain is the same as the set of column indices that remain. The minor of a matrix M is the determinant of some smaller square matrix, cut down from M by removing one or more of its rows and columns. The principal minor is one where the indices of the deleted rows are the same as the indices of the deleted columns.

Definition 14. (P-matrix). A matrix $M \in \mathbb{R}^{n\times n}$ is a P-matrix if all its principal minors are positive. It is a P_0-matrix if its principal minors are nonnegative.

Theorem 2.7. *[64] Let $M \in \mathbb{R}^{n\times n}$. The following statements are equivalent:*

1. *M is a P-matrix.*

2. *M reverses the sign of any nonzero vector, i.e.:*

$$[z_i(Mz)_i \leqslant 0 \ \textit{for all} \ i] \Rightarrow [z = 0].$$

3. *All real eigenvalues of M and its principal submatrices are positive.*

Definition 15. (Copositive matrix). A matrix $M \in \mathbb{R}^{n\times n}$ is said to be a copositive matrix on the set K if

$$x^T Mx \geqslant 0, \text{ for all } x \in K.$$

It is said to be strictly copositive on K if $x^T Mx > 0$ for all $x \in K$, $x \neq 0$, that is, if there exists $c > 0$ such that

$$x^T Mx > c\|x\|^2, \text{ for all } x \in K.$$

A basic problem in optimization is the detection of copositive matrices. The use of copositive matrices has broad applications in many areas of applied mathematics. In the last decade, there has been an interest in copositivity due to its impact in optimization modeling [38], dynamical systems and control theory [95], complementarity problems [74], and graph theory [12, 69]. There are many other interesting references concerning the role of copositivity in the modeling and analysis of optimization problems.

We have $M \succ 0 \Rightarrow M$ is a P-matrix, $M \succcurlyeq 0 \Rightarrow M$ is a P_0-matrix and a copositive matrix on $\mathbb{R}^n_+$.

One usually considers copositivity over convex sets [92]. Yet even in this case copositivity is hard to characterize. Many more matrix classes which are useful in complementarity theory exist [63].

2.1.4 Conic Optimization

In this section, we describe linear programming over convex cones in finite dimensional spaces, following [84].

Definition 16. (Linear cone). The linear cone, or positive orthant, is the set

$$\{x \in \mathbb{R}^n : x_k \geqslant 0, \ \ k = 1, \ldots, n\}.$$

Definition 17. (Quadratic cone). The quadratic cone, or Lorentz cone, or second order cone, is the set

$$\{x \in \mathbb{R}^n : x_1 \geqslant \sqrt{x_2^2 + \cdots + x_n^2}\}.$$

Let $\mathbb{S}^n$ denote the Euclidean space of $n \times n$ symmetric matrices of $\mathbb{R}^n \times \mathbb{R}^n$, with the inner product

$$\langle X, Y \rangle := \text{trace}\, XY = \sum_{i=1}^{n} \sum_{j=1}^{n} x_{ij} y_{ij},$$

where X, Y are two matrices with respective entries x_{ij}, y_{ij}, $i, j = 1, \ldots, n$.

Definition 18. (Semidefinite cone). The semidefinite cone is the set

$$\{X \in \mathbb{S}^n : x^T X x \geqslant 0, \ \forall x \in \mathbb{R}^n\}.$$

Note that if $K = \mathbb{R}^n$ is interpreted as a cone, then its dual $K' = \{0\}$ is the zero cone, which reduces to the zero vector of $\mathbb{R}^n$.

- *Linear Matrix Inequality:*

A linear matrix inequality, abbreviated by LMI, is a constraint

$$F_0 + \sum_{k=1}^{n} x_k F_k \geqslant 0,$$

on a vector $x \in \mathbb{R}^n$, where matrices $F_k \in \mathbb{S}^m$, $k = 0, 1, \ldots, n$ are given.

- *Primal/Dual Conic Problems:*

Conic programming is linear programming in a convex cone K : it is the problem of minimizing a linear function over the intersection of K with an affine subspace:

$$\begin{aligned} p^\star = \inf \ & c'x \\ \text{s.t.} \ & Ax = b \\ & x \in K \end{aligned} \tag{2.12}$$

where the infimum is with respect to a vector $x \in \mathbb{R}^n$ to be found, and the given problem data consists of a matrix $A \in \mathbb{R}^{m \times n}$, a vector $b \in \mathbb{R}^m$ and a vector $c \in \mathbb{R}^n$.

The feasibility set $\{x \in \mathbb{R}^n : Ax = b, \ \ x \in K\}$ is not necessarily closed, this is why in general we speak of an infimum, not a minimum.
If $K = \mathbb{R}^n$, then problem (2.12) is not interesting since either $p^\star = 0$ or $p^\star = +\infty$ or $p^\star = -\infty$. If K is the linear cone, then solving problem (2.12) is called linear programming (LP). If K is the quadratic cone, then this is called (convex) quadratic programming (QP). If K is the semidefinite cone,

then this is called (linear) semidefinite programming (SDP).

In mathematical programming, problem (2.12) is called the primal problem, and $p^\star$ denotes its infimum. The primal conic problem has a dual conic problem [28], which is:

$$\begin{aligned} d^\star \;\; &= \sup \;\; b'y \\ &\quad \text{s.t.} \;\; z = c - A'y \\ &\qquad\quad\; z \in K'. \end{aligned} \tag{2.13}$$

2.2 Solutions of Nonsmooth Systems

Our interest in this thesis is in studying a class of dynamical systems described by the variational inequalities

$$\dot{x}(t) \in f(x(t)) - \partial\varphi(x(t)), \;\; a.e. \;\; t \geqslant 0, \tag{2.14}$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$ is a given vector field, $x(t) \in \mathbb{R}^n$ denotes the state, and $\varphi : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is a given proper, convex and lower semicontinuous function.

More specifically, we focus on the particular case

$$\varphi = \psi_{\mathcal{S}},$$

where $\mathcal{S}$ is a given closed convex subset of $\mathbb{R}^n$ containing the origin and $\psi_{\mathcal{S}}$ is the indicator function of $\mathcal{S}$. Then, the differential inclusion (2.14) reads as

$$\dot{x}(t) \in f(x(t)) - \mathcal{N}_{\mathcal{S}}(x(t)), \;\; \text{a.e.} \;\; t \geqslant 0. \tag{2.15}$$

Inclusion (2.15) captures the class of complementarity systems studied in this thesis, but the framework of (2.14) is necessary for a broader class of complementarity systems such as the ones studied in [42, 146].

The formalism of system (2.15) with inclusion naturally allows us to describe dynamics constrained to evolve in set $\mathcal{S}$. Using the depiction in Figure 2.1, it is seen that, during the evolution of a trajectory, if $x(t)$ is in interior of $\mathcal{S}$, then $\mathcal{N}_{\mathcal{S}}(x(t)) = 0$ and the motion of the trajectory continues according to the differential equation $\dot{x}(t) = f(x(t))$. While $x(t)$ is on the boundary, we add a vector from the set $-\mathcal{N}_{\mathcal{S}}(x(t))$, which restricts the motion of the state trajectory in tangential direction on the boundary of the constraint set $\mathcal{S}$.

We focus on the particular class of constrained systems where the admissible set $\mathcal{S}$ is a cone, denoted by K. Let us give a theorem that specifies some conditions for the existence and uniqueness of solutions for such systems.

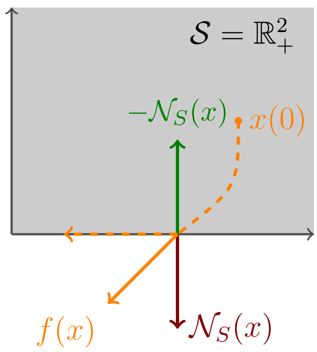

Figure 2.1 – State trajectories in constrained system with $S = \mathbb{R}^n_+$.

Theorem 2.8. *[41] Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a Lipschitz continuous vector field, and K a closed convex cone. Let $x_0 \in K$ be given. Then there exists a unique absolutely continuous function $x : [0, \infty) \to \mathbb{R}^n$ such that the differential inclusion (2.15) holds with $x(0) = x_0$ and*

$$x \text{ is right-differentiable on } [0, +\infty); \tag{2.16a}$$

$$x(t) \in K, \text{ for all } t \geqslant 0. \tag{2.16b}$$

2.2.1 Exploiting Complementarity Structure

In Section 1.3.2, we saw the connections between the dynamical system (2.15) and the complementarity systems, where one can see the later as a particular case of (2.15) with $\mathcal{S}$ being a closed convex cone. Having more structure on the set $\mathcal{S}$ allows us to get more insights into the system (2.15). In this section, we start with a description of the complementarity problem, which allows us to study the solutions of system (2.15) with $\mathcal{S}$ being a cone.

A complementarity problem refers to a system of equalities and inequalities, and due to its relevance in applications, this subject now has a rich mathematical theory, a variety of algorithms, and a wide range of applications in applied science and technology. They are widely used in many robotics tasks, like motion and manipulation, because of their ability to model nonsmooth behavior (e.g, contact dynamics). There is a great deal of practical interest in the development of robust and efficient algorithms for solving complementarity problems. A reference book for this topic is [63], see also [74].

Definition 19 (Complementarity Problem)**.** The complementarity problem consists in finding a vector in a finite-dimensional real vector space that satisfies a certain system of inequalities. Specifically, given $F : \mathbb{R}^n \to \mathbb{R}^n$, the complementarity problem, abbreviated CP, consists of finding a vector $\eta \in \mathbb{R}^n$ such that $\eta \geqslant 0$, $F(\eta) \geqslant 0$, $\eta^\top F(\eta) = 0$, written compactly as

$$0 \leqslant \eta \perp F(\eta) \geqslant 0. \tag{2.17}$$

This notation means that each component of η and $F(\eta)$ must be nonnegative, and both vectors must be perpendicular to each other, which translates to

$$\eta_i(F(\eta))_i = 0, \;\; \forall i \in \{1, \dots, n\}\,.$$

When $F(\eta) = M\eta + q$, for a matrix $M \in \mathbb{R}^{n\times n}$ and a vector $q \in \mathbb{R}^n$, i.e.

$$0 \leqslant \eta \perp M\eta + q \geqslant 0, \tag{2.18}$$

this is a linear complementarity problem denoted $\mathrm{LCP}(q, M)$, and the set of its solution is denoted $\mathrm{SOL}(q, M)$. Observe that if $q > 0$, the $\mathrm{LCP}(q, M)$ is always solvable with the zero vector being a trivial solution. Special instances of the linear complementarity problem can be found in the mathematical literature as early as 1940, but the problem received little attention until the mid 1960s at which time it became an object of study in its own right. Linear complementarity problems are broadly used in computational nonsmooth mechanics [33], and in applications including quadratic programming [124].

Similarly, for a given closed convex cone $K \subset \mathbb{R}^n$, the problem of finding $\eta \in K^\star$ such that

$$K^\star \ni \eta \perp F(\eta) \in K \tag{2.19}$$

is termed as a cone-complementarity problem, and for $F(\eta) = M\eta + q$, it is a linear cone-complementarity problem, denoted $\mathrm{LCCP}(q, M, K)$.

Based on discussions in [74, Chapter 2], the LCP (2.18) can be reformulated into three other frameworks: optimization problem, C-function, convex subdifferential.

- **Optimization problem:** First, problem (2.18) can be expressed as an optimization problem. It is easy to give the following proposition.

Proposition 2.9. *Let $q \in \mathbb{R}^n$ and $M \in \mathbb{R}^{n\times n}$. Vector $\eta \in \mathbb{R}^n$ is a solution of* $\mathrm{LCP}(q, M)$ *if and only if it is a solution to the following quadratic problem:*

$$\begin{aligned} &\min_{\eta\in\mathbb{R}^n} \; \eta^\top(M\eta + q) \\ &\textit{such that } M\eta + q \geqslant 0, \\ &\eta \geqslant 0, \end{aligned} \tag{2.20}$$

with an objective value of zero.

Similarly, for a given closed convex cone $K \subset \mathbb{R}^n$, one can also write the solution $\eta \in \mathbb{R}^n$ of LCCP(q, M, K) as the solution to the following optimization problem:

$$\begin{aligned} \min_{\eta \in \mathbb{R}^n} \; & \eta^\top (M\eta + q) \\ \text{such that } & M\eta + q \in K, \\ & \eta \in K^\star, \end{aligned} \tag{2.21}$$

with an objective value of zero.

• **C-function:** A second way of expressing the problem (2.18) is finding the root of a C-function.

Definition 20. A C-function is a function $f : \mathbb{R}^{2n} \to \mathbb{R}^n$ satisfying

$$f(a, b) = 0 \Leftrightarrow \langle a, b\rangle = 0, \;\; a, b \geqslant 0.$$

If f is a C-function, then LCP(q, M) is equivalent to finding $\eta \in \mathbb{R}^n$ such that $f(\eta, M\eta + q) = 0$.

One of well known C-functions is the min function which implies that $\eta \in \mathbb{R}^n$ is a solution of LCP(q, M) if and only if $\min(\eta, M\eta + q) = 0$.

• **Convex subdifferential:** A third way to express the problem (2.18) is through the subdifferential of the indicator function $\psi_{\mathbb{R}^n_+}$, which is defined as $\psi_{\mathbb{R}^n_+}(x) = 0$ if $x \in \mathbb{R}^n_+$ and $\psi_{\mathbb{R}^n_+}(x) = +\infty$ if $x \notin \mathbb{R}^n_+$.

Since the subdifferential of the indicator function $\partial\psi_{\mathbb{R}^n_+}(x)$ is equal to the normal cone $\mathcal{N}_{\mathbb{R}^n_+}(x)$, then we have the following equivalence [74, Proposition 1.1.3]:

$$\begin{aligned} 0 \leqslant \eta \perp \zeta \geqslant 0 &\Leftrightarrow \eta \in -\mathcal{N}_{\mathbb{R}^n_+}(\zeta) \\ &\Leftrightarrow \zeta \in -\mathcal{N}_{\mathbb{R}^n_+}(\eta). \end{aligned}$$

We give now the following result which is central in complementarity theory.

Theorem 2.10. *The* LCP(q, M) *has a unique solution for any* $q \in \mathbb{R}^n$ *if* $M \in \mathbb{R}^{n\times n}$ *is a P-matrix.*

A special subclass of P-matrices are the symmetric positive definite matrices.

For the analysis carried out in this thesis, it is important to know how the solution of an LCP, or LCCP in general, changes if we modify one of the parameters.

Proposition 2.11. *Given a closed convex cone $K \subset \mathbb{R}^n$ and a P-matrix M, let η denote the solution of* $\mathrm{LCCP}(q, M, K)$ *and η_α denote the solution of* $\mathrm{LCCP}(\alpha q, M, K)$*, for some $\alpha > 0$. Then, it holds that $\eta_\alpha = \alpha\eta$.*

Proof : Let $\eta \in \mathrm{LCCP}(q, M, K)$. Clearly, for each $\alpha > 0$,

$$\begin{aligned} \eta \in K^\star &\Leftrightarrow\ \alpha\eta \in K^\star \\ M\eta + q \in K &\Leftrightarrow\ \alpha(M\eta + q) = M(\alpha\eta) + (\alpha q) \in K \\ \eta^\top(M\eta + q) = 0 &\Leftrightarrow\ (\alpha\eta)^\top(M(\alpha\eta) + (\alpha q)) = 0. \end{aligned}$$

and hence $\alpha\eta \in \mathrm{LCCP}(\alpha q, M, K)$. Since the solution to such an LCCP are unique, it follows that $\eta_\alpha = \alpha\eta$. ♢

With these basic definitions, we introduce the following class of systems, the complementarity systems, for which we develop our main results of this thesis.

Definition 21 (Complementarity System)**.** A complementarity system consists of an ordinary differential equation coupled to complementarity conditions. Given a function $f : \mathbb{R}^n \to \mathbb{R}^n$ and a cone $K \subset \mathbb{R}^n$, a complementarity system is described by the following differential equation:

$$\begin{aligned} \dot{x} = f(x) + \eta \\ K^\star \ni \eta \perp x \in K. \end{aligned} \tag{2.22}$$

To draw connections with the system class (2.15), we use the basic result from convex analysis [74, Proposition 1.1.3]:

$$\eta \in -\mathcal{N}_K(x) \Longleftrightarrow K^\star \ni \eta \perp x \in K,$$

where the notation $K^\star \ni \eta \perp x \in K$ is the short hand for the three statements: i) $x \in K$, ii) $\eta \in K^\star$, and iii) $x^\top\eta = 0$.

Complementarity systems form a class of nonsmooth dynamical systems that is of use in mechanical and electrical engineering as well as in optimization and in other fields. A general way of coupling ordinary differential equations to complementarity conditions was proposed in [148] in 1996, and this work was extended in a later paper [149]. There has been significant work though in specific areas where combinations of differential equations with complementarity conditions arise.

Several works exist in the literature which deal with existence and numerical construction of the solution to system (2.22). A recent reference [41] contains results in this direction, along with pointers to earlier works. Motivated by these works, it is stipulated that the data of (2.22) satisfy the following assumption.

Assumption 1. Function $f : \mathbb{R}^n \to \mathbb{R}^n$ is locally Lipschitz continuous, $f(0) = 0$ and $K \subset \mathbb{R}^n$ is a closed convex cone.

The next statement concerns the sensitivity of the solution of an LCCP with respect to one of its parameters. The results given in [110, Section 2] and [125] focus on Lipschitz continuity of the solution to LCP problems, and they can be modified to get the following statement:

Proposition 2.12. *Consider system (2.22) under Assumption 1. Let $(x, \eta) : [0, \infty) \to \mathbb{R}^{2n}$ denote the solution with an admissible initial condition $x(0) \in K$. Then, there exists a constant $C > 0$ such that for each $t \geqslant 0$,*

$$\|\eta(t)\| \leqslant C\|f(x(t))\|. \tag{2.23}$$

2.2.2 Time Evolution

Before proceeding with next chapters which present our results, it is instructive to recall how a solution to (2.22) evolves with time, and the underlying optimization problem which may be solved to compute η.

For a fixed $s \geqslant 0$, if $x(s) \in \text{int}(K)$, then $\mathcal{N}_K(x(s)) = \{0\}$, and we let $\eta(s) = 0$. As a result, for some $\varepsilon > 0$ and $t \in [s, s+\varepsilon)$, we have $\dot{x}(t) = f(x(t))$ and $x(t) \in \text{int}(K)$.

However, if for some $\bar{s} \geqslant 0$, we have that $x(\bar{s}) \in \text{bd}(K)$, the boundary of K, then we essentially compute $\eta(\bar{s})$ satisfying the relation[1]

$$K^\star \ni \eta(\bar{s}) \perp f(x(\bar{s})) + \eta(\bar{s}) \in \mathcal{T}_K(x). \tag{2.24}$$

If the boundary constraint remains active over an interval $[\bar{s}, \bar{s}+\varepsilon]$ for some $\varepsilon > 0$, that is, for each $t \in [\bar{s}, \bar{s}+\varepsilon]$, $x(t) \in \text{bd}(K)$, then $\eta(t)$ satisfies the complementarity relation in (2.24). We say that

$$\eta(t) \in \text{LCCP}(f(x(t)), I, \mathcal{T}_K(x(t))),$$

which is equivalently described as the solution to the optimization problem stated in (2.21).

In what follows, it is also important to recall how we interpret the solution to (2.22) if $x(0) = x_0 \notin K$. In such a case, we let

$$x_0^+ = \text{proj}_K(x_0) \tag{2.25}$$

and then propagate the solution with x_0^+, the projection of x_0 on K with respect to Euclidean norm. We can thus formally define the solution to (2.22) as follows:

[1]Note that, for a closed convex cone $K \subseteq \mathbb{R}^n$, for each $x \in K$, we denote the tangent cone to K at x by $\mathcal{T}_K(x)$ and $\mathcal{N}_K(x) = -\mathcal{T}_K(x)^\star$. Also, for $x \in K$, if $\eta \in -\mathcal{N}_K(x)$, then $\eta \in K^\star$, and hence $K^\star \subseteq \mathcal{T}_K(x)^\star$.

Definition 22. For a given initial condition $x_0 \in \mathbb{R}^n$ and an interval $[0, T]$, a solution to (2.22) is an absolutely continuous function $x : [0, T] \to \mathbb{R}^n$, such that $x(t) \in K$ for each $t > 0$, and $x_0^+ = \text{proj}_K(x_0)$.

Existence and uniqueness of solutions is a basic issue in the formulation of any dynamical system. From the point of view of mathematical programming, conditions that ensure existence and uniqueness of solutions are of interest because they can serve as a soundness test on a proposed model.

Under Assumption 1, there exists a unique solution to (2.22) in the sense of Definition 22. We denote by $x(t; x_0)$ the solution of (2.22), at time $t \geqslant 0$ starting with initial condition x_0 at time 0. Assumption 1 also guarantees that the origin is an equilibrium and $x(t; 0) = 0$ is the unique trivial solution starting from $x_0 = 0$. Indeed, with K being a closed convex cone, we have $0 \in K$. Under the condition $f(0) = 0$, we have $\eta(t) = 0$ and $\dot{x}(t) = 0$, for all $t \geqslant 0$.

2.3 Overview of Stability Analysis

Maximal monotone operators were first introduced in [114] and [153] and its definition is given as follows.

Definition 23. (Maximal monotone operator). Consider a set-valued mapping $\mathcal{M} : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$, that is $\mathcal{M}(x) \subseteq \mathbb{R}^n$ for each $x \in \mathbb{R}^n$. We say that $\mathcal{M}$ is monotone if it satisfies the following property:

$$\langle y_1 - y_0, x_1 - x_0 \rangle \geqslant 0, \quad \text{for all} \quad y_0 \in \mathcal{M}(x_0), y_1 \in \mathcal{M}(x_1).$$

We say that $\mathcal{M}$ is maximal monotone if no expansion of its graph is possible in $\mathbb{R}^n \times \mathbb{R}^n$ without destroying monotonicity. In other words, $\mathcal{M}$ is maximal monotone if it is monotone and in addition $\mathcal{M} = \mathcal{M}'$, for all monotone $\mathcal{M}' : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ such that $\text{graph}(\mathcal{M}) \subset \text{graph}(\mathcal{M}')$, where $\text{graph}(\mathcal{M}) = \{(x, y) \mid y \in \mathcal{M}(x)\}$.

Let us consider the following linear dynamical system, denoted by Γ, and described by the quadruple (A, B, C, D):

$$\Gamma : \begin{cases} \dot{x} = Ax + B\eta \\ y = Cx + D\eta \end{cases}$$

The linear system Γ is said to be passive if there exists a positive semi-definite function $V : \mathbb{R}^n \to \mathbb{R}_+$ such that the dissipation inequality

$$V(x(t)) - V(x(0)) \leqslant \int_0^t \langle \eta(s), y(s) \rangle \, ds \tag{2.26}$$

is satisfied along all solutions of Γ, for each $x(0) \in \mathbb{R}^n$ and each $t \geqslant 0$. A function V satisfying (2.26) is called a *storage* function.

We say that Γ is strictly passive if

$$V(x(t)) - V(x(0)) \leqslant \int_0^t \langle \eta(s), y(s) \rangle \, ds - \int_0^t \psi(x(s)) \, ds \tag{2.27}$$

for some positive definite function ψ.

Certain interconnections of dynamical systems and nonsmooth relations can be expressed as maximal monotone operators. In this context, let us consider the following dynamical system:

$$\dot{x} = Ax + B\eta \tag{2.28a}$$
$$y = Cx + D\eta \tag{2.28b}$$
$$\eta \in -\mathcal{M}(y) \tag{2.28c}$$

where $\mathcal{M}$ is a maximal monotone operator.

This system can be equivalently written in the following form:

$$\dot{x} \in -\mathcal{H}(x) := Ax - B(\mathcal{M} + D)^{-1}(Cx). \tag{2.29}$$

The following theorem asserts that $\mathcal{H}$ is maximal monotone if the linear system is passive and the set-valued mapping $\mathcal{M}$ is maximal monotone.

Theorem 2.13 ([44, Theorem 2])**.** *Suppose that*

1. *The quadruple (A, B, C, D) is passive with storage function $x \mapsto x^\top x$.*

2. *The mapping $\mathcal{M}$ is maximal monotone.*

3. *It holds that* $\operatorname{im} C \cap \operatorname{rint}(\operatorname{im}(\mathcal{M} + D)) \neq \emptyset$.

Then the mapping $\mathcal{H}$ is maximal monotone.

Let us now mention a lemma that gives necessary and sufficient conditions for a system being strictly passive.

Lemma 2.14 (Kalman-Yakubovich-Popov Lemma)**.** *System Γ is strictly passive with storage function $V(x) = x^\top Px$ if and only if there exist matrices $L \in \mathbb{R}^{n\times p}$ and $W \in \mathbb{R}^{p\times p}$, a positive scalar $\epsilon > 0$, and a symmetric positive semi-definite matrix $P \in \mathbb{R}^{n\times n}$ such that:*

$$\begin{cases} A^\top P + PA = -LL^\top - \epsilon P \\ B^\top P - C = -W^\top L^\top \\ D + D^\top = W^\top W. \end{cases}$$

In other words, the system Γ is passive if and only if the linear matrix inequalities

$$P = P^\top \geqslant 0, \quad \begin{bmatrix} A^\top P + PA + \epsilon P & PB - C^\top \\ B^\top P - C & -(D^\top + D) \end{bmatrix} \leqslant 0.$$

Moreover, $V(x) = x^\top P x$ defines a storage function if P is a solution to the above linear matrix inequalities.

Due to Theorem 2.13 and Lemma 2.14, one can write down sufficient conditions to check if the system is asymptotically stable, under the restriction that the quadruple (A, B, C, D) describes a passive system. The computational burden, in that case, boils down to finding a quadratic positive definite Lyapunov function which is numerically achieved by solving LMIs, as indicated in Lemma 2.14. In this thesis, we are basically interested in the stability analysis of dynamical systems where an ordinary differential equation is coupled with a maximal monotone relationship, but the system described the ordinary differential equation is neither assumed to be linear nor passive. Hence, one cannot use the aforementioned approach based on passivity and LMIs for stability analysis and computing Lyapunov functions. In fact, we observe that, in general the class of Lyapunov functions depends on the constraint set under consideration. For this reason, we introduce the following definition of Lyapunov functions for the class of systems (1.1).

Definition 24 (Constrained Lyapunov Function). System (1.1) has a continuously differentiable (global) Lyapunov function $V : \mathbb{R}^n \to \mathbb{R}$ with respect to $\mathcal{S}$ if

1. There exist class $\mathcal{K}_\infty$ functions $\underline{\alpha}$, $\overline{\alpha}$ such that

$$\underline{\alpha}(\|x\|) \leqslant V(x) \leqslant \overline{\alpha}(\|x\|), \quad \forall\, x \in \mathcal{S};$$

2. There exists a class $\mathcal{K}$ function α such that

$$\langle \nabla V(x), f(x) \rangle \leqslant -\alpha(\|x\|), \ \forall\, x \in \operatorname{int}(\mathcal{S}), \tag{2.30a}$$
$$\langle \nabla V(x), f(x) + \eta_x \rangle \leqslant -\alpha(\|x\|), \ \forall\, x \in \operatorname{bd}(\mathcal{S}), \tag{2.30b}$$

where $-\eta_x$ is the projection of $f(x)$ on $\mathcal{N}_{\mathcal{S}}(x)$, such that $f(x) + \eta_x \in \mathcal{T}_{\mathcal{S}}(x)$.

In the next chapter, we address our first main question on the existence of Lyapunov functions in the sense of Definition 24, while restricting ourselves to the case where $\mathcal{S}$ is a positive orthant.

3

Stability Analysis: Converse Result

The stability analysis of constrained systems of the form (1.1) using Lyapunov functions has received considerable attention in the literature, and in this chapter we address the converse question when the constraint is described by a closed convex cone, so that the dynamics are equivalently expressed by a complementarity system. Since the state of such systems essentially evolves in a closed convex cone, often chosen to be the positive orthant, it is naturally desirable to consider Lyapunov functions which are positive definite over the positive orthant; the functions satisfying this latter property are called *copositive* functions. The need to search for such functions for stability analysis of complementarity systems was presented as an open problem in [43]. The papers [80, 79, 42] investigate sufficient stability conditions for linear complementarity systems, or conewise linear systems [93] in terms of copositive Lyapunov functions. The paper [79] also provides examples of systems where a positive definite Lyapunov function does not exist, but the system is nonetheless asymptotically stable and it admits a copositive Lyapunov function.

While these existing works have shown the utility of enlarging the search space of Lyapunov functions from positive definite to copositive functions, and cone-copositive functions when considering systems with state trajectories constrained to a cone rather than the positive orthant, none of the existing works has adressed the converse question:

Does every exponentially stable complementarity system admit a cone-copositive Lyapunov function?

The objective of this chapter is to answer this question in the affirmative by constructing a Lyapunov function as a functional of the solution trajectories, thereby concluding that one does not need to go beyond cone-copositive functions to find Lyapunov functions for complementarity systems.

3.1 Overview

Converse stability results for dynamical systems have been studied for a long time in the control community, see the recent survey article [97]. Moreover, due to discontinuities in the vector field at the boundary of the constraint set (which can be seen as an example of constrained switching), establishing the existence of Lyapunov functions within cone-copositive functions becomes difficult.

In this chapter, we establish an existence result for Lyapunov function, that is, if the system is exponentially stable then there exists a Lyapunov function, with certain properties, for such system.

There exist several results in the literature on converse Lyapunov theorems for systems where the vector fields are discontinuous, see [65, 111] for switched systems, and [42] for complementarity systems. The results in [65, 111] use linearity of the flows, and the results in [42] are restricted to the class of complementarity systems where the right-hand side is Lipschitz continuous (and hence not discontinuous). Some other articles have also showed a converse Lyapunov theorem for other classes of systems, see [76, 39].

Here, we study the converse result where the flow maps are not necessarily linear, and the complementarity relations may induce discontinuities in the vector field. In essence, we generalize the converse results on differential inclusions presented in [55, 147]. An essential difference compared to these results is that our system does not satisfy the regularity assumptions imposed in those works, and instead of strong stability (with respect to all possible vector fields in the differential inclusion), we address weak stability, that is, there exists a vector field in the differential inclusion for which the equilibrium is stable. Moreover, the structure of the system only allows construction over the admissible domain, which is a closed convex cone in our case.

This chapter is structured as follows. In Section 3.2, we describe the appropriate notions of stability which are to be adapted with respect to the constrained domain, and discuss some interesting properties that may arise due to the presence of constraints. In Section 3.3, we introduce the definition of cone-copositive Lyapunov functions. In Section 3.4, we present our main result: we establish an existence result for Lyapunov function and we carried out its technical proof all along the rest of this section. In Section 3.5, we show the existence of homogeneous Lyapunov functions which can be useful for numerical computation.

3.2 Stability Notions

During recent years, the concepts of stability of dynamical systems have evolved, either by modifying old ideas or by creating new ones. We define as in [79, 80] the stability of the origin for the system of our interest:

$$\begin{aligned} \dot{x} &= f(x) + \eta \\ K^{\star} \ni \eta &\perp x \in K. \end{aligned} \tag{3.1}$$

It is stable if small perturbations of the initial condition at the origin lead to solutions remaining in the neighborhood of the origin for all forward times:

Definition 25 (Stability)**.** The origin is stable in the sense of Lyapunov if for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$x_0 \in K, \|x_0\| \leqslant \delta \Rightarrow \|x(t, x_0)\| \leqslant \varepsilon, \quad \forall t \geqslant 0$$

where $x(t, x_0)$ denotes the solution at time t with initial condition x_0.

The origin is locally asymptotically stable if it is stable in the sense of Lyapunov and there exists $\beta > 0$ such that

$$x_0 \in K, \|x_0\| \leqslant \beta \Rightarrow \lim_{t \to +\infty} \|x(t, x_0)\| = 0.$$

The origin is globally asymptotically stable if the latter implication holds for arbitrary $\beta > 0$.

The origin is globally exponentially stable if there exists $c_0 > 0$ and $\alpha > 0$ such that

$$\|x(t, x_0)\| \leqslant c_0 e^{-\alpha t} \|x_0\|, \text{ for every } x_0 \in K.$$

The interpretation of stability is that the trajectories starting from points in the neighborhood of an equilibrium point remain close to that equilibrium point. Lyapunov's direct method allows us to check the stability of an equilibrium point without solving the differential equation of the system. This interesting and useful method had great influence on the development of the modern theory of stability of motion.

Compared to the conventional definitions of stability for unconstrained dynamical systems, our domain of interest is reduced to the set K in system (3.1). Also, the vector field jumps instantaneously at the boundaries of the set K, which may have an impact on the stability of the system. The following examples motivate why it is not enough to analyze stability just by looking at the vector field f in (3.1).

Example 4 (Constraints make the system stable, even if the unconstrained system is unstable)**.** Let $f(x) = Ax$ with $A = \begin{bmatrix} -1 & -2 \\ -1 & -1 \end{bmatrix}$, and $K = \mathbb{R}^2_+$. Matrix A is not Hurwitz stable since one of its eigenvalues is in the right-half complex plane. However, constrained system (3.1) is globally asymptotically stable, see our later Example 8 in Section 4.3.1 for a proof based on a Lyapunov function.

Example 5 (Constraints make the system unstable, even if the unconstrained system is stable)**.** Let $f(x) = Ax$ with $A = \begin{bmatrix} -1.5 & -1 \\ 2 & 1 \end{bmatrix}$, and $K = \mathbb{R}^2_+$. The matrix A is Hurwitz but the constrained system (3.1) is unstable because on the x_2-axis, the vector field is pointing away from the origin.

Note that in the interior of K, system (3.1) follows the dynamics $\dot{x} = f(x)$.

The first example, however, shows that even if the constrained system is globally asymptotically stable, it is not possible to work with a Lyapunov function for the unconstrained system. In Example 4, the unconstrained system does not admit a positive definite function with negative definite time derivative over the entire state space. Consequently, one has to enlarge the search for Lyapunov functions to functions which are positive definite only on the admissible domain.

The second example shows that even if one can find a Lyapunov function for the unconstrained system, it may not correspond to a Lyapunov function for the constrained system. Thus, the search of Lyapunov functions for the constrained system needs to be investigated differently from the unconstrained system.

3.3 Lyapunov Functions with Constraints

Based on the above notions, one has to adapt the notion of Lyapunov functions when analyzing the stability of complementarity systems. It is thus of interest to introduce *cone-copositive* functions:

Definition 26 (Cone-copositivity and copositivity). Let $K \subseteq \mathbb{R}^n$ be a closed convex cone. A real-valued function $h : \mathbb{R}^n \to \mathbb{R}$ is said to be cone-copositive with respect to K, if $h(x) \geqslant 0$ for each $x \in K$. When $K = \mathbb{R}^n_+$, we simply say that h is copositive.

Positive definite functions are obviously cone-copositive, regardless of the cone under consideration. However, in general, when the cone K is fixed, positive definite functions only form a subclass of the functions which are cone-copositive with respect to K. With this function class, the following definition of Lyapunov functions for (3.1) provides more flexibility:

Definition 27 (Cone-copositive Lyapunov Function). System (3.1) has a continuously differentiable (global) cone-copositive Lyapunov function $V : \mathbb{R}^n \to \mathbb{R}$ with respect to K if

1. There exist class $\mathcal{K}_\infty$ functions[1] $\underline{\alpha}$, $\overline{\alpha}$ such that

$$\underline{\alpha}(\|x\|) \leqslant V(x) \leqslant \overline{\alpha}(\|x\|), \quad \forall\, x \in K;$$

2. There exists a class $\mathcal{K}$ function α such that

$$\langle \nabla V(x), f(x) \rangle \leqslant -\alpha(\|x\|), \quad \forall\, x \in \mathrm{int}(K), \tag{3.2a}$$
$$\langle \nabla V(x), f(x) + \eta \rangle \leqslant -\alpha(\|x\|), \quad \forall\, x \in \mathrm{bd}(K), \tag{3.2b}$$

   where $\eta \in \mathrm{LCCP}(f(x), I, \mathcal{T}_K(x))$.

Condition (3.2) is splitted into two parts because the complementarity variable resulting from an LCCP takes nonzero value only on the boundary of the cone K.

Note that we require the inequalities to hold only for a particular selection of η. This aspect of our definition is in contrast with several existing works dealing with the existence of Lyapunov functions for differential inclusions [55, 147].

3.4 Existence Result

Our main result appears below. The proof of the following theorem is a rather lengthy and technical affair and it is carried out in the remainder of this section.

Theorem 3.1. *Under Assumption 1, if the origin is globally exponentially stable for system (3.1), then there exists a continuously differentiable cone-copositive Lyapunov function.*

3.4.1 Proof of the Existence Result

To prove Theorem 3.1, we start with the following lemma.

[1] A function $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ is said to be of class $\mathcal{K}$ if it is continuous, it satisfies $\alpha(0) = 0$, and it is increasing everywhere on its domain. It is said to be of class $\mathcal{K}_\infty$ if it is, in addition, unbounded.

Lemma 3.2. *If Assumption 1 holds and the origin is globally exponentially stable for system (3.1), then there exists a globally Lipschitz function $\widehat{f}$: $\mathbb{R}^n \to \mathbb{R}^n$ such that the system*

$$\begin{aligned} \dot{x} =& \widehat{f}(x) + \eta \\ K^\star \ni \eta \perp & x \in K \end{aligned} \tag{3.3}$$

has a globally exponentially stable equilibrium. If (3.3) admits a continuously differentiable cone-copositive Lyapunov function $\widehat{V}$, then $\widehat{V}$ is a Lyapunov function for (3.1).

Proof : For f locally Lipschitz in (3.1), there exists a continuous positive definite function $\beta : \mathbb{R}^n \to \mathbb{R}_+$, such that $\beta(x)f(x)$ is globally Lipschitz on $\mathbb{R}^n \setminus \{0\}$, [55, Lemma 4.10]. Set $\widehat{f}(x) := \beta(x)f(x)$ in (3.3). We first prove the second item: if $\widehat{V}$ is a continuously differentiable Lyapunov function for (3.3), then there exists a class $\mathcal{K}$ function $\widehat{\gamma}$ such that $\langle \nabla \widehat{V}, \beta(x)f(x)\rangle \leqslant -\widehat{\gamma}(\|x\|)$ for $x \in \operatorname{int}(K)$ and $\langle \nabla \widehat{V}, \beta(x)f(x) + \eta\rangle \leqslant -\widehat{\gamma}(\|x\|)$ for $x \in \operatorname{bd}(K)$. By choosing a class $\mathcal{K}$ function γ such that $\gamma(\|x\|) < \frac{1}{\beta(x)}\widehat{\gamma}(\|x\|)$, and using Proposition 2.11, it follows that $V = \widehat{V}$ is a continuously differentiable Lyapunov function for (3.1).

To prove the first item, we need to show that the origin of (3.3) is globally exponentially stable. Let $z \in [0, \infty)$ be a solution to (3.3), and let $\rho(t) = \int_0^t \beta(z(s))\, ds$. Using Proposition 2.11 and the chain rule for differentiation, it follows that $x(t) = z(\rho^{-1}(t))$ is a solution of (3.1). Thus, for every solution z of (3.3), there exists a solution x of (3.1) such that $z(t) = x(\rho(t))$. Lyapunov stability of the origin of (3.3) thus follows by inspection. Suppose that there exists a solution $\overline{z}$ such that $\overline{z}(t)$ does not converge to the origin as $t \to \infty$, then $\lim_{t\to\infty} \rho(t) = +\infty$. Let $\overline{x}$ be a solution to (3.1) such that $\overline{z}(t) = \overline{x}(\rho(t))$ and since (3.1) is asymptotically stable, we have $\lim_{t\to\infty} \overline{x}(\rho(t)) = 0$, which is a contradiction. Hence, $\overline{z}$ converges to the origin as well. ◇

Based on Lemma 3.2, it can be assumed for the proof of Theorem 3.1, without loss of generality, that f in (3.1) is a globally Lipschitz continuous vector field with modulus L and this assumption is assumed to hold in the remainder of this section. Note that, in Theorem 3.1, we assume the origin to be globally exponentially stable and our proof (appearing next) indeed uses that property. It remains to be seen if the proof can be adapted to the case where the origin is only asymptotically stable.

For the proof of Theorem 3.1, we construct the Lyapunov function for

(3.1) by introducing a function $V : \mathbb{R}^n \to \mathbb{R}$, defined as

$$V(z) = \int_0^\infty \|x(\tau; \mathrm{proj}_K(z))\|^{\frac{2L}{\alpha}+1} \, d\tau, \tag{3.4}$$

where $x(\tau; \overline{z})$ denotes the solution to system (3.1) at time $\tau \geqslant 0$ with $x(0^+) = \overline{z} \in K$. Note that V is defined for each $z \in \mathbb{R}^n$ and not just for $z \in K$. When $z \notin K$, the term $x(\tau; \mathrm{proj}_K(z))$ can be interpreted as the solution obtained by projecting the initial condition on K, and then propagating it continuously according to the system vector field. Thus, for $\tau > 0$, we have $x(\tau; \mathrm{proj}_K(z)) = x(\tau; z)$ for each $z \in \mathbb{R}^n$.

Step 1: *Bounds on Solutions*

The following lemma demonstrates the continuity of solutions with respect to the initial conditions and plays an important role in the remainder of the proof.

Lemma 3.3. *Let L be the Lipschitz modulus of f. If x and $\widehat{x}$ are two solutions to system (3.1) that satisfy $x(0) = z \in K$ and $\widehat{x}(0) = \widehat{z} \in K$, then it holds that, for each $\tau > 0$,*

$$\|x(\tau; z) - \widehat{x}(\tau; \widehat{z})\| \leqslant e^{L\tau} \|z - \widehat{z}\| \tag{3.5a}$$

and for some $C > 0$,

$$\|x(\tau; z)\| \geqslant e^{-C\tau} \|z\|. \tag{3.5b}$$

Proof : It will be assumed without loss of generality that $z \in K$ and $\widehat{z} \in K$ since $\| \mathrm{proj}_K(z) - \mathrm{proj}_K(\widehat{z})\| \leqslant \|z - \widehat{z}\|$. By definition of the solution to (3.1) and monotonicity of the normal cone operator, it follows that, for each $y \in K$,

$$\left\langle \frac{dx}{dt}(t) - f(x(t)), y - x(t) \right\rangle \geqslant 0$$

and similarly, for each $\widehat{y} \in K$,

$$\left\langle \frac{d\widehat{x}}{dt}(t) - f(\widehat{x}(t)), \widehat{y} - \widehat{x}(t) \right\rangle \geqslant 0,$$

where we have suppressed the dependence of x and $\widehat{x}$ on the initial condition for brevity. Letting $y = \widehat{x}(t) \in K$, and $\widehat{y} = x(t) \in K$, we have:

$$\left\langle \frac{dx}{dt}(t) - f(x(t)), \widehat{x}(t) - x(t) \right\rangle \geqslant 0$$

and,

$$\left\langle \frac{d\widehat{x}}{dt}(t) - f(\widehat{x}(t)), x(t) - \widehat{x}(t) \right\rangle \geqslant 0.$$

By adding the last two inequalities, we get the following:

$$\left\langle \frac{d}{dt}(x(t) - \widehat{x}(t)), x(t) - \widehat{x}(t) \right\rangle \leqslant \langle f(x(t)) - f(\widehat{x}(t)), x(t) - \widehat{x}(t) \rangle \,,$$

or equivalently,

$$\frac{d}{dt}\|x(t) - \widehat{x}(t)\|^2 \leqslant 2 \langle f(x(t)) - f(\widehat{x}(t)), x(t) - \widehat{x}(t) \rangle \,.$$

Because of the Lipschitz continuity assumption, $\|f(x(t)) - f(\widehat{x}(t))\| \leqslant L\|x(t) - \widehat{x}(t)\|$, and hence,

$$\frac{d}{dt}\|x(t) - \widehat{x}(t)\|^2 \leqslant 2L\|x(t) - \widehat{x}(t)\|^2.$$

The bound in (3.5a) now follows by integrating both sides, or invoking the so-called comparison lemma [98, Lemma 3.4]. To get the bound in (3.5b), we make use of Proposition 2.12 which ensures that there exists a constant $C_\eta > 0$ such that $|\eta| \leqslant C_\eta |f(x(t))|$. We therefore get

$$\begin{aligned}\left| \frac{d}{dt}\|x(t)\|^2 \right| &= 2 \left| \langle x(t), f(x(t)) + \eta \rangle \right| \\ &\leqslant 2\|x(t)\| \|f(x(t)) + \eta\| \\ &\leqslant 2\|x(t)\| (L\|x(t)\| + C_\eta L \|x(t)\|) \\ &\leqslant 2L(1 + C_\eta)\|x(t)\|^2.\end{aligned}$$

In particular, $\frac{d}{dt}\|x(t)\|^2 \geqslant -2L(1 + C_\eta)\|x(t)\|^2$, and hence, the inequality in (3.5b) follows by taking $C = L(1 + C_\eta)$. ♢

To show that V satisfies item 1) of Definition 27, let us first use the bound in (3.5b) from Lemma 3.3, so that

$$\begin{aligned}V(z) &\geqslant \int_0^\infty e^{-(2L+\alpha)C\tau/\alpha} \| \operatorname{proj}_K(z)\|^{\frac{2L}{\alpha}+1} \, d\tau \\ &\geqslant \underline{C} \| \operatorname{proj}_K(z)\|^{\frac{2L}{\alpha}+1},\end{aligned}$$

for some $\underline{C} > 0$. Also, exponential stability of the origin implies that $\|x(\tau; z)\| \leqslant c_0 e^{-\alpha\tau} \| \operatorname{proj}_K(z)\|$ and hence there exists $\overline{C} > 0$ such that

$$\begin{aligned}V(z) &\leqslant c_0 \int_0^\infty e^{-(2L+\alpha)\tau} \| \operatorname{proj}_K(z)\|^{\frac{2L}{\alpha}+1} \, d\tau \\ &\leqslant \overline{C} \, \| \operatorname{proj}_K(z)\|^{\frac{2L}{\alpha}+1}.\end{aligned}$$

Step 2: *Local Lipschitz Continuity of V*

To show that V is locally Lipschitz continuous, we need the following two properties [57]:

- V is continuous; and
- its Dini subderivative[2] satisfies

$$DV(z;v) \leqslant \phi(z)\|v\| \tag{3.6}$$

for every $v \in \mathbb{R}^n$, every $z \in \mathbb{R}^n$, and some locally bounded function $\phi : \mathbb{R}^n \to \mathbb{R}$, with $\phi(z) > 0$ for $z \neq 0$.

The continuity of V follows directly from Lemma 3.3 as the exponential bound on the solutions of the system makes V a composition of continuous functions. These properties can again be shown using Lemma 3.3. Fix $v \in \mathbb{R}^n$. Consider a sequence of initial conditions $\widehat{z}_k = z + \varepsilon_k v$. We get

$$\begin{aligned} DV(z;v) &\leqslant \liminf_{\varepsilon_k \to 0} \frac{V(z+\varepsilon_k v) - V(z)}{\varepsilon_k} \\ &= \liminf_{\varepsilon_k \to 0} \frac{1}{\varepsilon_k}\left(\int_0^\infty (\|\widehat{x}_k(\tau;\widehat{z}_k)\|^{\frac{2L}{\alpha}+1} - \|x(\tau;z)\|^{\frac{2L}{\alpha}+1})d\tau\right). \end{aligned} \tag{3.7}$$

where $\widehat{x}_k(\tau;\widehat{z}_k)$ is the notation of $x(\tau;\widehat{z}_k)$.

Using the mean-value theorem, for each $s \geqslant 0$, there exists $\xi(s)$ between $\|\widehat{x}_k(s;\widehat{z}_k)\|$ and $\|x(s;z)\|$ such that

$$\begin{aligned} &\|\widehat{x}_k(s;\widehat{z}_k)\|^{\frac{2L}{\alpha}+1} - \|x(s;z)\|^{\frac{2L}{\alpha}+1} \\ \leqslant &\left| \|\widehat{x}_k(s;\widehat{z}_k)\|^{\frac{2L}{\alpha}+1} - \|x(s;z)\|^{\frac{2L}{\alpha}+1} \right| \\ = &\left| \xi(s)^{2L/\alpha}(\|\widehat{x}_k(s;\widehat{z}_k)\| - \|x(s;z)\|) \right| \\ \leqslant &|\xi(s)|^{2L/\alpha}\|\widehat{x}_k(s;\widehat{z}_k) - x(s;z)\|. \end{aligned} \tag{3.8}$$

It follows from Lemma 3.3 that $\|\widehat{x}_k(s;\widehat{z}_k) - x(s;z)\| \leqslant e^{Ls}\varepsilon_k\|v\|$. Substituting these bounds in (3.7), we get

$$\begin{aligned} DV(z;v) &\leqslant \liminf_{\varepsilon_k \to 0} \frac{1}{\varepsilon_k}\left(\int_0^\infty |\xi(s)|^{2L/\alpha} e^{Ls}\varepsilon_k\|v\|\,ds\right) \\ &\leqslant \|v\|\int_0^\infty e^{Ls}|\xi(s)|^{2L/\alpha}\,ds. \end{aligned}$$

Due to the exponential stability assumption, $\|\xi(s)\| \leqslant \widehat{c}\,e^{-\alpha s}\|z\|$, for some $\widehat{c} > 0$, then we have

$$DV(z;v) \leqslant \|v\|\int_0^\infty \widehat{c}\,e^{-Ls}\|z\|^{2L/\alpha}\,ds.$$

[2]The Dini subderivative of V at x in the direction v is defined as

$$DV(x;v) := \liminf_{w \to v, \varepsilon \to 0^+} \frac{V(x+\varepsilon w) - V(x)}{\varepsilon}.$$

Hence we choose

$$\phi(z) = \widehat{c}\,\|z\|^{2L/\alpha} \int_0^\infty e^{-Ls}\, ds,$$

so that the bound (3.6) is seen to hold. Thus, V is locally Lipschitz continuous.

Step 3: *Infinitesimal Decrease in V*

As the next step, we show that the function V decreases along the system vector field. In what follows, we will denote the right-hand side of (3.1) by $F(z)$, so that

$$F(z) \in f(z) - \mathcal{N}_K(z).$$

The function V in (3.4) is differentiable almost everywhere because it is locally Lipschitz continuous. We next show that the Dini subderivative of V, along $F(z)$ is negative definite.

Lemma 3.4. *For the function $V : \mathbb{R}^n \to \mathbb{R}$ in (3.4), and $z \in K$,*

$$DV(z; F(z)) \leqslant -\|z\|^{\frac{2L}{\alpha}+1}. \tag{3.9}$$

Proof : [Proof of Lemma 3.4] To prove (3.9), we need a bound on $V(z) - V(z + tF(z))$ for $t \geqslant 0$ sufficiently small. We will get the desired bounds by rewriting the difference as

$$V(z + tF(z)) - V(z) = \Big[V(z + tF(z)) - V(x(t; z))\Big] + \Big[V(x(t; z)) - V(z)\Big] \tag{3.10}$$

and getting a bound on each of the two difference terms on the right-hand side. The first term $V(z + tF(z)) - V(x(t; z))$ can be analyzed from the following lemma:

Lemma 3.5. *For $t > 0$ sufficiently small, it holds that*

$$\|z + tF(z) - x(t; z)\| \leqslant o(t) \tag{3.11}$$

for each $z \in K$.

The proof of Lemma 3.5 will follow momentarily. Using the estimate (3.11), and the inequalities (3.7) and (3.8), we get

$$\begin{aligned} V(z + tF(z)) - V(x(t; z)) &\leqslant C_\phi \|z + tF(z) - x(t; z)\| \\ &= o(t), \end{aligned}$$

for a fixed $z \in K$, and some $C_\phi > 0$. For the second term on the right-hand side of (3.10), it follows from the definition of V in (3.4), with $x(0) = z$, that

$$V(z) \geqslant \int_0^t \|x(\tau; z)\|^{\frac{2L}{\alpha}+1} d\tau + \int_0^\infty \|x(\tau; x(t; z))\|^{\frac{2L}{\alpha}+1} d\tau$$

and hence

$$V(x(t; z)) - V(z) \leqslant -\int_0^t \|x(\tau; z)\|^{\frac{2L}{\alpha}+1} d\tau. \tag{3.12}$$

Substituting the bounds from (3.11) and (3.12) in (3.10), we get

$$\begin{aligned} &\liminf_{t \to 0^+} \frac{V(z + tF(z)) - V(z)}{t} \\ \leqslant\ &\liminf_{t \to 0^+} \frac{-\int_0^t \|x(\tau; z)\|^{\frac{2L}{\alpha}+1} d\tau}{t} \\ =\ &\liminf_{t \to 0^+} -\|x(t; z)\|^{\frac{2L}{\alpha}+1} = -\|z\|^{\frac{2L}{\alpha}+1}, \end{aligned}$$

and hence the Dini subderivative of V is negative definite for almost every $z \in K$. ♢

Proof : [Proof of Lemma 3.5] By definition, the solution x of system (3.1), with $x(0) = z \in K$, satisfies

$$\langle \dot{x}(t) - f(x(t)), x(t) - y \rangle \leqslant 0, \quad \forall\, y \in K, \tag{3.13}$$

for almost all $t \geqslant 0$. For $h > 0$ small enough, introduce the function $\widetilde{z} : [0, h] \to \mathbb{R}^n$ given by

$$\widetilde{z}(t) := z + tF(z) = z + tf(z) + t\eta_z,$$

where η_z is such that $\eta_z = 0$ for $z \in \operatorname{int}(K)$ and $\eta_z \in \operatorname{LCCP}(f(z), I, \mathcal{T}_K(z))$ for $z \in \operatorname{bd}(K)$. It is readily checked that $\widetilde{z}(t) \in K$ for all $t \in [0, h]$, and $\dot{\widetilde{z}} = \frac{d}{dt}\widetilde{z}(t) = F(z) = f(z) + \eta_z$. From the definition of $F(z)$, it follows that $\left\langle f(z) - \dot{\widetilde{z}}\,, \widetilde{y} - z \right\rangle \leqslant 0$, for all $\widetilde{y} \in K$, or equivalently,

$$\left\langle f(z) - \dot{\widetilde{z}}\,, \widetilde{y} - \widetilde{z}(t) \right\rangle \leqslant \langle F(z) - f(z), \widetilde{z}(t) - z \rangle\,, \forall\, \widetilde{y} \in K.$$

For $t > 0$ small enough, we have $(x(t) - \widetilde{z}(t) + t\widetilde{z}(t)) \in K$. Since K is a cone, we can take $\widetilde{y} = \frac{1}{t}\,(x(t) - \widetilde{z}(t) + t\widetilde{z}(t)) \in K$ to get

$$\left\langle f(z) - \dot{\widetilde{z}}\,, x(t) - \widetilde{z}(t) \right\rangle \leqslant t\,\langle \eta_z, \widetilde{z}(t) - z \rangle\,.$$

Taking $y = \widetilde{z}(t)$ in (3.13), and adding it to the last inequality, we get

$$\left\langle \dot{x}(t) - \dot{\widetilde{z}}, x(t) - \widetilde{z}(t) \right\rangle \leqslant \langle f(x(t)) - f(z), x(t) - \widetilde{z}(t) \rangle + t \langle \eta_z, \widetilde{z}(t) - z \rangle .$$

To bound the terms on the right-hand side, we observe that

$$\begin{aligned} \|f(z) - f(x(t))\| &\leqslant L\|z - x(t)\| \\ &\leqslant L\|z - \widetilde{z}(t)\| + L\|\widetilde{z}(t) - x(t)\| \\ &\leqslant Lt\|F(z)\| + L\|\widetilde{z}(t) - x(t)\|. \end{aligned}$$

Using Proposition 2.12, there is some constant C_η such that

$$\langle \eta_z, \widetilde{z}(t) - z \rangle = \langle \eta_z, tF(z) \rangle \leqslant t \, \|\eta_z\| \, \|F(z)\| \leqslant C_\eta t \|F(z)\|.$$

Consequently, we get

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|x(t;z) - \widetilde{z}(t)\|^2 = \left\langle \dot{\widetilde{z}} - \dot{x}(t), \widetilde{z}(t) - x(t) \right\rangle \\ &\leqslant \|f(z) - f(x(t))\| \cdot \|\widetilde{z}(t) - x(t)\| + C_\eta t^2 \|F(z)\| \\ &\leqslant L\|z - x(t)\| \cdot \|\widetilde{z}(t) - x(t)\| + C_\eta t^2 \|F(z)\| \\ &\leqslant Lt\|F(z)\|\|\widetilde{z}(t) - x(t)\| + L\|\widetilde{z}(t) - x(t)\|^2 + C_\eta t^2 \|F(z)\| \\ &\leqslant C_1 \|\widetilde{z}(t) - x(t)\|^2 + C_{2,z} t^2 \end{aligned}$$

where we used Young's inequality for the product term $Lt\|F(z)\| \cdot \|\widetilde{z}(t) - x(t)\|$, and chose $C_1 = (L + 0.5L^2)$ and $C_{2,z} = \max\{0.5\|F(z)\|^2, C_\eta \|F(z)\|\}$. Solving the differential inequality, and using the fact that, $\widetilde{z}(0) = x(0)$, we get

$$\|\widetilde{z}(t) - x(t)\|^2 \leqslant 2C_{2,z} \int_0^t \exp(2C_1(t - s)) s^2 \, ds.$$

Solving the integral on the right, we get

$$\|\widetilde{z}(t) - x(t)\|^2 \leqslant C_{3,z} \, t^3 + o(t^3),$$

for some $C_{3,z} > 0$, whence the estimate in (3.11) follows. ◇

Step 4: *Regularization of V*

The final step is to regularize V so that we obtain a continuously differentiable Lyapunov function.

Lemma 3.6. *Under Assumption 1, if the origin is globally exponentially stable for system (3.1), then there exists a continuously differentiable cone-copositive Lyapunov function.*

Proof : [Proof of Lemma 3.6] Using the function V in (3.4) as a template, we introduce

$$\begin{aligned} V_\sigma(z) &:= \int_{\mathbb{R}^n} V(z-y)\psi_\sigma(y)dy \\ &= \int_{\mathbb{R}^n} V(\mathrm{proj}_K(z-y))\psi_\sigma(y)dy \end{aligned}$$

where ψ_σ, $\sigma \in (0,1)$, is the so-called mollifier that satisfies: $\psi_\sigma \in \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R}_+)$, $\mathrm{supp}(\psi_\sigma) \subset \mathbb{B}(0,\sigma)$, and $\int_{\mathbb{R}^n} \psi_\sigma(y)dy = 1$. It follows from standard texts in functional analysis, see for example [30, Proposition 4.21], that V_σ is continuously differentiable and for every $\varepsilon > 0$ and a compact set $\mathcal{U}_c$, there exists $\overline{\sigma} > 0$, such that for every $\sigma \in (0, \overline{\sigma})$, we get $|V(x) - V_\sigma(x)| < \varepsilon$ for each $x \in \mathcal{U}_c$. Next, we show that $\langle \nabla V_\sigma(z), F(z)\rangle$ approximates $DV(z, F(z))$, for $z \in K$. Indeed, for a given $y \in \mathbb{R}^n$, and $z \in K$, let $\bar{z}_y = \mathrm{proj}_K(z-y)$. It then follows that[3]

$$\begin{aligned} \langle \nabla V_\sigma(z), F(z)\rangle &= \int_{\mathbb{R}^n} \langle \nabla V(\bar{z}_y), F(\bar{z}_y)\rangle \, \psi_\sigma(y)dy \\ &\quad + \int_{\mathbb{R}^n} \langle \nabla V(\bar{z}_y), F(z) - F(\bar{z}_y)\rangle \, \psi_\sigma(y)dy \\ &\leqslant -\|z\|^{\frac{2L}{\alpha}+1} + \varepsilon + C\int_{\mathbb{B}(0,\sigma)} \|\nabla V(\bar{z}_y)\| \|y\| dy \end{aligned}$$

where the bound on the first integral is due to Lemma 3.4, and the bound on the second integral is obtained from the Lipschitz continuity of f and that of η given in Proposition 2.12. Thus, on each compact set excluding the origin, we can find a function V_σ such that $\langle \nabla V_\sigma(z), F(z)\rangle$ is negative definite.

Let us now consider $\{\mathcal{U}_i\}_{i\in\mathbb{N}}$ to be a locally finite open cover of $\mathbb{R}^n \setminus \{0\}$ with $\mathcal{U}_i$ bounded and $0 \notin \mathrm{cl}(\mathcal{U}_i)$, for each $i \in \mathbb{N}$. Let $\{\chi_i\}_{i\in\mathbb{N}}$ be a subordinated $\mathcal{C}^1$ partition of unity. For each $i \in \mathbb{N}$, and $\varepsilon_i > 0$, we can choose the function V_i such that $|V(x) - V_i(x)| < \varepsilon_i$, and $\langle \nabla V_i(x), F(x)\rangle$ is negative, for each $x \in \mathrm{cl}(\mathcal{U}_i)$. Let $\overline{V} : \mathbb{R}_+ \to \mathbb{R}_+$ be such that $\overline{V}(0) = 0$ and $\overline{V}(x) := \sum_{i\in\mathbb{N}} \chi_i(x)V_i(x)$ for $x \neq 0$, then following the analysis in [55, Pages 106-108], it is seen that $\overline{V}$ is a cone-copositive Lyapunov function which is $\mathcal{C}^1$ on $\mathbb{R}^n \setminus \{0\}$, and continuous at $\{0\}$. Finally, to achieve differentiability at the origin, we can introduce a positive definite function $\beta : \mathbb{R}_+ \to \mathbb{R}_+$ with $\beta'(s) > 0$ for each $s > 0$ such that $W(x) = \beta(\overline{V}(x))$ is a continuously differentiable cone-copositive Lyapunov function with respect to K. $\diamondsuit$

[3] Since V is locally Lipschitz, its gradient ∇V exists almost everywhere and the value of the integral on the right-hand side is not affected by the value of ∇V on a set of Lebesgue measure zero.

Remark 3.7. The construction given in the proof of Lemma 3.6 actually gives a $\mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R})$ Lyapunov function. This regularization technique is inspired by [55], and has also been used for smoothening of locally Lipschitz Lyapunov functions for hybrid systems [78, Chapter 7] and switched systems [66].

3.5 Homogeneous case

For numerical purposes, it is useful to show the existence of homogeneous Lyapunov functions. We show that the previous developments can be generalized to construct a homogeneous Lyapunov function when the vector field f in the system description (3.1) is homogeneous.

The next two statements are generalizations of results given in [140].

Proposition 3.8. *Under Assumption 1, if the origin is locally exponentially stable for system* (3.1) *with f homogeneous, then it is also globally exponentially stable.*

Proof : We first show that if $x : [0, \infty) \to K$ is a solution that satisfies (3.1) starting with initial condition x_0, then for each $\lambda \geqslant 0$ and $t \geqslant 0$, the function $y(t) = \lambda x(\lambda^{d-1} t)$ is also a solution to system (3.1) starting with initial condition λx_0. It follows by inspection that $y(t) \in K$, for each $t \geqslant 0$. Noting that for each $z \in K$, and $\lambda > 0$, there exists $\overline{z} \in K$ such that $z = \lambda \overline{z}$, we get

$$\begin{aligned} &\langle \dot{y}(t) - f(y(t)), z - y(t) \rangle \\ &= \left\langle \lambda^d \dot{x}(\lambda^{d-1} t) - \underbrace{f(\lambda x(\lambda^{d-1} t))}_{=\lambda^d f(x(\lambda^{d-1} t))}, z - \lambda x(\lambda^{d-1} t) \right\rangle \\ &= \lambda^d \left\langle \dot{x}(\lambda^{d-1} t) - f(x(\lambda^{d-1} t)), \lambda \overline{z} - \lambda x(\lambda^{d-1} t) \right\rangle \\ &= \lambda^{d+1} \left\langle \dot{x}(\lambda^{d-1} t) - f(x(\lambda^{d-1} t)), \overline{z} - x(\lambda^{d-1} t) \right\rangle \geqslant 0, \end{aligned}$$

and hence $\dot{y}(t) - f(y(t)) \in -\mathcal{N}_K(y(t))$ for almost every $t \geqslant 0$.

Since the origin is locally exponentially stable, there is an open set relative to K, say $\mathcal{R}_0$, such that for each $x(0) \in \mathcal{R}_0$, the corresponding solution x converges to the origin. For an initial condition $y(0) \notin \mathcal{R}_0$, there is a constant $\lambda > 0$ such that $y(0) = \lambda x(0)$, with $x(0) \in \mathcal{R}_0$. Since the solutions are unique, the above reasoning shows that the solution starting from $y(0)$ stays within a bounded set and converges to the origin. ♢

The next result allows us to construct a homogeneous Lyapunov function under local exponential stability. The proof is inspired from [140].

Proposition 3.9. *Consider dynamical system (3.1) with f homogeneous and the origin locally exponentially stable. Let $W \in \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R})$ be a cone-copositive Lyapunov function for (3.1). Let $a \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R})$ be such that*

$$a = \begin{cases} 0 & \text{on } (-\infty, 1], \\ 1 & \text{on } [2, \infty), \end{cases} \tag{3.14}$$

and $\nabla a(s) \geqslant 0$, for each $s \in \mathbb{R}$. Let k be a positive integer. Then the function

$$\overline{W}(x) = \begin{cases} \int_0^\infty \frac{1}{\lambda^{k+1}} (a \circ W)(\lambda x)\, d\lambda & \text{if } x \in \mathbb{R}^n \backslash \{0\}, \\ 0 & \text{if } x = 0, \end{cases} \tag{3.15}$$

is a cone-copositive Lyapunov function of class $\mathcal{C}^{k-1}$ on $\mathbb{R}^n \backslash \{0\}$, and it satisfies

$$\overline{W}(sx) = s^k \overline{W}(x)$$

for all $x \in \mathbb{R}^n \backslash \{0\}$ and $s > 0$.

Proof : The key ingredient required for applying the construction of [140] is to show that if f is homogeneous of degree $d \geqslant 1$, then

$$\mathrm{LCCP}(f(\lambda x), I, \mathcal{T}_K(\lambda x)) = \lambda^d\, \mathrm{LCCP}(f(x), I, \mathcal{T}_K(x)),$$

that is the nonsmooth multiplier η respects the same homogeneity as the function $f(\cdot)$. This indeed follows from Proposition 2.11.

The function $\overline{W}$ is well defined since we have $W(x) \to +\infty$ as $\|x\| \to +\infty$ and vanishes at 0. Besides, we can find two numbers $\underline{a} > 0$ and $\overline{a} > 0$ such that $W(\lambda x) \leqslant 1$, for $\|x\| \in [0.5, 2]$, $\lambda \leqslant \underline{a}$, and $W(\lambda x) \geqslant 2$, for $\|x\| \in [0.5, 2]$, $\lambda \geqslant \overline{a}$. Then, for all $x \in \mathbb{R}^n$ satisfying $\|x\| \in [0.5, 2]$, we have

$$\begin{aligned} \overline{W}(x) &= \int_0^{\underline{a}} \frac{1}{\lambda^{k+1}} \underbrace{(a \circ W)(\lambda x)}_{=0}\, d\lambda + \int_{\underline{a}}^{\overline{a}} \frac{1}{\lambda^{k+1}} (a \circ W)(\lambda x)\, d\lambda \\ &\quad + \int_{\overline{a}}^{\infty} \frac{1}{\lambda^{k+1}} \underbrace{(a \circ W)(\lambda x)}_{=1}\, d\lambda \\ &= \int_0^\infty \frac{1}{\lambda^{k+1}} (a \circ W)(\lambda x)\, d\lambda + \left[-\frac{1}{k\lambda^k} \right]_{\overline{a}}^{\infty} \\ &= \int_0^\infty \frac{1}{\lambda^{k+1}} (a \circ W)(\lambda x)\, d\lambda + \frac{1}{k\overline{a}^k}. \end{aligned}$$

It is obvious that $\overline{W}$ is $\mathcal{C}^1$ on the set $\left\{x \middle| \, \|x\| \in \left[\frac{1}{2}, 2\right]\right\}$. So we have

$$\frac{\partial \overline{W}}{\partial x_i}(x) = \int_0^\infty \frac{\lambda}{\lambda^{k+1}} \nabla a(W(\lambda x)). \frac{\partial W}{\partial x_i}(\lambda x)\, d\lambda.$$

By the homogeneity of f and since η satisfies $\eta_{\lambda x} = \lambda^d \eta_x$, we obtain

$$\left\langle \nabla \overline{W}(x), f(x) + \eta_x \right\rangle = \int_0^\infty \frac{1}{\lambda^{d+k+1}} \nabla a(W(\lambda x)). \\ \langle \nabla W(\lambda x), f(\lambda x) + \eta_{\lambda x} \rangle \, d\lambda. \quad (3.16)$$

Since $\nabla a(s) > 0$ for some $s \in (1, 2)$ and W is a Lyapunov function then, for $\frac{1}{2} < \|x\| < 2$, the right-hand side is negative.

Homogeneity of $\overline{W}$ follows by a change of variable of integration. Therefore, we get $\overline{W}$ is $\mathcal{C}^1$ on $\mathbb{R}^n \backslash \{0\}$ and cone-copositive Lyapunov function with respect to K. ♢

In this chapter, we addressed the stability analysis for a class of complementarity systems using the method of Lyapunov functions : we established an existence result for cone-copositive Lyapunov function for exponentially stable complementarity system. Besides, we showed the existence of homogeneous Lyapunov function which is useful for numerical computation in the next chapter.

4

Computation of Lyapunov Functions

In this chapter, our target is to address the computational aspects of the Lyapunov functions of Chapter 3. While working with homogeneous vector fields, we prove that we can restrict our search to rational homogeneous functions. Moreover, one can adapt the algorithms from the literature on copositive programming to compute these rational functions. Then the question addressed in this chapter is the following one:

> If there exists a rational homogeneous cone-copositive Lyapunov function for a stable complementarity system, how can we construct it?

The answer to this question essentially boils down to finding certain polynomials which satisfy some nonnegativity condition, which is a challenging problem numerically. Such questions have received a lot of attention in modern developments in the field of real algebraic geometry [136, 141] which provide certificates of positivity of (polynomial) functions with Positivstellensätze relying on *sums-of-squares* (SOS) decompositions. Since it has been observed in [134, 53, 128] that finding SOS decompositions is equivalent to semidefinite programming (SDP) or linear matrix inequalities (LMI), numerical tools based on SOS optimization have been developed extensively over the past two decades to compute Lyapunov functions, see e.g. [128, 135, 85, 52].

We explore, in this chapter, two possible routes for designing algorithms for the search of Lyapunov functions.

4.1 Overview

We propose computationally tractable algorithms for finding the Lyapunov functions. The interesting aspect of our problem lies in computing Lyapunov functions which satisfy certain inequalities over a given set. For example, in linear complementarity systems, one needs to check the positivity of a function over the positive orthant only, and if the function we seek is of the form

$x^\top Px$, then finding such a function boils down to finding a *copositive* matrix P that satisfies certain inequalities. However, checking whether a given matrix is copositive is an NP-hard problem [22]. The papers [36, 37, 123], [71] propose algorithms for detecting copositivity of a matrix or tensor. Moreover, we will show with the help of an example that, even in the case of linear complementarity systems, such functions cannot be computed by solving a linear set of equations, as is done for unconstrained linear systems. Another challenging aspect of these problems is that, when dealing with conic constraints which are unbounded sets, there are no readily available Positivstellensatz that guarantee SOS decompositions of a positive polynomial over the sets of our interest. The field of copositive programming has been an active area of research over the past decade which addresses some of these challenges [21]. In computing the Lyapunov functions for complementarity systems which evolve on unbounded cones with positivity constraints, we are faced with similar challenges.

Motivated by such questions, we propose two approaches for computing homogeneous cone-copositive Lyapunov functions numerically. The first one is a *discretization method* which is based on finding an inner approximation of the cone of cone-copositive polynomials by using simplicial partitions and evaluating inequalities over a set of points taken on the simplex. It is shown that, as the partition gets finer, we can approximate any cone-copositive polynomial function. The second approach is an *SOS method* where we show that the positivity of polynomial over the given cone can be checked by expressing it as an SOS function. By increasing the degree of the approximating SOS polynomial, we again obtain a hierarchy of SDP problems to compute the desired Lyapunov function. Then, we derive the corresponding algorithms for those two techniques, which can be seen as an adaptation of tools available in the literature on polynomial optimization. The constraint set K that we first take is $\mathbb{R}^n_+$, the positive orthant in $\mathbb{R}^n$. The illustration of some academic examples is provided using standard Matlab toolboxes.

After that, we extend those ideas to study more general constraint sets $\mathcal{S}$ (conic sets and semi-algebraic sets) and how our earlier algorithms can be adapted for these broader class of sets. For conic constraints, we provide the discretization algorithm based on simplicial partitioning of a simplex. And for semi-algebraic constraints, we use the second method based on SOS decomposition of the Lyapunov function.

This chapter is organized as follows. In Section 4.2, by putting some structure on the system vector field, such as homogeneity, and using the appropriate density results, we prove the existence of a cone-copositive Lyapunov function which can be expressed as a rational of homogeneous polynomials. This later class of functions is seen to be particularly amenable for numerical computation. In Sections 4.3 and 4.4, we provide two types of algorithms (discretization method and SOS method) for precisely that purpose. These algorithms consist of a hierarchy of either linear or semidefinite optimization problems for computing the desired Lyapunov function. We study the following three cases of constraint sets, by increasing degree of generality:

- The positive orthant;
- Polyhedral cones;
- Semi-algebraic sets.

For each case of constraint sets, we give examples to illustrate our approach, by using the YALMIP toolbox in Matlab.

4.2 Polynomial Approximation

For the class of numerical algorithms that we propose in the next sections, it is important to show that the cone-copositive Lyapunov functions of (3.1) can actually be approximated by polynomial functions. Among the existing results in this direction, it is seen that the existence of polynomial Lyapunov functions has been shown under certain restrictions only. In [129], the authors use generalizations of the Weierstrass approximation theorem for nonlinear systems with smooth vector fields to show existence of polynomial Lyapunov functions on compact sets for exponentially stable systems. In the case of switched systems, the existence of polynomial Lyapunov functions has been proven in [111] when the solution maps (parameterized by time) are linear functions of the initial condition. Such methods cannot be generalized here because our vector fields are not even continuous, and even with f linear in (3.1), the resulting solution maps for the complementarity systems are nonlinear and hence nonconvex. As an example of this last observation, we consider the following example:

Example 6 (Constraints make the solution space nonlinear)**.** Let $f(x) = Ax$ with $A = \left[\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right]$ and $K = \mathbb{R}^2_+$ and let $x_1(0) = (a, 0)^\top$ and $x_2(0) = (0, b)^\top$. Let $x_i : \mathbb{R} \to \mathbb{R}^2$ be the solution starting with initial condition $x_i(0)$, $i = 1, 2$, and z denote the solution starting with initial condition $x_1(0) + x_2(0)$. It can be checked that $z(t)$ is not equal to $x_1(t) + x_2(t)$, for any $t > 0$ because we have $x_1(t) = x_1(0)$ for $t \geqslant 0$, $x_2(t) = e^{At}x_2(0) = \left[\begin{smallmatrix} \cos(t) & \sin(t) \\ -sin(t) & \cos(t) \end{smallmatrix}\right]\left[\begin{smallmatrix} 0 \\ b \end{smallmatrix}\right] = (b\sin(t), b\cos(t))$ which gives

$$x_1(t) + x_2(t) = (a + b\sin(t), b\cos(t)),$$

but we have

$$\begin{aligned} z(t) = e^{At}(x_1(0)+x_2(0)) &= \begin{bmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} \\ &= (a\cos(t)+b\sin(t), -a\sin(t)+b\cos(t)), \end{aligned}$$

which is not equal to $x_1(t) + x_2(t)$ for $a, b \neq 0$.

These discussions and the example suggest that it may not be possible to find a homogeneous polynomial approximation to the Lyapunov function proposed in Theorem 3.1 in Chapter 3. Due to lack of any known results on density of homogeneous polynomials in the class of differentiable functions, we enlarge our search to rational functions whose numerator and denominator are homogeneous polynomials. For such functions, we have the following density result [9, Lemma 2.1]:

Proposition 4.1. *Let $W \in \mathcal{C}^1(\mathbb{R}^n; \mathbb{R}_+)$ be a homogeneous function of degree d and $\epsilon > 0$ be a given scalar. There exist an even integer r and a homogeneous polynomial p of degree $r + d$, such that*

$$\max \left\{ \max_{x \in S^{n-1}} \left| \widetilde{W}(x) \right|, \max_{x \in S^{n-1}} \left\| \nabla \widetilde{W}(x) \right\| \right\} \leqslant \epsilon$$

where S^{n-1} denotes the unit sphere in $\mathbb{R}^n$ and $\widetilde{W}(x) = W(x) - \frac{p(x)}{\|x\|^r}$.

With such a rational function in hand which approximates the homogeneous function from Proposition 3.9 (in terms of value and gradient) to desired accuracy, one can establish the existence of a rational homogeneous cone-copositive Lyapunov function.

4.3 Discretization and Copositive Functions

In the previous section, we motivated the need for computing cone-copositive homogeneous Lyapunov functions for the class of constrained dynamical systems (3.1). Proposition 4.1 suggests that for a certain class of complementarity systems, we can reduce our search of Lyapunov functions to the space of rational polynomial functions, where the denominator has a certain structure. By fixing the denominator, we reformulate our problem as finding the numerator in the form of a polynomial which satisfies certain inequalities.

We carry out the steps by specifying the inequalities that must be satisfied, and the algorithms using convex optimization methods that can be implemented for computing such functions.

Just as a quick motivation for what follows, we remark that contrary to unconstrained linear systems, the following example shows that copositive Lyapunov functions cannot be simply obtained by solving a linear equation, and hence there is a need to develop tools for computing them.

Example 7 (Copositive Lyapunov functions are not obtained by solving linear equations)**.** Let $K = \mathbb{R}^2_+$ and $A = \begin{bmatrix} -1 & -2 \\ -1 & -1 \end{bmatrix}$. Let $H = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ the identity matrix which is copositive on cone K. By solving the equation $A^\top G + GA = -H$, we obtain $G = \begin{bmatrix} -1 & \frac{3}{4} \\ \frac{3}{4} & -\frac{1}{4} \end{bmatrix}$ which is not copositive. On the other hand, if we take for example the copositive matrix $\widetilde{H} = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$, we obtain the copositive matrix $\widetilde{G} = \begin{bmatrix} 1 & -\frac{1}{4} \\ -\frac{1}{4} & \frac{3}{4} \end{bmatrix}$ by solving $A^\top \widetilde{G} + \widetilde{G}A = -\widetilde{H}$.

This example shows that for a given matrix A, we can have a copositive matrix H without the existence of G copositive verifying $A^\top G + GA = -H$, but with existence of some $\widetilde{G}$ such that $-A^\top \widetilde{G} - \widetilde{G}A$ is copositive.

We now establish the inequalities which will be used in our algorithms to find copositive homogeneous Lyapunov functions. We restrict our attention to full-dimensional polyhedral cones, that is, $K = \{x \in \mathbb{R}^n | Fx \geqslant 0\}$ with non-empty interior. By using Proposition 4.1, let

$$V(x) = \frac{h(x)}{(\sum_{i=1}^n x_i^2)^r} = \frac{h(x)}{\|x\|_2^{2r}} \tag{4.1}$$

where r is a nonnegative integer, and $h(\cdot)$ is a homogeneous polynomial of degree at least $2r + 1$, copositive on K. Here, we used the notation that $x = (x_1, x_2, \ldots, x_n)^\top \in K$.

As we know, finding such Lyapunov function is equivalent to finding V that satisfies the inequalities:

$$V(x) = \frac{h(x)}{\|x\|_2^{2r}} \geqslant 0, \quad \forall x \in K \backslash \{0\} \tag{4.2a}$$

$$-\langle \nabla V(x), f(x) + \eta \rangle \geqslant 0, \quad \forall x \in K \backslash \{0\} \tag{4.2b}$$

where

$$\begin{aligned} & -\langle \nabla V(x), f(x) + \eta \rangle \\ &= \frac{-\|x\|_2^2 \langle \nabla h(x), f(x) + \eta \rangle + 2rh(x) \langle x, f(x) + \eta \rangle}{\|x\|_2^{2(r+1)}} \end{aligned}$$

with $\eta = \mathrm{LCCP}(f(x), I, \mathcal{T}_K(x))$. The numerator is denoted by

$$s(x) := -\|x\|_2^2 \langle \nabla h(x), f(x) + \eta \rangle + 2rh(x) \langle x, f(x) + \eta \rangle$$

which is a homogeneous polynomial if h and f are homogeneous polynomials. So we have

$$V(x) = \frac{h(x)}{\|x\|_2^{2r}} \geqslant 0, \ \ \forall x \in K \backslash \{0\} \tag{4.3a}$$

$$-\langle \nabla V(x), f(x) + \eta \rangle = \frac{s(x)}{\|x\|_2^{2(r+1)}} \geqslant 0, \ \ \forall x \in K \backslash \{0\}. \tag{4.3b}$$

Thus, finding a copositive V for system (3.1) with the structure imposed in Proposition 4.1 boils down to finding h and s such that

$$h(x) \geqslant 0, \ \ \forall x \in K \backslash \{0\} \tag{4.4a}$$

$$s(x) \geqslant 0, \ \ \forall x \in K \backslash \{0\}. \tag{4.4b}$$

Since η is nonzero only on the boundary of K, we replace the second inequality in (4.4) by inequalities with respect to each face of polyhedron K. Let $S_i := \{x \in K \mid (Fx)_i = 0\}$, $i \in \{1, \ldots, n_K\}$ denote the faces of K. Let

$$s_i(x) = -\|x\|_2^2 \langle \nabla h(x), f(x) + \eta_i \rangle + 2rh(x) \langle x, f(x) + \eta_i \rangle$$

for all $x \in S_i$ where $\eta_i = \mathrm{LCCP}(f(x), I, \mathcal{T}_K(x))$. In the interior of K, we have $\eta = 0$ so let

$$s_0(x) = -\|x\|_2^2 \langle \nabla h(x), f(x) \rangle + 2rh(x) \langle x, f(x) \rangle .$$

Consequently, the inequalities used for finding V can be rewritten as follows:

$$h(x) \geqslant 0, \ \ \forall x \in K \backslash \{0\} \tag{4.5a}$$

$$s_0(x) \geqslant 0, \ \ \forall x \in \mathrm{int}(K \backslash \{0\}) \tag{4.5b}$$

$$s_i(x) \geqslant 0, \ \ \forall x \in S_i, \ \ i \in \{1, \ldots, n_K\} . \tag{4.5c}$$

4.3.1 Copositive functions with $K = \mathbb{R}_+^n$

Let us assume $K = \mathbb{R}_+^n$. The basic idea behind the discretization methods is to select a certain number of points in the cone $\mathbb{R}_+^n$ and evaluate the inequalities (4.5) with a certain polynomial function parameterized by finitely many unknowns. This allows us to construct an inner approximation of copositive polynomials with respect to cone $\mathbb{R}_+^n$.

Copositivity plays a role in quadratic optimization, where the set of copositive matrices can be used to obtain relaxations on the unknown optimal value. Many discrete optimization problems can be formulated as (the dual of

a) linear program over the copositive cone [137]. Contrarily to positive semidefiniteness, copositivity of a matrix is a property that cannot be checked by means of its eigenvalues, and it is considerably harder to check copositivity of a matrix than semidefiniteness. More precisely, deciding whether a given matrix is copositive is an NP-hard problem [22].

In the literature, several algorithms to check copositivity of a matrix have been presented and formulated. There exist algorithms that uses discretization methods [36, 37], [71], and a moment relaxation hierarchy [123]. Here, we restrict ourselves to discretization schemes and generalize the existing algorithms for arbitrary polynomials (not necessarily quadratic functions).

To describe this discretization algorithm, let us first consider the convex cone of copositive polynomials

$$\mathcal{C} := \left\{ g \in \mathbb{R}^d[x] \;\middle|\; \begin{array}{l} g \text{ is homogeneous and} \\ g(x) \geqslant 0 \text{ for all } x \in \mathbb{R}^n_+ \end{array} \right\}, \tag{4.6}$$

where $\mathbb{R}^d[x]$ denotes the ring of polynomials of degree d, over the field of reals, in $x \in \mathbb{R}^n$.

We will establish an inner approximation of $\mathcal{C}$ based on simplicial partitions inside cone $\mathbb{R}^n_+$. To do so, we first need to introduce *tensors*, which generalize the notion of a matrix, and will be used for compact representation of polynomials of our interest.

Definition 28. A tensor $\mathcal{A}$ of order d over $\mathbb{R}^n$ is a multilinear form

$$\begin{array}{ccc} \underbrace{\mathbb{R}^n \times \mathbb{R}^n \times \cdots \times \mathbb{R}^n}_{d \text{ times}} & \to & \mathbb{R} \\ (x^1, x^2, \ldots, x^d) & \mapsto & \mathcal{A}[x^1, x^2, \ldots, x^d] \end{array}$$

where

$$\mathcal{A}[x^1, x^2, \ldots, x^d] = \sum_{i_1=1}^{n} \sum_{i_2=1}^{n} \sum_{i_d=1}^{n} a_{i_1,i_2,\ldots,i_d} x^1_{i_1} \cdots x^d_{i_d}$$

and $a_{i_1,i_2,\ldots,i_d}$ corresponds to a real number from a table with n^d entries, indexed by $i_1, i_2, \ldots, i_d \in \{1, \ldots, n\}$. We say that $\mathcal{A}$ is symmetric if

$$a_{i_1,i_2,\ldots,i_d} = a_{j_1,j_2,\ldots,j_d}$$

whenever $i_1 + i_2 + \cdots + i_d = j_1 + j_2 + \cdots + j_d$, for all possible permutations $i_1, i_2, \ldots, i_d$ and $j_1, j_2, \ldots, j_d$ of $\{1, \ldots, n\}$.

A matrix $A \in \mathbb{R}^{n \times n}$ describes a tensor of order 2 over $\mathbb{R}^n$, also called a quadratic form, where the coefficients of the quadratic form belong to a table with n^2 entries $a_{i,j}$ with $i, j = \{1, \ldots, n\}$. A general homogeneous polynomial $g \in \mathbb{R}^d[x]$, with $d \geqslant 2$, can be written as

$$g(x) = g(x_1, \ldots, x_n) = \sum_{\substack{i=(i_1,\ldots,i_n)\\ i_1+\cdots+i_n=d}} a_i x_1^{i_1} \cdots x_n^{i_n}.$$

Using the tensor representation, g can also be compactly written in the form

$$g(x) = G[\underbrace{x, x, \ldots, x}_{d \text{ times}}] \tag{4.7}$$

where G is a symmetric tensor. The following lemma shows an equivalent expression for copositivity which we will consider all along this section.

Lemma 4.2. *Consider a homogeneous polynomial $g \in \mathbb{R}^d[x]$ of degree d and let $\|\cdot\|$ denote any norm in $\mathbb{R}^n$. We have*

$$g \in \mathcal{C} \iff g(x) \geqslant 0 \text{ for all } x \in \mathbb{R}^n_+ \text{ with } \|x\| = 1.$$

Proof : [$\Rightarrow$] is obvious. [$\Leftarrow$]: Take $x \in \mathbb{R}^n_+$ with $\|x\| \neq 1$. If $\|x\| = 0$ then $x = 0$ and $g(0) = 0$ because of the homogeneity of g. If $\|x\| > 0$ then $\tilde{x} := \frac{x}{\|x\|}$ fulfills $\|\tilde{x}\| = 1$, therefore $g(x) = g(\|x\|\tilde{x}) = \|x\|^d g(\tilde{x}) \geqslant 0$, for all $x \in \mathbb{R}^n_+$ which means $g \in \mathcal{C}$. ♢

If we choose the 1-norm, then the set $\Delta^S := \left\{x \in \mathbb{R}^n_+ | \; \|x\|_1 = 1\right\}$ is the standard simplex. Because of Lemma 4.2, copositivity of a homogeneous polynomial g is then expressed as

$$g(x) \geqslant 0 \text{ for all } x \in \Delta^S.$$

Our goal is to discretize the simplex Δ^S and obtain a hierarchy of linear inequalities with respect to the discretization points which allow us to approximate the set $\mathcal{C}$.

Definition 29. Let Δ be a simplex in $\mathbb{R}^n$ defined as Definition 2. A family $\mathcal{P}_m := \{\Delta^1, \ldots, \Delta^m\}$ of simplices satisfying

$$\Delta = \bigcup_{i=1}^{m} \Delta^i \text{ and } \operatorname{int} \Delta^i \cap \operatorname{int} \Delta^j = \emptyset \text{ for } i \neq j$$

is called a simplicial partition of Δ.

Definition 30. For a simplicial partition $\mathcal{P}_m = \{\Delta^1, \ldots, \Delta^m\}$ of a simplex Δ, where $v_1^k, \ldots, v_p^k$ denote the vertices of simplex Δ^k, the maximum diameter of a simplex in $\mathcal{P}_m$ is defined as

$$\delta(\mathcal{P}_m) := \max_{k\in\{1,\ldots,m\}} \max_{i,j\in\{1,\ldots,p\}} \|v_i^k - v_j^k\|.$$

Thus the diameter of the partition quantifies the distance between vertices in each simplex contained in the partition.

For a given partition $\mathcal{P}_m = \{\Delta^1, \ldots, \Delta^m\}$ of Δ^S and a homogeneous polynomial g defined as in (4.7), let us consider the set Q^k, which contains all the vertices of Δ^k, and moreover, let the set $\mathcal{I}^d_{\mathcal{P}_m}$ be defined as

$$\mathcal{I}^d_{\mathcal{P}_m} = \left\{ g \in \mathbb{R}^d[x] \,\middle|\, \begin{array}{l} G[q_1, q_2, \ldots, q_d] \geqslant 0, \\ \{q_1, q_2, \ldots, q_d\} \in Q^k, k = 1, \ldots, m \end{array} \right\}. \tag{4.8}$$

The following proposition shows that $\left\{\mathcal{I}^d_{\mathcal{P}_l}\right\}_{l\in\mathbb{N}}$ is a sequence of inner approximation which approximates the cone of copositive polynomials under the condition that the diameter of the simplicial partition goes to zero.

Proposition 4.3. *Let $\{\mathcal{P}_l\}_{l\in\mathbb{N}}$ be a sequence of simplicial partitions of Δ^S such that $\delta(\mathcal{P}_l) \to 0$. Then, we have*

$$\operatorname{int}\mathcal{C} \subseteq \bigcup_{l\in\mathbb{N}} \mathcal{I}^d_{\mathcal{P}_l} \subseteq \mathcal{C}, \ \ \textit{and hence } \mathcal{C} = \overline{\bigcup_{l\in\mathbb{N}} \mathcal{I}^d_{P_l}}.$$

Proposition 4.3 ensures that if we construct a hierarchy of linear programs by making the partition finer, we can find a rational Lyapunov function for homogeneous systems if the origin is exponentially stable.

To prove Proposition 4.3, we need the following two lemmas. The first one gives us sufficient conditions for copositivity and the second one a necessary condition for strict copositivity.

Lemma 4.4. *Consider the set of vectors, $V_{\mathcal{P}} := \{v_1, \cdots, v_p\}$, and let $\Delta = \operatorname{conv}\{v_1, \ldots, v_p\}$. If*

$$G[v_{i_1}, v_{i_2}, \ldots, v_{i_d}] \geqslant 0 \ \textit{for all} \ i_1, i_2, \ldots, i_d \in \{1, \ldots, p\}, \tag{4.9}$$

then $g(x) = G[\underbrace{x, x, \ldots, x}_{d \text{ times}}] \geqslant 0$ for all $x \in \Delta$.

Proof : For each point $x \in \Delta$, we can represent it in the affine hull of Δ by its uniquely determined barycentric coordinates $\lambda = (\lambda_1, \ldots, \lambda_p)$ with respect to Δ i.e.

$$x = \sum_{j=1}^{p} \lambda_j v_j \ \text{ with } \ \sum_{j=1}^{p} \lambda_j = 1.$$

This gives

$$g(x) = G[x, x, \ldots, x]$$

$$= G\Big[\sum_{i_1=1}^{p} \lambda_{i_1} v_{i_1}, \sum_{i_2=1}^{p} \lambda_{i_2} v_{i_2}, \dots, \sum_{i_d=1}^{p} \lambda_{i_d} v_{i_d}\Big]$$
$$= \sum_{i_1,i_2,\dots,i_d=1}^{p} G[v_{i_1}, v_{i_2}, \dots, v_{i_d}]\lambda_{i_1}\lambda_{i_2}\dots\lambda_{i_d}.$$

For $x \in \Delta$, we have $\lambda_i \geqslant 0$, and by the assumption (4.9), we get $g(x) \geqslant 0$ for all $x \in \Delta$. ♢

Lemma 4.5. *Let $g \in \mathbb{R}^d[x]$ be strictly copositive and homogeneous. Then there exists $\epsilon > 0$ such that for any finite simplicial partition $\mathcal{P}_m = \{\Delta^1, \dots, \Delta^m\}$ of Δ^S with $\delta(\mathcal{P}_m) \leqslant \epsilon$, we have $\forall k = 1, \dots, m$, and $i_1, i_2, \dots, i_d \in \{1, \dots, |Q^k|\}$,*

$$G[v^k_{i_1}, v^k_{i_2}, \dots, v^k_{i_d}] > 0,$$

where $v^k_1, v^k_2, \dots \in Q^k$, the set containing the vertices of the simplex Δ^k.

Proof : We have by assumption that g is strictly copositive which means that the tensor form $G[x^1, x^2, \dots, x^d]$ is strictly positive on the diagonal of $\Delta^S \times \Delta^S \times \dots \times \Delta^S \subset \mathbb{R}^{nd}$. By continuity, for every $x^i \in \Delta^S$, there exists $\epsilon_{x^i} > 0$ such that, for $j = 1, \dots, d$,

$$\|x^i - x^j\| \leqslant \epsilon_{x_i} \Rightarrow G[x^1, x^2, \dots, x^d] > 0.$$

Since G is uniformly continuous on the compact set $\Delta^S \times \dots \times \Delta^S$, it follows that $\epsilon := \inf_{x^i \in \Delta^S} \epsilon_{x^i}$ is strictly positive.

Let $\mathcal{P}_m = \{\Delta^1, \dots, \Delta^m\}$ be a simplicial partition of Δ^S with $\delta(\mathcal{P}_m) \leqslant \epsilon$. Let Δ^k with $k = 1, \dots, m$ be an arbitrary simplex, and v^k_i, $i = 1, \dots, |Q^k|$ arbitrary vertices of Δ^k. Then, for $i, j = 1, \dots, |Q^k|$, we have $\|v^k_i - v^k_j\| < \epsilon$, and therefore $G[v^k_{i_1}, v^k_{i_2}, \dots, v^k_{i_d}] > 0$ for all $i_1, i_2, \dots, i_d \in \{1, \dots, |Q^k|\}$, so the statement is proved. ♢

Proof : [Proof of Proposition 4.3] Take $g \in \operatorname{int} \mathcal{C}$, which means that g is strictly copositive. Lemma 4.5 implies that there exists $l_0 \in \mathbb{N}$, such that $g \in \mathcal{I}^d_{\mathcal{P}_{l_0}}$. Then $g \in \bigcup_{l \in \mathbb{N}} \mathcal{I}^d_{\mathcal{P}_l}$, and $\operatorname{int} \mathcal{C} \subseteq \bigcup_{l \in \mathbb{N}} \mathcal{I}^d_{\mathcal{P}_l}$.

Next, for proving $\bigcup_{l \in \mathbb{N}} \mathcal{I}^d_{\mathcal{P}_l} \subseteq \mathcal{C}$, we have to show that $\mathcal{I}^d_{\mathcal{P}_l} \subseteq \mathcal{C}$ for some $l \in \mathbb{N}$. So take $g \in \mathcal{I}^d_{\mathcal{P}_l}$ for some $l \in \mathbb{N}$. To prove $g \in \mathcal{C}$, it is sufficient to prove nonnegativity of $g(x)$ for $x \in \Delta^S$. Let us choose an arbitrary $x \in \Delta^S$, then $x \in \Delta^k$ for some $\Delta^k \in \mathcal{P}_l$. By direct use of Lemma 4.4, we get $g(x) = G[x, x, \dots, x] \geqslant 0$ for all $x \in \Delta^S$.

Lastly, since $\mathcal{C} = \overline{\operatorname{int} \mathcal{C}}$, we get $\mathcal{C} = \overline{\bigcup_{l \in \mathbb{N}} \mathcal{I}^d_{\mathcal{P}_l}}$. ♢

The pseudocode which allows us to compute Lyapunov function based on the method of discretization of simplices is given in Algorithm 1.

Algorithm 1: Discretization method in $\mathbb{R}^n_+$

Input: vector field f, maximum degree $d_{\max}$ (resp. $r_{\max}$) of the numerator (resp. denominator) of Lyapunov function, minimum diameter of the simplical partition ϵ.

Output: either a copositive Lyapunov function V, or an error message.

```
Δ^S ⟵ {x ∈ ℝ^n_+ | ‖x‖_1 = 1}
δ ⟵ 1
while δ > ϵ do
    Q^ℓ ⟵ vertices of simplex Δ^ℓ of a simplicial partition {Δ^1, ..., Δ^m}
      of Δ^S with diameter δ
    forall r = 0, 1, 2, ..., r_max do
        forall d = 1, 2, ..., d_max do
            forall ℓ = 1, 2, ..., m do
                h ⟵ homogeneous polynomial of degree d and n variables
                  with unknown coefficients
                forall i = 1, 2, ..., |Q^ℓ| do
                    x_i ⟵ v_i ∈ Q^ℓ
                    η_{x_i} ⟵ LCCP(f(x_i), I, T_{ℝ^n_+}(x_i))
                    s_k(x_i) ← −‖x_i‖_2^2 ⟨∇h(x_i), f(x_i) + η_{x_i}⟩ +
                    2r h(x_i) ⟨x_i, f(x_i) + η_{x_i}⟩, k = 0, ..., n
                end
                forall j = 1, 2, ..., binom(|Q^ℓ|, d) do
                    Q^ℓ_j ⟵ j^th combination of d vertices in Q^ℓ
                    Solve the LP problem in the coefficients of h
                      corresponding to the constraints H[q_1, ..., q_d] ⩾ 0 and
                      S_k[q_1, ..., q_d] ⩾ 0 where H, S_k denote the tensors of
                      h, s_k and {q_1, ..., q_d} ∈ Q^ℓ_j, k = 0, ..., n
                    if the LP problem is feasible then
                        return V(x) = h(x) / ‖x‖_2^{2r}
                    end
                end
            end
        end
    end
    δ ⟵ δ/2
end
display("Lyapunov function not found")
```

Now, we give examples for computing copositive polynomial Lyapunov functions for complementarity systems by implementing our discretization method. In our examples, the YALMIP toolbox in Matlab is used to input the LP problems and solve them with the conic solver MOSEK.

Example 8 (Quadratic Lyapunov function by the discretization method)**.** Consider system (3.1) with $f(x) = Ax$ and $A = \left[\begin{smallmatrix} -1 & -2 \\ -1 & -1 \end{smallmatrix}\right]$ and $K = \mathbb{R}^2_+$. We apply the discretization method on the standard simplex

$$\Delta^S := \left\{ x \in \mathbb{R}^2_+ |\ \|x\|_1 = 1 \right\}$$

of Algorithm 1 with $\epsilon = 0.1$. Starting with a degree 2 polynomial, we solve for its coefficients at the vertices of Δ^S. This procedure in Algorithm 1 is applied by partitioning all the simplices at each step by a factor of half and solving certain inequalities at the vertices of resulting simplices. For this example, we found

$$V(x) = x_1^2 + x_1 x_2 + x_2^2, \tag{4.10}$$

in four iterations, that is, Algorithm 1 terminates with $\delta = 1/16$.

Example 9 (Cubic Lyapunov function by the discretization method)**.** Consider system (3.1) with $K = \mathbb{R}^2_+$ and

$$f(x) = \begin{bmatrix} -x_1^2 - 2x_2^2 + x_1 x_2 \\ -x_1^2 - x_2^2 + 2x_1 x_2 \end{bmatrix}. \tag{4.11}$$

Applying the discretization method of Algorithm 1 by taking $\epsilon = 0.1$, after 4 iterations with $\delta = \frac{1}{16}$, we obtain

$$V(x) = x_1^3 + \frac{3}{2} x_1 x_2^2 + \frac{3}{2} x_2 x_1^2 + \frac{1}{2} x_2^3. \tag{4.12}$$

4.3.2 Polyhedral Conic Constraints

For what will follow here is that we will extend the previous ideas developed in Subsection 4.3.1 to study more general constraint sets (conic sets) and how our earlier algorithm can be adapted for these broader class of sets. We provide the discretization algorithm where a hierarchy of linear programs is constructed for the search of the desired function.

We let the following assumptions be imposed for the numerical computation of Lyapunov functions of the form (4.1).

(A1) The function $f : \mathbb{R}^n \to \mathbb{R}^n$ is locally Lipschitz continuous, it satisfies $f(0) = 0$, and it is homogeneous.

(A2) The constraint set is a closed polyhedral cone, that is

$$K = \{x \in \mathbb{R}^n \mid Fx \geqslant 0\}$$

for some matrix $F \in \mathbb{R}^{m\times n}$.

Polynomial inequalities:

Let $S_i := \{x \in K \mid (Fx)_i = 0\}$, denote the faces of K, for $i \in \{1, \dots, m\}$. We recall that finding a Lyapunov function V in the form (4.1) boils down to find h such that

$$h(x) \geqslant 0, \;\; \forall x \in K \tag{4.13a}$$
$$s_0(x) \geqslant 0, \;\; \forall x \in \mathrm{int}(K) \tag{4.13b}$$
$$s_i(x) \geqslant 0, \;\; \forall x \in S_i, \;\; i \in \{1, \dots, m\}\,. \tag{4.13c}$$

where

$$s_0(x) = -\|x\|_2^2 \langle \nabla h(x), f(x)\rangle + 2rh(x)\langle x, f(x)\rangle\,,$$
$$s_i(x) = -\|x\|_2^2 \langle \nabla h(x), f(x) + \eta_i\rangle + 2rh(x)\langle x, f(x) + \eta_i\rangle\,, \quad x \in S_i.$$

Algorithm description:

The method consists of discretizing simplex and constructing an inner approximation of cone-copositive polynomials with respect to cone K. The basic idea behind our algorithm for systems with conic sets, and homogeneous vector fields, is the same as the one developed in Subsection 4.3.1, which consists of checking inequalities (4.13) only for finitely many points over a simplex.

Because of the conic structure of K, we get two nice properties that are desirable for implementing an algorithm (the second property is seen in Subsection 4.3.1):

- Let $\mathcal{O}_j$, $j = 1, \dots, 2^n$, denote the orthants of $\mathbb{R}^n$, and let $K_j := \mathcal{O}_j \cap K$, $S_{ij} := \mathcal{O}_j \cap S_i$, for $i = 1, \cdots, m$. Then, each K_j and S_{ij} is a closed convex polyhedral cone.

- For a homogeneous polynomial $h \in \mathbb{R}^d[x]$ of degree d, it holds that

$$h(x) \geqslant 0, \; \forall\, x \in K \iff h(x) \geqslant 0, \; \forall\, x \in K, \; \|x\| = 1. \tag{4.14}$$

As a result of these properties, it is convenient to introduce the simplices obtained by intersecting the cones K_j or S_{ij} with the set $\{x \in \mathbb{R}^n \mid \|x\| = 1\}$, that is,

$$\Sigma_j := \{x \in K_j | \; \|x\|_1 = 1\}\,, \quad \Sigma_{ij} := \{x \in S_{ij} | \; \|x\|_1 = 1\}\,.$$

We next reduce the task to checking the inequalities on a finite number of points in each of the simplex Σ_j and Σ_{ij}.

Because of equivalence in (4.14), positivity of a homogeneous polynomial h is then expressed as

$$h(x) \geqslant 0 \text{ for all } x \in \bigcup_{j=1}^{2^n} \Sigma_j \cup \Big(\cup_{i=1}^{m} \Sigma_{ij} \Big).$$

Simplex Discretization: Our goal is to discretize the simplex Σ and obtain a hierarchy of linear inequalities with respect to the discretization points which allow us to find the desired function.

We use the same results mentioned in Subsection 4.3.1, which lead to the following algorithm for computing the cone-copositive Lyapunov function with respect to K, of the form (4.1), satisfying the inequalities (4.13). An algorithm, similar to Algorithm 1, can also be worked out for the generic cone case, and below we provide the main steps involved in this procedure:

1. Take $h \in \mathbb{R}[x]$, homogeneous of degree d, and fix $r \in \mathbb{N}$.
2. For each orthant $\mathcal{O}_j$, $j = 1, \dots, 2^n$, compute the sets $K_j = K \cap \mathcal{O}_j$ and for each $i = 1, \dots, m$, let $S_{ij} = S_i \cap \mathcal{O}_j$.
3. Identify the simplices $\Sigma_j \subset K_j$, and $\Sigma_{ij} \subset S_{ij}$ which are non-empty.
4. For each nonempty simplex $\Sigma \in \{\Sigma_j\} \cup \{\Sigma_{ij}\}$, $j = 1, \dots, 2^n$, $i = 1, \dots, m$,
   (a) Compute a simplicial partitioning of the set Σ, denoted by $\{\Delta^1, \dots, \Delta^{\bar{\ell}}\}$, and let Q^ℓ be the corresponding set of vertices of Δ^ℓ.
   (b) For each set of d vertices $\{q_1, \dots, q_d\} \in Q^\ell$, solve the LP problem in the coefficients of h corresponding to the constraints
   $$H[q_1, \dots, q_d] \geqslant 0, \quad \text{and } S_k[q_1, \dots, q_d] \geqslant 0 \tag{4.15}$$
   where H, S_k denote the tensors of h, s_k, $k = 0, \dots, m$ and $\{q_1, \dots, q_d\} \in Q^\ell$.
   (c) If (4.15) is infeasible, refine partition, and check (4.15) again.
5. Iterate by increasing d and r.

As an illustration of our algorithm, we compute a quadratic Lyapunov function using the discretization method.

Example 10. Consider system (3.1) with $f(x) = Ax$ and $A = \left[\begin{smallmatrix} -1 & -2 \\ -1 & -1 \end{smallmatrix}\right]$ and $K = \{x \in \mathbb{R}^2 \,|\, Cx \geqslant 0\}$, with $C = \left[\begin{smallmatrix} -0.25 & 1 \\ 1 & -0.25 \end{smallmatrix}\right]$. We apply the discretization method on the three simplices that correspond to $K_j = K \cap \mathcal{O}_j$,

$$\Sigma_1 = \text{conv}([1,0]^\top, [0,1]^\top), \Sigma_2 = \text{conv}([1,0]^\top, [0.8,-0.2]^\top)$$
$$\Sigma_3 = \text{conv}([0,1]^\top, [-0.2,0.8]^\top),$$

and the two simplices which correspond to the two faces of the cone reduce to a singleton, that is,

$$\Sigma_{12} = [0.8, -0.2], \quad \Sigma_{23} = [-0.2, 0.8].$$

Solving the resulting inequalities, we obtain

$$V(x) = 2.9x_1^2 + x_1x_2 + x_2^2, \tag{4.16}$$

which indeed satisfies the inequalities in (4.13).

4.4 SOS Method

A commonly employed tool for checking the positivity of a polynomial is to write it in the form of a sum-of-squares of other polynomials. While testing positivity is a computationally hard problem, the question of finding an SOS decomposition of a polynomial is actually a semidefinite program [134, 53, 128]. The crux of such ideas can be found in [127] and its application to copositivity is sketched in [128].

While computing Lyapunov functions V using inequalities (4.5), we notice that we are faced with two problems, which prevent us from using conventional SOS techniques. The first problem is that there is no readily available Positivstellensatz for unbounded domains like cones. The second problem is that our Lyapunov functions are not necessarily SOS because a SOS polynomial is in particular positive definite but our systems require searching for a Lyapunov function beyond positive definite functions. To overcome these problems, we study a technique for finding polynomials that satisfy (4.5).

4.4.1 Conic Constraints with Pólya's Positivstellensatz

We assume here that $K = \mathbb{R}^n_+$. The basic idea is to get rid of the constraint $x \in \mathbb{R}^n_+$. We let $x_i = y_i^2$, $i \in \{1, \cdots, n\}$ be the change of variable where y^2 is the short-hand for $(y_1^2, \ldots, y_n^2)$. Clearly we have

$$h(x) > 0, \;\; \forall x \in \mathbb{R}^n_+ \iff h(y^2) > 0, \;\; \forall y \in \mathbb{R}^n.$$

Then, the inequalities (4.5) are rewritten as follows

$$P_h(y) := h(y^2) > 0, \;\; \forall y \in \mathbb{R}^n \tag{4.17a}$$
$$P_{s_0}(y) := s_0(y^2) > 0, \;\; y_i \neq 0, \forall i \tag{4.17b}$$
$$P_{s_i}(y) := s_i(y^2) > 0, \;\; y_i = 0, \;\; i \in \{1, ..., n\} \tag{4.17c}$$

where h, s_o, s_i are homogeneous polynomials.

Next, we define the polynomials

$$P_h^{(d)}(y) := \|y\|^{2d} P_h(y) \tag{4.18a}$$
$$P_{s_0}^{(d)}(y) := \|y\|^{2d} P_{s_0}(y) \tag{4.18b}$$
$$P_{s_i}^{(d)}(y) := \|y\|^{2d} P_{s_i}(y) \tag{4.18c}$$

where d is an integer. It is obvious that inequalities (4.17) are satisfied if and only if

$$P_h^{(d)}(y) > 0, \;\; \forall y \in \mathbb{R}^n \tag{4.19a}$$
$$P_{s_0}^{(d)}(y) > 0, \;\; y_i \neq 0, \forall i \tag{4.19b}$$
$$P_{s_i}^{(d)}(y) > 0, \;\; y_i = 0, \;\; i \in \{1, ..., n\}\,. \tag{4.19c}$$

Proposition 4.6. *For the homogeneous copositive functions h, s_0 and s_i, $i \in \{1, \ldots, n\}$, there exists $d \in \mathbb{N}^*$ sufficiently large such that the polynomials $P_h^{(d)}$, $P_{s_0}^{(d)}$ and $P_{s_i}^{(d)}$ are SOS.*

Proof : We will carry out the proof only for h and it will be similar for the other polynomials. Let

$$K_n^d := \left\{ h \in \mathbb{R}[x] | \; P_h^{(d)} \;\text{ SOS} \right\}$$
$$C_n^d := \left\{ h \in \mathbb{R}[x] | \; P_h^{(d)} \;\text{ has positive coefficients} \right\}.$$

We notice that $C_n^d \subseteq K_n^d$ because if $P_h^{(d)}(y)$ has only positive coefficients then the polynomial $P_h(y) = h(y^2)$ is SOS and since $P_h(y)$ is multiplied by $\|y\|^{2d}$, it follows that $P_h^{(d)}(y)$ is SOS. So we just need to prove that $P_h^{(d)}(y)$ has positive coefficients.

The copositivity of h is equivalent to the positivity of P_h. And since h is homogeneous, this will be also equivalent to the positivity of P_h on the unit ball which means $P_h(y) > 0$, $\forall y \in \mathbb{R}^n$, $\sum_{i=1}^n y_i^2 = 1$. By substituting y_i^2 by z_i, we obtain $P_h(z) > 0$, $\forall z \geqslant 0$, $\sum_{i=1}^n z_i = 1$.

Let us now recall Pólya's Theorem, see [105] and [133] for the proof.

Theorem 4.7. (Pólya's Theorem) *Let $f \in \mathbb{R}[x]$ be homogeneous. If $f > 0$ on the simplex $\{x \geqslant 0 | \ \sum_{i=1}^{n} x_i = 1\}$, then there exists a sufficiently large $l \in \mathbb{N}$ for which the polynomial $(\sum_{i=1}^{n} x_i)^l f(x)$ has all its coefficients nonnegative.*

Applying this Theorem to the homogeneous polynomial $P_h(z)$, we obtain that for sufficiently large $d \in \mathbb{N}$, all the coefficients of the polynomial $P_h^{(d)}(z) = (\sum_{i=1}^{n} z_i)^d P_h(z)$ are positive. Then $P_h^{(d)}$ is SOS in view of the fact that $C_n^d \subseteq K_n^d$. ♢

To sum up this section, the foregoing result allows us to write an algorithm to compute the polynomials $P_h^{(d)}$, $P_{s_0}^{(d)}$ and $P_{s_i}^{(d)}$ in the form of SOS. The result is then used to get a homogeneous copositive Lyapunov function.

The pseudocode based on SOS decomposition is given in Algorithm 2 given below. In addition to the procedure outlined in this subsection, we use the YALMIP command `solvesos` to model and solve the SOS optimization problem: It computes the unknown coefficients h_i that we associate with the polynomial $h \in \mathbb{R}^q[x]$, while minimizing $\sum h_i^2$, under the constraint that $P_h^{(d)}(x)$, $P_{s_k}^{(d)}(x)$, $k = 0, \dots, n$ must be SOS for some $d \in \mathbb{N}$.

Now, we give examples for computing copositive polynomial Lyapunov functions for complementarity systems by implementing our SOS method. In our examples, the YALMIP toolbox in Matlab is used to input the SOS optimization problems and solve them with the conic solver MOSEK.

Example 11 (Quadratic Lyapunov function by SOS method)**.** Consider system (3.1) with $f(x) = Ax$ and $A = \begin{bmatrix} -1 & 10 \\ 0 & -2 \end{bmatrix}$ and $K = \mathbb{R}_+^2$. Following Algorithm 2, we express positivity condition on desired polynomials by requiring them to be SOS, and use the YALMIP command `solvesos` which calls a semidefinite solver to yield the desired coefficients. We obtain

$$V(x) = 0.1x_1^2 + 0.1916x_1x_2 + 1.1137x_2^2. \tag{4.21}$$

Example 12 (A copositive quadratic Lyapunov function that is not positive definite)**.** Consider system (3.1) with $f(x) = Ax$ and $A = \begin{bmatrix} -1 & -3 & -2 \\ -5 & 1 & -1 \\ 3 & -10 & -2 \end{bmatrix}$ and $K = \mathbb{R}_+^3$. Applying the SOS method of Algorithm 2 we obtain

$$V(x) = 2.3234x_1^2 + 3.6729x_1x_2 + 1.7352x_2^2 + 1.1273x_1x_3 + 2.6769x_2x_3 + 1.2820x_3^2. \tag{4.22}$$

This polynomial is not positive definite since one of the eigenvalues of the corresponding matrix is negative. The unit level set of this polynomial Lyapunov function is shown in Figure 4.1.

Algorithm 2: SOS Approximations of Lyapunov Functions

Input: vector field f, maximum degree $q_{\max}$ (resp. $r_{\max}$) of the numerator (resp. denominator) of Lyapunov function, maximum degree $d_{\max}$ for expressing homogeneous polynomials
Output: either a copositive Lyapunov function V, or an error message.
forall $r = 1, 2, ..., r_{\max}$ **do**
  forall $q = 1, 2, ..., q_{\max}$ **do**
    1. $h \longleftarrow$ homogeneous polynomial of degree q and n variables with unknown coefficients h_i
      $s_0(x) = -\|x\|_2^2 \langle \nabla h(x), f(x) \rangle + 2rh(x) \langle x, f(x) \rangle$
    forall $k = 1, 2, ..., n$ **do**
      $\eta_k(x) \longleftarrow \mathrm{LCCP}(f(x), I, \mathcal{T}_{\mathbb{R}^n_+}(x))$, for $x \in S_k$
      $s_k(x) = -\|x\|_2^2 \langle \nabla h(x), f(x) + \eta_k \rangle$
        $+2rh(x) \langle x, f(x) + \eta_k \rangle$
    end
    2. $P_h(y) \longleftarrow h(y^2)$
      $P_{s_k}(y) \longleftarrow s_k(y^2)$, $k = \{0, \ldots, n\}$
    3. **forall** $d = 0, ..., d_{\max}$ **do**
      $P_h^{(d)}(y) \longleftarrow \|y\|^{2d} P_h(y)$
      $P_{s_k}^{(d)}(y) \longleftarrow \|y\|^{2d} P_{s_k}(y)$, $k = \{0, \ldots, n\}$
      SOLVESOS $\Big(\mathrm{SOS}(P_h^{(d)}), \mathrm{SOS}(P_{s_0}^{(d)}), \mathrm{SOS}(P_{s_k}^{(d)}),$
        $\sum_i h_i^2, [\,], [h_i]\Big)$
      4. **if** *the SOS program is feasible* **then**
        return $V(x) = \frac{h(x)}{\|x\|_2^{2r}}$
      end
    end
  end
end
display("Lyapunov function not found")

4.4.2 Generic Constraints

For what follows in this subsection, we study more general constraint sets (semi-algebraic sets). We use the SOS decomposition where a hierarchy of semidefinite programs is constructed to compute Lyapunov functions as a SOS polynomial.

Let the constraint sets in this subsection be denoted by $\mathcal{S}$ and let the following assumption hold.

Assumption 2. The set $\mathcal{S}$ is convex, it contains the origin $\{0\}$, and it is

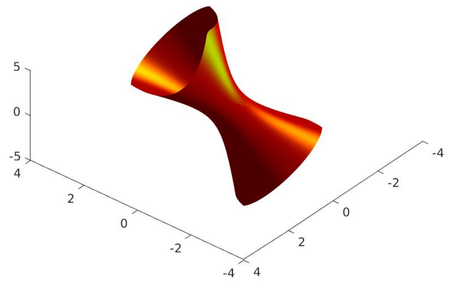

Figure 4.1 – Non-convex unit level set of the quadratic Lyapunov function V in Example 12.

described as

$$\mathcal{S} := \{x \in \mathbb{R}^n \,|\, g_i(x) \geqslant 0, i = 1, \dots, M\} \tag{4.23}$$

for some continuously differentiable functions $g_i : \mathbb{R}^n \to \mathbb{R}$. Furthermore, the gradients $\nabla g_i(x) \neq 0$ in some neighborhood of the set $\{x \in \mathbb{R}^n \,|\, g_i(x) = 0\}$.

We present a numerical approach to deal with sets of the form $\mathcal{S}$ in (4.23) where g_i are not necessarily linear.

One of the difficulties in checking the Lyapunov conditions is that the corresponding inequality has to be checked for $\eta \in -\mathcal{N}_{\mathcal{S}}(x)$ for all $x \in \mathcal{S}$, which is not feasible in general. In the previous section, we only need to check the inequalities at finitely many points at which η could be obtained as a solution to an optimization problem, but this works only under the conic structure of $\mathcal{S}$. For more general sets without conic structure, it is of interest to obtain Lyapunov functions without having to solve for η.

One way to avoid computation of η is to impose certain assumption on the gradient of Lyapunov function and provide sufficient conditions which can be checked independently of η. We then use these conditions to compute Lyapunov functions using a semidefinite program based on SOS decomposition.

Sufficient Conditions:

With the aforementioned motivation, we first provide a set of inequalities as a sufficient condition for checking asymptotic stability, which are independent of η and use the information of the gradients of the generating functions g_i, $i = 1, \dots, M$.

Proposition 4.8 (Sufficient Conditions)**.** *Consider the system*

$$\dot{x} = f(x) + \eta \tag{4.24a}$$

$$\eta \in -\mathcal{N}_{\mathcal{S}}(x), \tag{4.24b}$$

under Assumption 2. Assume that there exists a continuously differentiable $V(\cdot)$ that satisfies the following conditions:

- *$V(0) = 0$, and $\underline{\alpha}(\|x\|) \leqslant V(x) \leqslant \overline{\alpha}(\|x\|)$ for every $x \in \mathcal{S}$, and some class $\mathcal{K}$ functions $\underline{\alpha}$, $\overline{\alpha}$.*
- *$\langle f(x), \nabla V(x)\rangle \leqslant -\alpha(\|x\|)$, for every $x \in \mathcal{S}$, and some positive definite function α.*
- *If x is such that $g_i(x) = 0$, for some $i \in \{1, \cdots, M\}$, then*

$$\langle \nabla g_i(x), \nabla V(x)\rangle \leqslant 0.$$

Then V is a Lyapunov function for system (4.24) and the origin is globally asymptotically stable.

Proof : Consider a function $V \in \mathcal{C}^1(\mathbb{R}^n, \mathbb{R})$ that satisfies the listed conditions. We show that these conditions guarantee that V is Lyapunov function for system (4.24) when $\mathcal{S}$ is described by (4.23) under assumption 2. To see this, we first introduce the set $J(x)$ which defines the set of active constraints, that is,

$$J(x) = \{i \in \{1, \dots, M\} \,|\, g_i(x) = 0\}. \tag{4.25}$$

Then, the set-valued mapping $\mathcal{N}_{\mathcal{S}}$ is defined as

$$\mathcal{N}_{\mathcal{S}}(x) = \begin{cases} 0, & \text{if } x \in \text{int}(\mathcal{S}), \\ \left\{ \sum_{j\in J(x)} \lambda_j \nabla g_j(x); \;\; \lambda_j \leqslant 0 \right\}, & \text{if } J(x) \neq \emptyset, \\ \emptyset, & \text{if } x \notin \mathcal{S}. \end{cases}$$

Thus, if $x \in \text{int}(\mathcal{S})$, then $\eta = 0$, and

$$\langle \nabla V(x), f(x)\rangle \leqslant -\alpha(\|x\|), \quad x \in \text{int}(\mathcal{S}).$$

When x is such that $J(x) \neq \emptyset$, we have that

$$\eta = -\sum_{j\in J(x)} \lambda_j \nabla g_j(x), \;\; \text{for some} \;\; \lambda_j \leqslant 0,$$

which follows

$$\langle \eta, \nabla V(x)\rangle = -\left\langle \sum_{j\in J(x)} \lambda_j \nabla g_j(x), \nabla V(x) \right\rangle \leqslant 0,$$

since we have $\langle \nabla g_j(x), \nabla V(x) \rangle \leqslant 0$, for $j \in \{1, \cdots, M\}$ and $\lambda_j \leqslant 0$. Thus, for each $x \in \mathcal{S}$, and $\eta \in -\mathcal{N}_{\mathcal{S}}(x)$, we have shown that

$$\langle \nabla V(x), f(x) + \eta \rangle \leqslant -\alpha(\|x\|)$$

which completes the proof. ♢

Sum-of-Squares Decomposition:

We now present a numerical approach to compute the Lyapunov function which satisfies the conditions of Proposition 4.8. The three conditions can be actually listed as positivity constraints on the function V and its gradient ∇V. One way to ensure the positivity is to write the function as a sum-of-squares, which boils down to a semidefinite program. The basic idea behind computing the Lyapunov function for system (4.24) under Assumption 2 is to find a Lyapunov function where the three positivity constraints in Proposition 4.8 can be written as sum-of-squares.

We focus our attention on convex semi-algebraic sets, which are basically described by the intersection of the sublevel sets of finitely many polynomial inequalities. That is, in the definition of the set $\mathcal{S}$ in (4.23), we introduce the following assumption:

Assumption 3. The set $\mathcal{S}$ in (4.23) is compact and the function $g_i \in \mathbb{R}[x]$, for every $i = 1, \ldots, M$.

For such sets, we can implement the following procedure to compute V in the form of sum-of-squares:

1. Let $V \in \mathbb{R}[x]$ of degree $d \in \mathbb{N}$.
2. For each $x \in \mathcal{S}$, let
$$V(x) = \sigma_0(x) + \sum_{i=1}^{M} \sigma_i(x) g_i(x).$$
for some SOS polynomials $\sigma_0, \cdots, \sigma_M$.
3. For each $x \in \mathcal{S}$, if $J(x) = \emptyset$, let
$$-\langle \nabla V(x), f(x) \rangle = \chi_0(x) + \sum_{i=1}^{M} \chi_i(x) g_i(x).$$
for some SOS polynomials $\chi_0, \cdots, \chi_M$.

4. For each $x \in \mathcal{S}$, $J(x) \neq \emptyset$, let, for each $j \in J(x)$,

$$-\langle \nabla V(x), \nabla g_j(x) \rangle = \chi_{j,0}(x) + \sum_{i \notin J(x)} \chi_{j,i}(x) g_i(x) + \sum_{i \in J(x)} \varphi_{j,i}(x) g_i(x), \tag{4.26}$$

for some SOS polynomials $\chi_{j,i}$, whereas $\varphi_{j,i} \in \mathbb{R}[x]$ are not necessarily sum-of-squares.

5. Iterate by increasing d, the degree of V.

An important question to consider, in the implementation of this algorithm, is whether one can always find SOS decomposition of a positive polynomial on a semialgebraic set. One possible answer to this question comes from Putinar's Positivstellensatz theorem (Theorem 2.6).

A direct application of this result to our problem suggests that, if system (4.24) admits a polynomial Lyapunov function which satisfies the conditions in Proposition 4.8, then the hierarchy of semidefinite programs constructed in our algorithm (by increasing the degree d of the search function) is guaranteed to find us a Lyapunov function.

To compute V with such a parameterization, one may use the YALMIP toolbox in Matlab to solve the underlying semidefinite program.

Example 13. As an illustration of the foregoing algorithm, we consider an academic example in $\mathbb{R}^2$ with two constraints. Let $g_1(x) = x_1 - x_2^2$, and $g_2(x) = 1 - x_1$. These two functions describe the compact semi-algebraic set $\mathcal{S}$ in (4.24), and we take vector field f to be

$$f(x) = \begin{pmatrix} -x_1^2 \\ 0 \end{pmatrix}.$$

Based on the algorithm, a Lyapunov function for this example is

$$V(x) = x_1^2 + x_2^2,$$

which indeed satisfies the conditions listed in Proposition 4.8. Note that the system without constraints, that is, $\dot{x} = f(x)$ is only stable, but not asymptotically stable. However, the constrained system is asymptotically stable since, within the set $\mathcal{S}$, $x_1 = 0$ implies $x_2 = 0$.

The examples seen in Sections 4.3 and 4.4 just provide an illustration of two classes of algorithms primarily used for checking positivity or copositivity of polynomials, and how they can be used for computing Lyapunov functions with constrained dynamics. The survey article [23] provides an overview of these methods, along with some other techniques, which appear in general in the literature on checking copositivity. Further questions such as using other algorithms or comparing computational complexity of different methods require further investigation.

5

Ensemble Approximations for Constrained Systems

In the theory of dynamical systems, studying the evolution of state trajectories, both qualitatively and quantitatively, is a common occurrence. For ordinary differential equations, with a *fixed* initial condition described by a point in the finite-dimensional vector space, the tools for analyzing the behavior of trajectories are widely available. However, for many applications, it is of interest to consider the evolution of dynamical systems when the initial condition is described by distribution of mass over some set in the state space. This chapter explores this latter direction for a particular class of nonsmooth dynamical systems. If we consider a probability measure to describe the distribution of the initial conditions of a dynamical system, then the time evolution of this initial probability measure with respect to underlying dynamics is the object of our interest.

5.1 Overview

For an autonomous dynamical system described by an ordinary differential equation (ODE) with Lipschitz continuous vector field, the time evolution of this measure is described by a linear partial differential equation (PDE) called the Liouville equation or the continuity equation, see e.g. [152, Section 5.4]. The solution to the Liouville equation, that is the probability measure describing the distribution at time t, is the pushforward or image measure of the initial probability measure through the flow map at time t. Lipschitz continuity of the vector field ensures that the flow map of the ODE is invertible, which in turn ensures that the pushforward measure is the unique solution to the Liouville equation. This approach of associating the continuity equation with finite dimensional ODEs has found relevance in numerical optimal control [103, 88] as well as in several control-theoretic problems [16, 32, 31].

When the vector field is not Lipschitz continuous, then the study of the evolution of the initial distribution is more involved. The first occurrence of continuity equations corresponding to nonsmooth ODEs occurs in [68]. Continuity equations corresponding to one-sided Lipschitz vector fields have been studied in [25, 26]. In [11], the authors consider less regular ODEs and study uniqueness of solutions for (Lebesgue) almost-all initial conditions by using the Liouville equation.

The dynamical systems for which we want to study the evolution of probability measures (describing the distribution of states) are the constrained systems described by the differential inclusion

$$\dot{x} \in f(x) - \mathcal{N}_{\mathcal{S}}(x) \tag{5.1}$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$ is a Lipschitz continuous function and $\mathcal{S} \subset \mathbb{R}^n$ is a time-varying closed convex set.

For the constrained system (5.1), when the initial condition $x(0) \in \mathcal{S}$ is given, the question of existence and uniqueness of solution to system (5.1) has already been well-established in the literature, and the origins of such works can be found in [119], see [72] for a recent exposition. However, if we consider the initial conditions described by a probability measure, then the evolution of this measure under the dynamics of (5.1) has received very little attention in the literature. One can study such problems by considering stochastic versions of (5.1) by adding a diffusion term on the right-hand side. Such systems first came up in the study of variational inequalities arising in stochastic control [18], and in the literature, we can find results on existence and uniqueness of solutions in appropriate function space. In [51], this is done by considering Yosida approximations of the maximal monotone operator, whereas [19] provides a proof based on time-discretization of system (5.1). These approaches have been generalized for prox-regular set $\mathcal{S}$ in [20], and the case where the drift term contains Young measures [49, 50]. One could also, in principle, formulate a partial differential equation with set-valued elements and study the solutions of such equations under appropriate hypothesis, which is the case in [24] but it is not clear how to derive the corresponding set-valued partial differential equation for system (5.1) and whether the resulting inclusion would satisfy the necessary hypothesis for well-posedness. Different from these approaches, and inspired by the fact that the evolution of a probability measure for single-valued dynamical system is described by the Liouville equation, it is natural to ask whether the evolution of a probability measure under the dynamics of system (5.1) can be studied using the Liouville equation. To the best of authors' knowledge, this approach has only been adopted in [67], where the authors consider system of form (5.1) without the drift term $f(\cdot)$. Since the right-hand side of (5.1) is set-valued, it is not immediately clear how the divergence term in the Liouville equation is to be interpreted. In [67], the authors consider approximations to the solutions of Liouville equation associated with (5.1), which are similar to time-stepping algorithm. That is, a time-discretization technique is introduced which is based on projecting the density function on to the constraint set with respect to the Wasserstein metric.

In this chapter, we consider a different route for computing the approximate solution of system (5.1) in the space of probability measures. Inspired by the concepts presented in [11], our basic idea is to consider Lipschitz approximations of system (5.1). The particular approximations that we work with are the ones obtained by *Yosida-Moreau* regularization and are parameterized by a positive scalar converging to zero. We can then associate a single-valued Liouville equation to each of these approximants, and establish convergence of the resulting sequence of measures. Unlike [67], our approach for numerically solving the Liouville equation does not depend upon discretization in time, or space for that matter. Instead, we use functional discretization: we choose a family of test functions (the monomials) on which the evolution measure and the associated moments are then approximated numerically by a hierarchy of semidefinite programs. Furthermore, we also show that the support of the sequence of measures converges (with respect to the Hausdorff distance) to the support of the pushforward measure for the nonsmooth system. These analytical results allow us to get an approximation of the actual solution.

Since the pushforward measure, at each time instant t, is an infinite-dimensional object, it can be challenging to approximate it numerically. One possibility – that we do not explore here – could to use Monte-Carlo probabilistic algorithms. Instead, we investigate a purely deterministic approach: in order to get a quantitative measure of the distribution of state at any time instant, which involves building a hierarchy of moments defined by the action of a finite Borel measure on polynomial test functions, and encoding the positivity constraints on moment matrix by using sum-of-squares (SOS) decomposition. This technique, called moment-SOS hierarchy [86] has been used in a successful manner in several engineering problems and it is based on the decomposition into sum-of-squares of nonnegative polynomials and it encodes the moments of nonnegative measures on compact basic semi-algebraic set. The SOS method, seen in Chapter 4, is only a special case, that focuses on the dual only, of the moment-SOS hierarchy. For our purposes, the moment-SOS hierarchy allows us to approximate numerically the moments (up to some finite order) associated with the pushforward measure. Also, using the recent developments on approximating the support of a measure with the Christoffel-Darboux kernel [104], we can approximate the support of the pushforward measure, and hence the trajectories corresponding to a certain initial distribution.

This chapter is structured as follows. In Section 5.2, we formalize the problem and introduce the basic mathematical elements necessary for doing so. In Section 5.3, we construct Lipschitz approximations of our initial dynamical system. In Sections 5.4 and 5.5, we study certain properties of the sequence of measures associated with approximations constructed in Section 5.3. Numerical aspects for approximating the moments, and support, of the probability measure describing the evolution of system dynamics are also discussed in Sections 5.4 and 5.5. We illustrate our results with the help of an academic example in Section 5.6.

5.2 Problem Formulation

5.2.1 Evolution of Ensembles

Let us consider the time-varying ODE

$$\dot{z}(t) = g(t, z(t)), \quad z(0) = z_0, \tag{5.2}$$

over a given time interval $[0, T]$, where $g : [0, T] \times \mathbb{R}^n \to \mathbb{R}^n$ is a given vector field and $z(t) \in \mathbb{R}^n$ is the state. For each $t \in [0, T]$, let us consider the flow map $G_t : \mathbb{R}^n \to \mathbb{R}^n$, so that the mapping $z_0 \mapsto G_t(z_0)$ provides the value of state trajectory of (5.2) at time t, and moreover it satisfies

$$\partial_t G_t(z_0) = g(t, G_t(z_0)), \quad G_0(z_0) = z_0, \quad (t, z_0) \in [0, T] \times \mathbb{R}^n. \tag{5.3}$$

In this chapter, we consider the evolution of dynamical systems when the initial condition is defined probabilistically. In particular, we use the notation $z(0) \sim \xi_0$ to mean that $z(0)$ is a random variable whose law is a given probability measure, or density function $\xi_0 \in \mathcal{P}(\mathbb{R}^n)$, where $\mathcal{P}(\mathcal{S})$ denotes the set of probability measures supported on $\mathcal{S}$.

This model allows to capture an initial spatial distribution of particles. To define the corresponding density function at time $t \geqslant 0$, denoted by $\xi_t \in \mathcal{P}(\mathbb{R}^n)$, we consider the pushforward or image measure of ξ_0 through the flow map $G_t(\cdot)$. That is, let

$$\xi_t := G_t \sharp \xi_0, \tag{5.4}$$

so that, for every Borel subset $B \subset \mathbb{R}^n$, it holds that

$$\xi_t(B) = \xi_0(\{z \in \mathbb{R}^n : G_t(z) \in B\}).$$

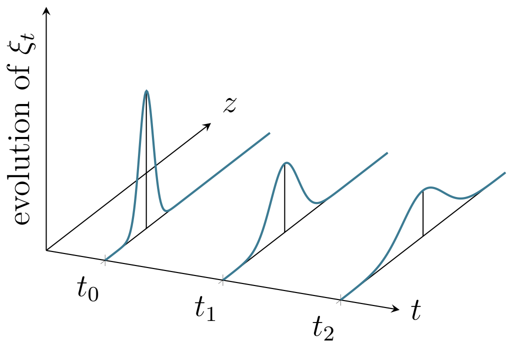

Figure 5.1 – Evolution of probability measure ξ_t w.r.t. time and space.

The evolution of ξ_t is described by the following PDE, called the continuity or *Liouville* equation:

$$\partial_t \xi_t + \operatorname{div}(\xi_t g) = 0, \tag{5.5}$$

with the initial condition:

$$\xi|_{t=0} = \xi_0. \tag{5.6}$$

The Liouville equation (5.5) should be understood in the sense of distributions, i.e.

$$\int_{\mathbb{R}^n} \left(\partial_t v(t,z) + \partial_z v(t,z) \cdot g(t,z)\right) d\xi_t(z) = 0$$

for all continuously differentiable functions v from $\mathbb{R}_+ \times \mathbb{R}^n$ to $\mathbb{R}$. The equivalence between the solutions of ODE (5.2) and PDE (5.5), is established in the following result, see e.g. [152, Theorem 5.34]:

Theorem 5.1. *For each $t \in [0,T]$, let $G_t : \mathbb{R}^n \to \mathbb{R}^n$ be a diffeomorphism so that (5.3) holds. Given $\xi_0 \in \mathcal{P}(\mathbb{R}^n)$, let ξ_t be defined as in (5.4). Then, ξ_t is the unique solution of the Liouville equation (5.5)-(5.6) over the time interval $[0,T]$.*

The importance of the Liouville PDE relies on its linearity in the probability measure ξ_t, whereas the Cauchy ODE is nonlinear in the state trajectory $z(t)$. This PDE governs the time evolution of a measure transported by the flow of a nonlinear dynamical system. The nonlinear dynamics is then replaced by a linear equation on measures. It is important to note that, in Theorem 5.1, the equivalence is established under the assumption that G_t is a diffeomorphism for each $t \in [0,T]$, which in particular requires that the flow map G_t is invertible. ODEs with Lipschitz vector fields have this property, but when the vector field is not Lipschitz continuous in state variable, the backward invertibility assumption may not hold, or the flow map G_t may itself not be uniquely defined.

5.2.2 Ensembles of Constrained System

In this chapter, we are interested in studying a class of dynamical systems described by the variational inequalities

$$\dot{z}(t) \in f(t, z(t)) - \mathcal{N}_{\mathcal{S}(t)}(z(t)), \quad z(0) \sim \xi_0, \tag{5.7}$$

over an interval $[0, T]$ for some given $T > 0$, where $f : [0, T] \times \mathbb{R}^n \to \mathbb{R}^n$ is a given vector field, $\mathcal{S} : [0, T] \rightrightarrows \mathbb{R}^n$ a compact and convex-valued mapping.

We emphasize that, in (5.7), $\xi_0 \in \mathcal{P}(\mathcal{S}(0))$ is a probability measure that specifies the distribution of the initial state. For each $t \in [0, T]$, let us denote the flow map by $F_t : \mathcal{S}(t) \to \mathcal{S}(t)$, so that $z_0 \mapsto F_t(z_0)$ is the value at time t of the state trajectory of (5.7) with $z(0) = z_0$. Given this random initial condition, the state at each time t can also be interpreted as a random variable in $\mathcal{S}(t)$, i.e. $z(t) \sim \xi_t \in \mathcal{P}(\mathcal{S}(t))$ defined by $\xi_t := F_t \sharp \xi_0$. However, unlike Lipschitz continuous ODEs, the mapping F_t is not invertible in general. An example illustrating this fact is given next.

Example 14 (Flow map not invertible)**.** Let $f(z) = Az$ with $A = \left[\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right]$ and $\mathcal{S} = \mathbb{R}^2_+$ and let z_0 be a given initial condition, with angle θ_0. For $t \leqslant \theta_0$, we have $z(t) = F_t(z_0) = e^{At} z_0 = \left[\begin{smallmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{smallmatrix}\right] z_0$. And for $t \geqslant \theta_0$, we have $z(t) = [|z_0| \ 0]^\top$. For example if $z_0 = [1 \ 1]^\top$, it holds $\theta_0 = \frac{\pi}{4}$ and then for $t \geqslant \theta_0$, we have $z(t) = \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = [\sqrt{2} \ 0]^\top$. The flow map reads

$$z(t) = F_t(z_0) = \begin{cases} e^{At} z_0 & \text{if } t \leqslant \theta_0 \\ [|z_0| \ 0]^\top & \text{if } t \geqslant \theta_0. \end{cases}$$

Indeed, as we can observe, the flow map is not invertible since given a state $z(t)$ for a given time $t \geqslant \theta_0$, it is not possible to retrieve the initial condition z_0.

As a consequence of Example 14, it is seen that the flow map associated with dynamical system (5.7) is not necessarily invertible, and hence the conditions of Theorem 5.1 are not satisfied in general for such systems. On the other hand, for each $t \in [0, T]$, the forward flow map F_t is well-defined and therefore the solution $\xi_t := F_t \sharp \xi_0$ exists and is uniquely defined. However, it is not possible to write down the evolution equation for ξ_t, like Liouville equation for smooth ODEs, due to nonsmooth set-valued dynamics in (5.7). Recent literature in this direction deals with such problems, either by studying partial differential equations with set-valued mappings [24] or by introducing an approximation based on time discretization [67]. In this chapter, our goal is to find alternate methods based on functional discretization with monomial basis to approximate the measure ξ_t and propose computational algorithms to calculate such approximations numerically.

5.2.3 Problem Formulation

We consider the dynamical system (5.7) with flow map $F_t : \mathbb{R}^n \to \mathbb{R}^n$. For a given $\xi_0 \in \mathcal{P}(\mathcal{S}(0))$, since there is no direct derivation of the PDE for characterizing the evolution of $\xi_t := F_t \sharp \xi_0$, we compute an approximation of ξ_t as follows:

- Construct a sequence of ODEs with Lipschitz continuous right-hand sides which approximate the solution of ODE (5.7) for a fixed initial condition. This construction is based on a regularization of (5.7), and results in a sequence parameterized by a scalar $\lambda > 0$.
- Exploit the regularity of the approximating ODE to construct a sequence of measures $\xi_t^\lambda := F_t^\lambda \sharp \xi_0$.
- When λ tends to 0, prove that ξ_t^λ converges to $\xi_t := F_t \sharp \xi_0$ in the weak-star topology. In particular, all finite order moments of ξ_t^λ converge to the moments of ξ_t.
- When λ tends to 0, prove the convergence of the support of ξ_t^λ to the support of ξ_t in the Hausdorff metric.

From a computational viewpoint, the by-product of the above results is that, for a fixed $\lambda > 0$, one can invoke efficient numerical methods for computing moments associated with the probability measure ξ_t^λ and the support of ξ_t^λ. This allows us to compute an approximation of the moments and support of ξ_t associated with nonsmooth system (5.7).

5.3 Lipschitz Approximation

The first step in our analysis is to compute an approximation of the solutions of (5.7) by using Moreau-Yosida regularization. The development carried out here is inspired by [35]. We introduce a sequence of approximate solutions, the so-called Moreau-Yosida approximants $\{z_\lambda\}_{\lambda>0}$, which are obtained by solving the following ODE parameterized by $\lambda > 0$:

$$\dot{z}_\lambda(t) = f(t, z_\lambda(t)) - \frac{1}{\lambda}(z_\lambda(t) - \operatorname{proj}(z_\lambda(t), \mathcal{S}(t))), \quad z_\lambda(0) = z_0 \in \mathcal{S}(0) \quad (5.8)$$

over the interval $[0, T]$, where $\text{proj}(z, \mathcal{S})$ is the (unique) Euclidean projection of vector z onto convex set $\mathcal{S}$. It is observed that, for each $\lambda > 0$, the right-hand side of (5.8) is (globally) Lipschitz continuous, and therefore, there exists a continuously differentiable trajectory $z_\lambda : [0, T] \to \mathbb{R}^n$ such that (5.8) holds for every $t \in [0, T]$. The relation between the solution of the inclusion (5.7) and the approximants $\{z_\lambda\}_{\lambda>0}$ holds under the following assumptions:

Assumption 4. There exists a constant $L_f > 0$ such that, for each $t \in [0, T]$,

$$|f(t, z)| \leqslant L_f(1 + |z|), \quad \forall z \in \mathbb{R}^n$$
$$|f(t, z_1) - f(t, z_2)| \leqslant L_f|z_1 - z_2|, \quad \forall z_1, z_2 \in \mathbb{R}^n.$$

Assumption 5. The mapping $\mathcal{S} : [0, T] \rightrightarrows \mathbb{R}^n$ is closed and convex-valued for each $t \in [0, T]$, and $\mathcal{S}(\cdot)$ varies in a Lipschitz continuous manner with time, that is, there exists a constant $L_\mathcal{S} \geqslant 0$, such that

$$d_H(\mathcal{S}(t_1), \mathcal{S}(t_2)) \leqslant L_\mathcal{S}|t_1 - t_2|, \quad \forall t_1, t_2 \in [0, T].$$

The notation $d_H(A, B)$ means the Hausdorff distance between sets A and B, that is,

$$d_H(A, B) := \max \left\{ \sup_{y \in A} d(y, B), \ \sup_{x \in B} d(x, A) \right\} \tag{5.9}$$

where $d(x, A)$ denotes the Euclidean distance between vector x and set A.

Theorem 5.2. *Under Assumptions 4–5, consider the sequence of solutions $\{z_\lambda\}_{\lambda>0}$ to parameterized ODE (5.8) on an interval $[0, T]$. Then, as $\lambda \to 0$, the sequence converges uniformly to a Lipschitz continuous function $z : [0, T] \to \mathbb{R}^n$, the unique solution to the differential inclusion (5.7).*

The proof of this theorem is discussed in the remainder of this section.

Proof : The basic idea of the proof is to show that the sequence $\{z_\lambda\}_{\lambda>0}$ satisfies bounds ensuring uniform convergence to a function $z(\cdot)$ solving (5.7). This development is carried out in three steps.

Step 1: Estimates on the sequence $\{z_\lambda\}_{\lambda>0}$. As a first step, to obtain bounds on the norm of $z_\lambda(.)$, let us begin by computing bounds on the norm of $\dot{z}_\lambda(\cdot)$ as stated in the following lemma.

Lemma 5.3. *For each $\lambda > 0$, it holds*

$$|\dot{z}_\lambda(t)| \leqslant 2L_f + L_f|z_\lambda(t)| + L_f \max_{0 \leqslant s \leqslant t} |z_\lambda(s)| + L_\mathcal{S}, \tag{5.10}$$

where $L_f, L_\mathcal{S}$ were introduced in Assumptions 4 and 5 respectively.

Proof : For each $\lambda > 0$, the dynamics for z_λ in (5.8) yield

$$\begin{aligned} |\dot{z}_\lambda(t)| &= |f(t, z_\lambda(t)) - \frac{1}{\lambda}(z_\lambda(t) - \text{proj}(z_\lambda(t), \mathcal{S}(t)))| \\ &\leqslant |f(t, z_\lambda(t))| + \frac{1}{\lambda}|z_\lambda(t) - \text{proj}(z_\lambda(t), \mathcal{S}(t))|. \end{aligned} \tag{5.11}$$

For the first term in the right-hand side of (5.11), we have that

$$|f(t, z_\lambda(t))| \leqslant L_f(1 + |z_\lambda(t)|). \tag{5.12}$$

For the second term in the right-hand side of (5.11), we introduce the function $d_\lambda(t) = \inf_{y \in \mathcal{S}(t)} |y - z_\lambda(t)|$, so that $d_\lambda(t) = d_{\mathcal{S}(t)}(z_\lambda(t))$. It is seen that $d_\lambda(t) = |z_\lambda(t) - \text{proj}(z_\lambda(t), \mathcal{S}(t))|$. So $\frac{1}{\lambda}|z_\lambda(t) - \text{proj}(z_\lambda(t), \mathcal{S}(t))| = \frac{1}{\lambda} d_\lambda(t)$. To obtain a bound on $d_\lambda(t)$, we compute the derivative of $d_\lambda^2(t)$:

$$\begin{aligned} \frac{d}{dt} d_\lambda^2(t) &= \frac{d}{dt} d_{\mathcal{S}(t)}^2(z_\lambda(t)) \\ &= \lim_{\epsilon \to 0} \frac{d_{\mathcal{S}(t+\epsilon)}^2(z_\lambda(t+\epsilon)) - d_{\mathcal{S}(t)}^2(z_\lambda(t))}{\epsilon} \\ &= \lim_{\epsilon \to 0} \frac{d_{\mathcal{S}(t+\epsilon)}^2(z_\lambda(t+\epsilon)) - d_{\mathcal{S}(t)}^2(z_\lambda(t+\epsilon))}{\epsilon} + \frac{d_{\mathcal{S}(t)}^2(z_\lambda(t+\epsilon)) - d_{\mathcal{S}(t)}^2(z_\lambda(t))}{\epsilon}. \end{aligned} \tag{5.13}$$

For the first term in the limit, we use that

$$\begin{aligned} & d_{\mathcal{S}(t+\epsilon)}^2(z_\lambda(t+\epsilon)) - d_{\mathcal{S}(t)}^2(z_\lambda(t+\epsilon)) \\ & \leqslant d_H(\mathcal{S}(t+\epsilon), \mathcal{S}(t)) \Big(d_{\mathcal{S}(t+\epsilon)}(z_\lambda(t+\epsilon)) + d_{\mathcal{S}(t)}(z_\lambda(t+\epsilon)) \Big) \\ & \leqslant |\epsilon| L_{\mathcal{S}} \Big(d_{\mathcal{S}(t+\epsilon)}(z_\lambda(t+\epsilon)) + d_{\mathcal{S}(t)}(z_\lambda(t+\epsilon)) \Big). \end{aligned} \tag{5.14}$$

For the second term in the limit, we first notice that

$$\begin{aligned} d_{\mathcal{S}(t)}^2(z_\lambda(t+\epsilon)) - d_{\mathcal{S}(t)}^2(z_\lambda(t)) &= d_{\mathcal{S}(t)}^2(z_\lambda(t) + \epsilon \dot{z}_\lambda(t)) - d_{\mathcal{S}(t)}^2(z_\lambda(t)) \\ &\quad + \Big(d_{\mathcal{S}(t)}(z_\lambda(t+\epsilon)) - d_{\mathcal{S}(t)}(z_\lambda(t) + \epsilon \dot{z}_\lambda(t)) \Big) \\ &\quad \Big(d_{\mathcal{S}(t)}(z_\lambda(t+\epsilon)) + d_{\mathcal{S}(t)}(z_\lambda(t) + \epsilon \dot{z}_\lambda(t)) \Big). \end{aligned}$$

Since $z_\lambda(.)$ is differentiable, $z_\lambda(t+\epsilon) = z_\lambda(t) + \epsilon \dot{z}_\lambda(t) + \mathcal{O}(\epsilon)$ and hence $d_{\mathcal{S}(t)}(z_\lambda(t+\epsilon)) - d_{\mathcal{S}(t)}(z_\lambda(t) + \epsilon \dot{z}_\lambda(t)) = \mathcal{O}(\epsilon)$. This implies that

$$d_{\mathcal{S}(t)}^2(z_\lambda(t+\epsilon)) - d_{\mathcal{S}(t)}^2(z_\lambda(t)) = d_{\mathcal{S}(t)}^2(z_\lambda(t) + \epsilon \dot{z}_\lambda(t)) - d_{\mathcal{S}(t)}^2(z_\lambda(t)).$$

And,

$$\lim_{\epsilon \to 0} \frac{1}{\epsilon} \Big[d_{\mathcal{S}(t)}^2(z_\lambda(t+\epsilon)) - d_{\mathcal{S}(t)}^2(z_\lambda(t)) \Big] = \langle \nabla d_{\mathcal{S}(t)}^2(z_\lambda(t)), \dot{z}_\lambda(t) \rangle$$

$$= 2\langle z_\lambda(t) - \text{proj}(z_\lambda(t), \mathcal{S}(t)), \dot{z}_\lambda(t)\rangle. \tag{5.15}$$

By substitution of (5.14) and (5.15) in equation (5.13), we obtain

$$\begin{aligned}\frac{d}{dt}d_\lambda^2(t) = 2d_\lambda(t)\dot{d}_\lambda(t) &\leqslant 2d_\lambda(t)\dot{z}_\lambda(t) + 2L_{\mathcal{S}}d_\lambda(t)\\ &\leqslant 2d_\lambda(t)\left(f(t, z_\lambda(t)) - \frac{1}{\lambda}d_\lambda(t)\right) + 2L_{\mathcal{S}}d_\lambda(t)\\ &\leqslant -\frac{2}{\lambda}d_\lambda^2(t) + 2d_\lambda(t)f(t, z_\lambda(t)) + 2L_{\mathcal{S}}d_\lambda(t).\end{aligned}$$

Dividing by $2d_\lambda(t)$, we get

$$\frac{d}{dt}d_\lambda(t) \leqslant -\frac{1}{\lambda}d_\lambda(t) + f(t, z_\lambda(t)) + L_{\mathcal{S}},$$

which implies that,

$$d_\lambda(t) \leqslant e^{-t/\lambda}d_\lambda(0) + \int_0^t e^{-(t-s)/\lambda}(f(s, z_\lambda(s)) + L_{\mathcal{S}})\ ds.$$

Or, $d_\lambda(0) = |z_0 - \text{proj}(z_0, \mathcal{S}(0))| = 0$ since $z_0 \in \mathcal{S}(0)$ and we have that f satisfies (5.12), then it follows

$$\frac{1}{\lambda}d_\lambda(t) \leqslant \frac{1}{\lambda}\int_0^t e^{-(t-s)/\lambda}(L_f + L_f|z_\lambda(s)| + L_{\mathcal{S}})\ ds. \tag{5.16}$$

And therefore, substituting (5.12) and (5.16) in (5.11), we get

$$\begin{aligned}|\dot{z}_\lambda(t)| &\leqslant L_f + L_f|z_\lambda(t)| + \frac{1}{\lambda}\int_0^t e^{-(t-s)/\lambda}(L_f + L_f|z_\lambda(s)| + L_{\mathcal{S}})\ ds\\ &\leqslant L_f + L_f|z_\lambda(t)| + \frac{L_f}{\lambda}\int_0^t e^{-(t-s)/\lambda}\ ds + \frac{L_f}{\lambda}\int_0^t e^{-(t-s)/\lambda}|z_\lambda(s)|\ ds\\ &\quad + \frac{L_{\mathcal{S}}}{\lambda}\int_0^t e^{-(t-s)/\lambda}\ ds.\end{aligned}$$

We have

$$\begin{aligned}\frac{L_f}{\lambda}\int_0^t e^{-(t-s)/\lambda}\ ds &= \frac{L_f}{\lambda}e^{-t/\lambda}\left[\lambda e^{s/\lambda}\right]_0^t = \frac{L_f}{\lambda}e^{-t/\lambda}\left(\lambda e^{t/\lambda} - \lambda\right)\\ &= L_f\left(1 - e^{-t/\lambda}\right) \leqslant L_f.\end{aligned}$$

Similarly,

$$\frac{L_{\mathcal{S}}}{\lambda}\int_0^t e^{-(t-s)/\lambda}\ ds \leqslant L_{\mathcal{S}}.$$

Besides, we have

$$\frac{L_f}{\lambda}\int_0^t e^{-(t-s)/\lambda}|z_\lambda(s)|\,ds \leqslant \underbrace{\frac{L_f}{\lambda}\int_0^t e^{-(t-s)/\lambda}\,ds}_{\leqslant L_f}\,.\max_{0\leqslant s\leqslant t}|z_\lambda(s)|$$
$$\leqslant L_f\max_{0\leqslant s\leqslant t}|z_\lambda(s)|.$$

The bound of $|\dot{z}_\lambda(t)|$ is then expressed as

$$|\dot{z}_\lambda(t)| \leqslant 2L_f + L_f|z_\lambda(t)| + L_f\max_{0\leqslant s\leqslant t}|z_\lambda(s)| + L_{\mathcal{S}}.$$

♢

Based on Lemma 5.3, let us now calculate $\frac{d}{dt}|z_\lambda(t)|^2$ for getting an estimate on $|z_\lambda(\cdot)|$. First, we observe that

$$\frac{d}{dt}|z_\lambda(t)|^2 = 2\langle z_\lambda(t), \dot{z}_\lambda(t)\rangle \leqslant 2|z_\lambda(t)||\dot{z}_\lambda(t)|. \tag{5.17}$$

Substituting (5.10) in (5.17) yields

$$\frac{d}{dt}|z_\lambda(t)|^2 \leqslant 2L_f|z_\lambda(t)|^2 + 2L_f|z_\lambda(t)|.\max_{0\leqslant s\leqslant t}|z_\lambda(s)| + (4L_f+2L_{\mathcal{S}})|z_\lambda(t)|.$$

Let $y_\lambda(t) = |z_\lambda(t)|^2$, so

$$\frac{d}{dt}y_\lambda(t) \leqslant 2L_f y_\lambda(t) + 2L_f\sqrt{y_\lambda(t)}.\max_{0\leqslant s\leqslant t}\sqrt{y_\lambda(s)} + (4L_f+2L_{\mathcal{S}})\sqrt{y_\lambda(t)}.$$

Since the right-hand side of this differential inequality results in a nonnegative and nondecreasing function, it follows that $y_\lambda(t) \leqslant \widehat{y}_\lambda(t)$, for all $t \in [0,T]$, where $\widehat{y}_\lambda$ satisfies

$$\begin{aligned}\frac{d}{dt}\widehat{y}_\lambda(t) &= 2L_f\widehat{y}_\lambda(t) + 2L_f\sqrt{\widehat{y}_\lambda(t)}.\sqrt{\widehat{y}_\lambda(t)} + (4L_f+2L_{\mathcal{S}})\sqrt{\widehat{y}_\lambda(t)}\\ &= 4L_f\widehat{y}_\lambda(t) + (4L_f+2L_{\mathcal{S}})\sqrt{\widehat{y}_\lambda(t)}.\end{aligned} \tag{5.18}$$

By using the substitution $v(t) = (\widehat{y}_\lambda(t))^{\frac{1}{2}}$ in (5.18), it yields

$$\dot{v}(t) = 2L_f v(t) + 2L_f + L_{\mathcal{S}}.$$

The solution of this differential equation is $v(t) = e^{2L_f t}v(0)+(e^{2L_f t}-1)\frac{(2L_f+L_{\mathcal{S}})}{2L_f}$. Consequently, $|z_\lambda(t)|^2 = y_\lambda(t) \leqslant \widehat{y}_\lambda(t) = v(t)^2$, and we obtain

$$|z_\lambda(t)| \leqslant e^{2L_f T}|z_\lambda(0)| + (e^{2L_f T}-1)\frac{(2L_f+L_{\mathcal{S}})}{2L_f}, \tag{5.19}$$

since $v(0) = (\widehat{y}_\lambda(t))^{\frac{1}{2}} = (y_\lambda(t))^{\frac{1}{2}} = |z_\lambda(0)|$. Hence, $|z_\lambda(t)|$ is bounded on the interval $[0,T]$, independently of λ.

Step 2: Extracting a converging subsequence. Based on the estimates in *Step 1*, there exists a subsequence of $z_\lambda(\cdot)$ which converges to $z(\cdot)$. More formally, the following statement is obtained.

Lemma 5.4. *There exists a subsequence $\{z_{\lambda_i}\}_{i\in\mathbb{N}}$ which converges uniformly to a Lipschitz continuous function $z(\cdot)$ on $[0,T]$.*

The proof of Lemma 5.4 is a consequence of the Arzelà-Ascoli theorem since the sequence $\{z_{\lambda_i}\}_{i\in\mathbb{N}}$ is continuously differentiable and $\{\dot{z}_{\lambda_i}\}_{i\in\mathbb{N}}$ is uniformly bounded. The limit function $z(\cdot)$ is also Lipschitz continuous in this case.

Step 3: Limit is a solution. To finish the proof of Theorem 5.2, we just need to show that the limit $z(\cdot)$ satisfies the differential inclusion (5.7). This particular step requires a variational inequality, which is stated in the following lemma.

Lemma 5.5. *If $\varphi : [0,T] \to \mathbb{R}^n$ is a continuous function that satisfies $\varphi(s) + \int_{t_1}^{s} f(r, z_\lambda(r))\, dr \in \mathcal{S}(s)$ for each $s \in [t_1,t_2]$, for $t_1, t_2 \in [0,T]$, then*

$$\int_{t_1}^{t_2} \Big\langle \varphi(s), \dot{z}(s) - f(s, z(s)) \Big\rangle\, ds \geqslant \frac{1}{2}\left(\left\| z(t_2) - \int_{t_1}^{t_2} f(r, z(r))\, dr \right\|^2 - \|z(t_1)\|^2 \right). \tag{5.20}$$

Proof : Let $\bar{z}_\lambda(s) := \operatorname{proj}(z_\lambda(s), \mathcal{S}(s))$; then $s \mapsto \bar{z}_\lambda(s)$ is a continuous mapping. Since $\varphi(s) + \int_{t_1}^{s} f(r, z_\lambda(r))\, dr \in \mathcal{S}(s)$ and λ is positive, it follows from the definition of the projections that

$$\begin{aligned} &\Big\langle \varphi(s) + \int_{t_1}^{s} f(r, z_\lambda(r))\, dr - \bar{z}_\lambda(s), \dot{z}_\lambda(s) - f(s, z_\lambda(s)) \Big\rangle \\ &= \frac{1}{\lambda} \Big\langle \varphi(s) + \int_{t_1}^{s} f(r, z_\lambda(r))\, dr - \bar{z}_\lambda(s), \bar{z}_\lambda(s) - z_\lambda(s) \Big\rangle \geqslant 0. \end{aligned}$$

Then

$$\Big\langle \varphi(s), \dot{z}_\lambda(s) - f(s, z_\lambda(s)) \Big\rangle \geqslant \Big\langle \bar{z}_\lambda(s) - \int_{t_1}^{s} f(r, z_\lambda(r))\, dr, \dot{z}_\lambda(s) - f(s, z_\lambda(s)) \Big\rangle,$$

which implies that,

$$\begin{aligned} &\int_{t_1}^{t_2} \Big\langle \varphi(s), \dot{z}_\lambda(s) - f(s, z_\lambda(s)) \Big\rangle\, ds \geqslant \\ &\qquad \int_{t_1}^{t_2} \Big\langle \bar{z}_\lambda(s) - \int_{t_1}^{s} f(r, z_\lambda(r))\, dr, \dot{z}_\lambda(s) - f(s, z_\lambda(s)) \Big\rangle\, ds. \end{aligned}$$

Since at the points where $z_\lambda(.)$ is differentiable, we have

$$\begin{aligned}\langle \bar{z}_\lambda(s), \dot{z}_\lambda(s) - f(s, z_\lambda(s))\rangle &= \langle \bar{z}_\lambda(s) - z_\lambda(s), \dot{z}_\lambda(s) - f(s, z_\lambda(s))\rangle \\ &\quad + \langle z_\lambda(s), \dot{z}_\lambda(s) - f(s, z_\lambda(s))\rangle \\ &= \underbrace{\frac{1}{\lambda}|\bar{z}_\lambda(s) - z_\lambda(s)|^2}_{\geqslant 0} + \langle z_\lambda(s), \dot{z}_\lambda(s) - f(s, z_\lambda(s))\rangle,\end{aligned}$$

it follows that,

$$\langle \bar{z}_\lambda(s), \dot{z}_\lambda(s) - f(s, z_\lambda(s))\rangle \geqslant \langle z_\lambda(s), \dot{z}_\lambda(s) - f(s, z_\lambda(s))\rangle,$$

and,

$$\int_{t_1}^{t_2} \Big\langle \varphi(s), \dot{z}_\lambda(s) - f(s, z_\lambda(s))\Big\rangle\, ds \geqslant \\ \int_{t_1}^{t_2} \Big\langle z_\lambda(s) - \int_{t_1}^{s} f(r, z_\lambda(r))\, dr, \dot{z}_\lambda(s) - f(s, z_\lambda(s))\Big\rangle\, ds.$$

We have

$$\begin{aligned}&\int_{t_1}^{t_2} \Big\langle z_\lambda(s) - \int_{t_1}^{s} f(r, z_\lambda(r))\, dr, \dot{z}_\lambda(s) - f(s, z_\lambda(s))\Big\rangle\, ds \\ &= \frac{1}{2}\left[\|z_\lambda(s) - \int_{t_1}^{s} f(r, z_\lambda(r))\, dr\|^2\right]_{t_1}^{t_2} \\ &= \frac{1}{2}\left(\|z_\lambda(t_2) - \int_{t_1}^{t_2} f(r, z_\lambda(r))\, dr\|^2 - \|z_\lambda(t_1)\|^2\right),\end{aligned}$$

hence, we obtain that

$$\int_{t_1}^{t_2} \Big\langle \varphi(s), \dot{z}_\lambda(s) - f(s, z_\lambda(s))\Big\rangle\, ds \geqslant \\ \frac{1}{2}\left(\|z_\lambda(t_2) - \int_{t_1}^{t_2} f(r, z_\lambda(r))\, dr\|^2 - \|z_\lambda(t_1)\|^2\right).$$

We take limits with respect to $\lambda \to 0$. Since $z_\lambda(.)$ converges pointwise to $z(.)$, we have $\Big\langle \varphi(s), \dot{z}_\lambda(s) - f(s, z_\lambda(s))\Big\rangle \longrightarrow \Big\langle \varphi(s), \dot{z}(s) - f(s, z(s))\Big\rangle$ for each $s \in [t_1, t_2]$, and $\|z_\lambda(t_2) - \int_{t_1}^{t_2} f(r, z_\lambda(r))\, dr\|^2 \longrightarrow \|z(t_2) - \int_{t_1}^{t_2} f(r, z(r))\, dr\|^2$, and $\|z_\lambda(t_1)\|^2 \longrightarrow \|z(t_1)\|^2$.

Therefore, this yields to

$$\int_{t_1}^{t_2} \Big\langle \varphi(s), \dot{z}(s) - f(s, z(s))\Big\rangle\, ds \geqslant \frac{1}{2}\left(\|z(t_2) - \int_{t_1}^{t_2} f(r, z(r))\, dr\|^2 - \|z(t_1)\|^2\right),$$

and Lemma 5.5 is then proved. ♢

We now complete the proof of Theorem 5.2 by showing that the limit of the converging subsequence $z(\cdot)$ satisfies $\dot{z}(t) \in f(t, z(t)) - \mathcal{N}_{\mathcal{S}(t)}(z(t))$ that is, $\langle \xi - z(t), \dot{z}(t) - f(t, z(t)) \rangle \geqslant 0$, for any $\xi \in \mathcal{S}(t)$ and for almost every $t \geqslant 0$. This is indeed the case, since for every $\xi \in \mathcal{S}(t)$, we can take a Lipschitz continuous function $\varphi : [t, T] \to \mathbb{R}^n$ such that, due to Lemma 5.5, we get

$$\int_{[t,t+\epsilon[} \langle \varphi(s), \dot{z}(s) - f(s, z(s)) \rangle \, ds \geqslant \frac{1}{2} \left(\|z(t+\epsilon) - \int_t^{t+\epsilon} f(r, z(r)) \, dr\|^2 - \|z(t)\|^2 \right),$$

and by letting $\varphi(s) = \xi - (\xi - \varphi(s))$, we obtain

$$\int_{[t,t+\epsilon[} \langle \xi, \dot{z}(s) - f(s, z(s)) \rangle \, ds - \int_{[t,t+\epsilon[} \langle \xi - \varphi(s), \dot{z}(s) - f(s, z(s)) \rangle \, ds$$
$$\geqslant \frac{1}{2} \Big\langle z(t+\epsilon) - \int_t^{t+\epsilon} f(r, z(r)) \, dr + z(t), z(t+\epsilon) - \int_t^{t+\epsilon} f(r, z(r)) \, dr - z(t) \Big\rangle,$$

which implies

$$\Big\langle \xi, z(t+\epsilon) - z(t) - \int_t^{t+\epsilon} f(s, z(s)) \, ds \Big\rangle - \int_t^{t+\epsilon} \langle \xi - \varphi(s), \dot{z}(s) - f(s, z(s)) \rangle \, ds$$
$$\geqslant \frac{1}{2} \Big\langle z(t+\epsilon) - \int_t^{t+\epsilon} f(r, z(r)) \, dr + z(t), z(t+\epsilon) - \int_t^{t+\epsilon} f(r, z(r)) \, dr - z(t) \Big\rangle.$$

From this, we get

$$\Big\langle \xi - \frac{1}{2} \left(z(t+\epsilon) - \int_t^{t+\epsilon} f(r, z(r)) \, dr + z(t) \right), z(t+\epsilon) - z(t) - \int_t^{t+\epsilon} f(s, z(s)) \, ds \Big\rangle$$
$$\geqslant \int_t^{t+\epsilon} \langle \xi - \varphi(s), \dot{z}(s) - f(s, z(s)) \rangle \, ds$$
$$\geqslant -\epsilon \max_{s \in [t,t+\epsilon[} |\xi - \varphi(s)| |\dot{z}(s) - f(s, z(s))|$$
$$\geqslant -\epsilon \max_{s \in [t,t+\epsilon[} |\xi - \varphi(s)| |\dot{z}(s)| - \epsilon \, L_f \max_{s \in [t,t+\epsilon[} |\xi - \varphi(s)| (1 + |z(s)|).$$

Since $z(\cdot)$ is Lipschitz continuous, $z(\cdot)$ is bounded on $[0, T]$ and differentiable almost everywhere on $[0, T]$. Hence, for almost every $t \in [0, T]$, where $z(\cdot)$ is differentiable, dividing the last inequality by ϵ, we get

$$\left\langle \xi - \frac{1}{2} \left(z(t+\epsilon) - \int_t^{t+\epsilon} f(r, z(r)) \, dr + z(t) \right), \frac{z(t+\epsilon) - z(t)}{\epsilon} - \frac{\int_t^{t+\epsilon} f(s, z(s)) \, ds}{\epsilon} \right\rangle$$
$$\geqslant -M \max_{s \in [t,t+\epsilon[} |\xi - \varphi(s)| - M L_f \max_{s \in [t,t+\epsilon[} |\xi - \varphi(s)|,$$

for some constant $M > 0$. Letting ϵ tend to zero, we get

$$\langle \xi - z(t), \dot{z}(t) - f(t, z(t)) \rangle \geqslant 0, \text{ for each } \xi \in \mathcal{S}(t),$$

and hence, $z(\cdot)$ satisfies the differential inclusion (5.7). ♢

Remark 5.6. In the literature, we can find several proofs of convergence of solutions obtained from Moreau-Yosida regularization to the solution of systems closely related to (5.7), see for example [35, 101, 121]. The proof technique adopted here closely follows the outline given in [35], but the difference here is that we add the Lipschitz perturbation $f(t, z)$ on the right-hand side of (5.7), which modifies certain calculations.

5.4 Convergence of Approximating Measures

Using the results from the previous section on the convergence of solutions for fixed initial condition, we now study the evolution of probability measures for system (5.7). As before, let us assume that $z(0)$ is a random variable whose law is a given probability measure $\xi_0 \in \mathcal{P}(\mathcal{S}(0))$. We recall that the flow map for system (5.7) is denoted by F_t, so that $t \mapsto z(t) := F_t(z_0)$ is the unique solution to (5.7).

For the Lipschitz approximation given in (5.8), consider the map $F_t^\lambda : \mathbb{R}^n \to \mathbb{R}^n$, so that $t \mapsto z_\lambda(t) := F_t^\lambda(z_0)$ defines the unique solution to (5.8). Since the right-hand side of (5.8) is Lipschitz continuous for each $\lambda > 0$, we can consider a sequence of probability measures $\xi_t^\lambda \in \mathcal{P}(\mathcal{S}(t))$ defined as

$$\xi_t^\lambda := F_t^\lambda \sharp \xi_0$$

for each $t \in [0, T]$ and $\lambda > 0$. From Theorem 5.1, it follows that ξ_t^λ satisfies the partial differential equation:

$$\partial_t \xi_t^\lambda + \operatorname{div}(\xi_t^\lambda f_t^\lambda) = 0 \tag{5.21}$$

in the sense of distributions, with the initial condition $\xi\big|_{t=0} = \xi_0$, and

$$f_t^\lambda(z) := f(t, z) - \frac{1}{\lambda}\Big(z - \operatorname{proj}(z, \mathcal{S}(t))\Big). \tag{5.22}$$

On the other hand, we do not know how to derive a meaningful PDE for ξ_t. However, in the sequel, we show that the probability measure ξ_t can be approximated by ξ_t^λ as $\lambda \to 0$. This way, a good numerical approximation of ξ_t^λ would also provide an approximation of ξ_t.

5.4.1 Weak-star Convergence

We first show convergence in the weak-star topology. This allows us to approximate the evolution of the moments of the measure ξ_t using the moments of ξ_t^λ. Given a measure ξ, we denote its support by $\mathrm{supp}(\xi)$, defined as the smallest closed set whose complement has zero measure with respect to ξ. Equivalently, it is the smallest closed set for which every point has a neighborhood of positive measure with respect to ξ.

Proposition 5.7. *Let $v : \mathbb{R}^n \to \mathbb{R}$ be a continuous function, and assume that ξ_0 has bounded support. Then,*

$$\lim_{\lambda\to 0} \int_{\mathbb{R}^n} v(z)\ d\xi_t^\lambda(z) = \int_{\mathbb{R}^n} v(z)\ d\xi_t(z). \tag{5.23}$$

Proof : By definition of the pushforward measure ξ_t^λ, it holds

$$\int_{\mathbb{R}^n} v(z)\ d\xi_t^\lambda(z) = \int_{\mathbb{R}^n} v(F_t^\lambda(y))\ d\xi_0(y) \tag{5.24}$$

for all continuous functions v. From Theorem 5.2, for each $t \in [0, T]$, we have $\lim_{\lambda\to 0} z_\lambda(t) = z(t)$, which is equivalent to

$$\lim_{\lambda\to 0} F_t^\lambda(y) = F_t(y), \quad \forall y \in \mathcal{S}(0).$$

Since v is any continuous function, this implies

$$\lim_{\lambda\to 0} v(F_t^\lambda(y)) = v(F_t(y)).$$

By assumption, $v \circ F_t^\lambda$ is bounded on the bounded set $\mathrm{supp}(\xi_0)$. This allows us to invoke Lebesgue's dominated convergence theorem to get

$$\lim_{\lambda\to 0} \int_{\mathbb{R}^n} v(F_t^\lambda(y))\ d\xi_0(y) = \int_{\mathbb{R}^n} v(F_t(y)) d\xi_0(y). \tag{5.25}$$

Hence, (5.24) and (5.25) yield

$$\lim_{\lambda\to 0} \int_{\mathbb{R}^n} v(z)\ d\xi_t^\lambda(z) = \int_{\mathbb{R}^n} v(F_t(y)) d\xi_0(y).$$

Using again the change of variables formula, we obtain

$$\lim_{\lambda\to 0} \int_{\mathbb{R}^n} v(z)\ d\xi_t^\lambda(z) = \int_{\mathbb{R}^n} v(z)\ d\xi_t(z)$$

for all continuous functions v on $\mathbb{R}^n$. Therefore, the equality in (5.23) is proved. $\diamondsuit$

Remark 5.8. In the proof of Proposition 5.7, the boundedness of $\mathrm{supp}(\xi_0)$ was used to invoke dominated convergence theorem. The result of Proposition 5.7 extends in some cases where $\mathrm{supp}(\xi_0)$ is unbounded. In particular, if it can be shown that there exists a function $g : [0, T] \times \mathbb{R}^n \to \mathbb{R}_{\geqslant 0}$ such that, for each $\lambda > 0$,

$$\left| F_t^\lambda(y) \right| \leqslant g(t, y), \quad t \in [0, T]$$

then the convergence in (5.23) holds for all continuous functions v which satisfy

$$\int_{\mathbb{R}^n} v(g(t, y)) d\xi_0(y) < \infty, \quad t \in [0, T].$$

5.4.2 Relations Describing Moments

An immediate consequence of Proposition 5.7 is that we can get a desired approximation of the moments of ξ_t by choosing appropriate test functions v. This amounts to computing the moments of ξ_t^λ. We will now explore numerical techniques which allow us to compute the solution of (5.21) by computing the desired moments.

Toward this end, we first observe that the Liouville equation (5.21) can be equivalently written as a linear PDE satisfied by the occupation measures

$$d\mu^\lambda := dt \; d\xi_t^\lambda, \quad \textit{with} \quad \mu_0^\lambda := \delta_0 \xi_0, \; \mu_T^\lambda := \delta_T \xi_T,$$

which is

$$\partial_t \mu^\lambda + \mathrm{div}(\mu^\lambda f_\lambda) + \mu_T^\lambda = \mu_0^\lambda \tag{5.26}$$

which again should be understood in the sense of distributions, i.e.

$$\int_{\mathbb{R}_+} \int_{\mathbb{R}^n} \left(\partial_t v(t, z) + \partial_z v(t, z) \cdot f_\lambda(t, z) \right) d\mu^\lambda(t, z) = \int_{\mathbb{R}_+} \int_{\mathbb{R}^n} v(t, z) (d\mu_T^\lambda(t, z) - d\mu_0^\lambda(t, z)),$$

for all continuously differentiable functions v.

We compute approximate moments of μ^λ by applying the moment-SOS hierarchy [86]. This method consists of minimizing a functional subject to the following constraints:

1. The Liouville equation (5.26) expressed in the sense of distributions, as a linear constraints on the moments of μ^λ and μ_T^λ.

2. Necessary linear matrix inequality (LMI) constraints based on the dual of Putinar's Positivstellensatz (Theorem 2.6).

We will see in the following how to formulate the Liouville equation (5.26) as a linear moment constraint.

Let g be a polynomial vector field defined as

$$g : \underbrace{(z_1, z_2, \ldots, z_n)}_{z} \in \mathbb{R}^n \mapsto (g_1, g_2, \ldots, g_n) \in \mathbb{R}^n,$$

and v be a monomial test function, with a maximum degree $d \in \mathbb{N}$, defined as

$$v : (t, z) \mapsto t^a z^b := t^a z_1^{b_1} z_2^{b_2} \cdots z_n^{b_n},$$

for all $(a, b) \in \mathbb{N}^{n+1}$, with $a + b_1 + b_2 + \cdots + b_n \leqslant d$. The maximal degree d is called the relaxation degree.

Besides, let us denote

$$y_{a-1,b} := \int_0^T \int_{\mathbb{R}^n} t^{a-1} z^b d\mu^\lambda(t, z) \tag{5.27}$$

and

$$y^T_{a,b} := \int_0^T \int_{\mathbb{R}^n} t^a z^b d\mu^\lambda_T(t, z), \tag{5.28}$$

$$y^0_{a,b} := \int_0^T \int_{\mathbb{R}^n} t^a z^b \; d\mu^\lambda_0(t, z). \tag{5.29}$$

Let e_i denote the vector whose only non-zero entry is equal to one at position i.

Proposition 5.9. *The Liouville equation (5.26) is equivalently expressed as:*

$$y^T_{a,b} - y^0_{a,b} = a y_{a-1,b} + \sum_{i=1}^n \int_0^T \int_{\mathbb{R}^n} b_i t^a z^{b-e_i} g_i(z) \; d\mu^\lambda(t, z) \tag{5.30}$$

which are linear constraints that link the moments of the initial measure, terminal measure and occupation measure.

Proof : Choosing $v(t, z) = t^a z^b$ as a monomial test function, the Liouville equation (5.26) is then written as

$$\langle \partial_t \mu^\lambda, v\rangle + \langle \operatorname{div}(\mu^\lambda g), v\rangle + \langle \mu^\lambda_T, v\rangle = \langle \mu^\lambda_0, v\rangle,$$

which implies

$$\int_0^T \int_{\mathbb{R}^n} (\partial_t v(t, z) + \partial_z v(t, z) \cdot g(z)) \; d\mu^\lambda(t, z) = \int_0^T \int_{\mathbb{R}^n} v(t, z) \; (d\mu^\lambda_T(t, z) - d\mu^\lambda_0(t, z)). \tag{5.31}$$

We have

$$\partial_t v(t,z) = a t^{a-1} z^b,$$

and

$$\partial_z v(t,z) = (b_1 t^a z_1^{b_1-1} z_2^{b_2} \cdots z_n^{b_n}, b_2 t^a z_1^{b_1} z_2^{b_2-1} \cdots z_n^{b_n}, \ldots, b_n t^a z_1^{b_1} z_2^{b_2} \cdots z_n^{b_n-1}).$$

Replacing $\partial_t v(t,z)$ and $\partial_z v(t,z)$ by their expressions in (5.31) yields

$$\int_0^T \int_{\mathbb{R}^n} (a t^{a-1} z^b + \sum_{i=1}^n b_i t^a z^{b-e_i} g_i(z)) \, d\mu^\lambda(t,z)$$
$$= \int_0^T \int_{\mathbb{R}^n} t^a z^b \, d\mu_T^\lambda(t,z) - \int_0^T \int_{\mathbb{R}^n} t^a z^b \, d\mu_0^\lambda(t,z)$$

which is the expected statement by using the notations (5.27), (5.28) and (5.29). ♢

5.4.3 Numerical Computation

Based on the result of Proposition 5.9, we now describe a numerical method for computing $y_{a,b}^T$. It is assumed that the initial measure μ_0 is given, which allows us to compute $y_{a,b}^0$. We next describe the main steps involved in writing a semidefinite program for calculating $y_{a,b}^T$ corresponding to the measure μ^λ. Note that, for each $\lambda > 0$, the measure μ^λ is supported on a subset of $\mathbb{R}^{n+1}$. In what follows, we provide some elements of construction for our algorithm for a finite Borel measure μ supported on $\mathbb{R}^p$.

Given a Borel probability measure μ and $\alpha \in \mathbb{N}^p$, we let

$$y_\alpha(\mu) = \int_{\mathbb{R}^p} z^\alpha d\mu(z),$$

where we recall that $z^\alpha := z^{\alpha_1} z^{\alpha_2} \ldots z_p^{\alpha_p}$. We consider the set $\{\alpha \in \mathbb{N}^p;\ \alpha_1 + \cdots + \alpha_p \leqslant d\}$ with graded lexicographic order, and denote it by $\mathbb{N}_d^p$; for example, with $p = 2$, $d = 2$, $\mathbb{N}_2^2 = \{(0,0),(1,0),(0,1),(2,0),(1,1),(0,2)\}$. The cardinality of $\mathbb{N}_d^p$ is $s(d) := \binom{p+d}{d}$, which is the number of monomials of degree at most d. The sequence $y = (y_\alpha(\mu))_{\alpha\in\mathbb{N}^p}$ therefore encodes the moments of the measure μ.

The moment matrix of degree d associated with a Borel measure μ, denoted by $M_d(\mu)$ is a matrix of dimension $s(d) \times s(d)$, whose rows and columns are indexed by monomials of degree at most d. For $\alpha, \beta \in \mathbb{N}_d^p$, the corresponding entry in $M_d(\mu)$ is defined by $(M_d(\mu))_{\alpha,\beta} := y_{\alpha+\beta}(\mu)$. As an example, once again with $p = 2$, $d = 2$, $M_2(\mu) \in \mathbb{R}^{6\times 6}$, and the element in second row ($\alpha = (1,0)$), third column ($\beta = (0,1)$), corresponds to $\int_{\mathbb{R}^2} z_1 z_2 \, d\mu(z)$.

To see an alternate representation of $M_d(\mu)$, let $b_d(z) := (z^\alpha)_{\alpha\in\mathbb{N}^p_d} \in \mathbb{R}[z]_d^{s(d)}$ denote the vector of monomials of degree less than or equal to d, with graded lexicographic order. If the sequence $\{y_\alpha\}_{\alpha\in\mathbb{N}^p}$ has a representing measure μ, i.e. $y_\alpha = \int_{\mathbb{R}^p} z^\alpha d\mu(z)$ for all $\alpha \in \mathbb{N}^p$, we can use the equivalent definition

$$M_d(\mu) := \int_{\mathbb{R}^p} b_d(z)b_d(z)^\top \, d\mu(z),$$

where the integral is understood entrywise. We can also define the localizing matrix of degree d with respect to a given $q(z) \in \mathbb{R}[z]$ by

$$M_{d-\lceil \deg(q)/2\rceil}(q\mu) := \int_{\mathbb{R}^p} q(z)b_d(z)b_d(z)^\top \, d\mu(z)$$

where $\lceil x\rceil$ denotes the smallest integer greater than x.

Assume that $X \subset \mathbb{R}^n$ is a compact basic semialgebraic set i.e.

$$X := \{z \in \mathbb{R}^n : p_k(z) \geqslant 0, \;\; k = 0, \ldots, n_X\}$$

for given $p_k \in \mathbb{R}[z]$, $k = 0, \ldots, n_X$. Let $p_0(z) = 1$ and let one of the inequalities $p_k(z) \geqslant 0$ be of the form $R - \sum_{i=1}^n z_i^2 \geqslant 0$ where R is a sufficiently large positive constant.

Theorem 5.10. *(Putinar's Theorem) The sequence of moments y has a representing measure supported on X if and only if $M_{d-\lceil \deg p_k/2\rceil}(p_k\mu)$, $k = 0, \ldots, n_X$ are positive semidefinite for all $d \in \mathbb{N}$.*

The above theorem represents the dual (moment) formulation to the Putinar theorem, namely SOS formulation, described in Theorem 2.6. The moment-SOS hierarchy, based on Theorem 5.10, allows us to compute approximate moments of the occupation measure and terminal measures. Recall that the moments of the initial measure are given since the initial measure is given. We fix a degree $d \in \mathbb{N}$ and we consider the linear constraint (5.30) linking moments of degree up to d, and subject to the constraints that the localizing matrices of the occupation measure and terminal measure, truncated to moments of degree up to d, are all positive semi-definite. This results in a finite-dimensional feasibility problem describe by linear matrix inequalities. The higher is the relaxation degree d, the better are the approximate moments, in the sense that when d tends to infinity, Theorem 5.10 and linear constraint (5.30) ensure that we have indeed moments of measures satisfying the Liouville equation.

The LMI constraints are automatically constructed by the `msdp` command in Gloptipoly for Matlab [87]. For more details about the LMI constraints, we refer the reader to [84, Section 3.3] or the two introductory chapters of [86].

5.5 Convergence of Support of Measures

For several applications, it is important to approximate the support of the measure ξ_t since it provides a probabilistic estimate of the state trajectories at time $t \in [0, T]$. Once again, our goal is to approximate the support of ξ_t by the support of ξ_t^λ where ξ_t^λ satisfies (5.21).

5.5.1 Hausdorff convergence of support

We first show that $\mathrm{supp}(\xi_t^\lambda)$ converges in the Hausdorff distance to $\mathrm{supp}(\xi_t)$.

Proposition 5.11. *For each $t \in [0, T]$, it holds*

$$\lim_{\lambda \to 0} d_H(\mathrm{supp}(\xi_t^\lambda), \mathrm{supp}(\xi_t)) = 0. \tag{5.32}$$

Proof : First, let $A_t^\lambda := \mathrm{supp}(\xi_t^\lambda)$ and $A_t := \mathrm{supp}(\xi_t)$. For proving that $\lim_{\lambda\to 0} d_H(A_t^\lambda, A_t) = 0$, we need to prove the following two limits:

$$\lim_{\lambda \to 0} \sup_{y_\lambda \in A_t^\lambda} d(y_\lambda, A_t) = 0, \tag{5.33}$$

and

$$\lim_{\lambda \to 0} \sup_{x \in A_t} d(x, A_t^\lambda) = 0. \tag{5.34}$$

For proving (5.33), we first observe that

$$\sup_{y_\lambda \in A_t^\lambda} d(y_\lambda, A_t) = \sup_{y_\lambda \in A_t^\lambda} \inf_{x \in A_t} |y_\lambda - x|,$$

and hence it needs to be shown that for every $y_\lambda \in A_t^\lambda$, there exists $x \in A_t$ such that $|x - y_\lambda|$ converges to zero as λ converges to zero. Since $y_\lambda \in A_t^\lambda$, there exists $z_0 \in \mathrm{supp}(\xi_0)$ such that $y_\lambda = F_t^\lambda(z_0)$. By choosing $x = F_t(z_0) \in A_t$, it follows from Theorem 5.2 that $\lim_{\lambda\to 0} F_t^\lambda(z_0) = F_t(z_0)$, or equivalently, $|x - y_\lambda|$ converges to 0 as $\lambda \to 0$.

For proving (5.34), we similarly observe that

$$\sup_{x \in A_t} d(x, A_t^\lambda) = \sup_{x \in A_t} \inf_{y_\lambda \in A_t^\lambda} |x - y_\lambda|.$$

Following the same idea as before, let us take $x \in A_t$, then there exists $z_0 \in \mathrm{supp}(\xi_0)$ such that $x = F_t(z_0)$. By choosing $y_\lambda = F_t^\lambda(z_0) \in A_t^\lambda$, it again follows from Theorem 5.2 that $|x - y_\lambda|$ converges to 0 as $\lambda \to 0$, and (5.34) is obtained. ♢

5.5.2 Approximation of support

Just like the approximation of moments, we can provide some numerical methods to approximate the support of the sequence of measures ξ_t^λ. By Proposition 5.11, by computing such an approximation for $\lambda > 0$ sufficiently small, we get an approximation of the support of the probability measure ξ_t for the original system.

The technique we present is based on approximating the support of a measure by the sublevel sets of a polynomial function. In particular, for a finite Borel measure μ, with non-singular moment matrix $M_d(\mu)$, we introduce the mapping

$$\mathbb{R}^n \ni x \mapsto \Lambda_{\xi,d}(x) := b_d(x)^\top M_d(\mu)^{-1} b_d(x) \in \mathbb{R},$$

which we call Christoffel-Darboux polynomial. Thus, the basic idea behind the construction of the support of the measure μ is to use the finite order moments, and show that the sublevel sets of the Christoffel-Darboux polynomial indeed converge to the actual support of μ. This technique has been proposed in [104] for stationary measures under certain hypothesis. Here, we show that by placing certain hypothesis on the initial measure ξ_0, the approximations ξ_t^λ obtained by the Liouville equation satisfy the required hypothesis, which allow us to approximate the support of ξ_t^λ by constructing the corresponding Christoffel-Darboux polynomial.

The following statement shows the existence of a sublevel set that approximates the support of the sequence of measures ξ_t^λ, when λ and $t \in [0, T]$ are fixed.

Proposition 5.12. *Let ξ_0 be absolutely continuous with respect to the Lebesgue measure and let us suppose that* $\operatorname{supp}(\xi_0)$ *is compact. For a fixed $\lambda > 0$, and $t \in [0, T]$, consider ξ_t^λ obtained by solving (5.21), and $M_{d,t}^\lambda(\xi_t^\lambda)$ the corresponding moment matrix of degree d. For every $\epsilon > 0$ (small enough), there exists $d \in \mathbb{N}$ (large enough) and $\gamma_d > 0$, such that the sublevel set*

$$S_{d,t}^\lambda := \{z \in \mathbb{R}^p \mid b_d(z)^\top M_{d,t}^\lambda(\xi_t^\lambda)^{-1} b_d(z) \leqslant \gamma_d\} \tag{5.35}$$

satisfies

$$d_H(S_{d,t}^\lambda, \operatorname{supp}(\xi_t^\lambda)) \leqslant \epsilon, \tag{5.36}$$

as $d \to +\infty$.

Proof : For each $\lambda > 0$ and $t \in [0, T]$, if we show that

- The set $\operatorname{supp}(\xi_t^\lambda)$ is compact and has nonempty interior.

- It holds that ξ_t^λ is absolutely continuous with respect to the Lebesgue measure.

then, the statement follows by applying [104, Theorem 3.11] to the measure ξ_t^λ.

The aforementioned properties basically follow from the fact that, for a fixed $t \in [0, T]$ and $\lambda > 0$, the mapping $F_t^\lambda : \mathbb{R}^n \to \mathbb{R}^n$ is a homeomorphism obtained from the solution of an ODE with Lipschitz continuous right-hand side (5.8). Let L^λ denote the (uniform with respect to time) Lipschitz constant for the mapping on the right-hand side of (5.8). One can readily show that for a pair of initial conditions y_0, z_0 and $y_t := F_t^\lambda(y_0)$, $z_t := F_t^\lambda(z_0)$, it holds that

$$|z_0 - y_0| \exp(-L^\lambda t) \leqslant |z_t - y_t| \leqslant |z_0 - y_0| \exp(L^\lambda t).$$

Using this estimate, and recalling that $\xi_0^\lambda = \xi_0$, it readily follows that $\operatorname{supp}(\xi_t^\lambda)$ is compact and has nonempty interior under the given hypothesis on ξ_0.

Absolute continuity of ξ_t^λ with respect to Lebesgue measure holds if ξ_t^λ is absolutely continuous with respect to ξ_0. The later indeed holds because for every measurable set A, Lipschitz continuity of F_t^λ implies that

$$\xi_0(A) = 0 \Rightarrow \xi_t^\lambda(A) = \xi_0((F_t^\lambda)^{-1}(A)) = 0, \tag{5.37}$$

whence the desired result follows. ♢

5.6 Illustrative Example

In this section, we give an example that illustrates the computation of the moments associated with ξ_t^λ of the regularized system (5.8) in the case where $f : \mathbb{R}^2 \to \mathbb{R}^2$, by applying the moment-SOS hierarchy [86].

Consider the constrained system (5.7) of Example 14 where $f(z) = Az$ with $A = \left[\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right]$ and $\mathcal{S} = \mathbb{R}^2_+$. Let us write the regularized system (5.8) in polar coordinates (r, θ) as follows:

$$\begin{cases} \dot{r}(t) = 0, \\ \dot{\theta}_\lambda(t) = -1 - \frac{1}{\lambda}(\theta_\lambda(t) - \operatorname{proj}(\theta_\lambda(t), \mathcal{S}(t))). \end{cases}$$

or equivalently:

$$\begin{cases} \dot{r}(t) = 0, \\ \dot{\theta}_\lambda(t) = -1 - \frac{1}{\lambda}(\theta_\lambda(t) - \max(\theta_\lambda(t), 0)). \end{cases} \tag{5.38}$$

Let $d = 4$ be the degree of relaxation, and let us choose different values of the regularization parameter $\lambda \in \{0.05, 0.1, 0.5\}$. We introduce the initial measure as a Dirac measure with respect to time product a uniform measure in $[0,1] \times [0, \frac{1}{2}]$ with respect to the state.

We calculate the moment of the initial measure to replace it directly in Liouville constraint (5.30), where the variables z_1 and z_2 in (5.30) are respectively r and θ. For all $(a, b_1, b_2) \in \mathbb{N}^3$, with $a+b_1+b_2 \leqslant d$, the moment of the initial measure is then given as

$$\begin{aligned}\int_0^T \int_{\mathbb{R}^n} t^a z_1^{b_1} z_2^{b_2}\ d\mu_0(t,z) &= \int_0^T \int_{\mathbb{R}^n} t^a z_1^{b_1} z_2^{b_2}\ \delta_0(dt)\lambda_{[0,1]}(dz_1)\lambda_{[0,\frac{1}{2}]}(dz_2)\\ &= 0^a \int_0^1 z_1^{b_1}\ dz_1 \int_0^{\frac{1}{2}} 2z_2^{b_2}\ dz_2\\ &= 0^a \frac{1}{b_1+1}\left(1^{b_1+1} - 0^{b_1+1}\right)\frac{2}{b_2+1}\left(\left(\frac{1}{2}\right)^{b_2+1} - 0^{b_2+1}\right).\end{aligned}$$

Then we apply the moment-SOS hierarchy [86] which allows us to approximate numerically the moments of the unknown occupation measure and terminal measure. For different values of the terminal time $T \in \{0, 0.1, 0.2, \dots, 1\}$, this gives us:

- The evolution of the moment $\int r(t)^2\ d\mu_T^\lambda$ as a function of time, which we observe numerically is a constant for different values of the regularization parameter λ.

- The evolution of the moment $\int \theta(t)^2\ d\mu_T^\lambda$ as a function of time for different values of the regularization parameter λ, which is illustrated on Figure 5.2.

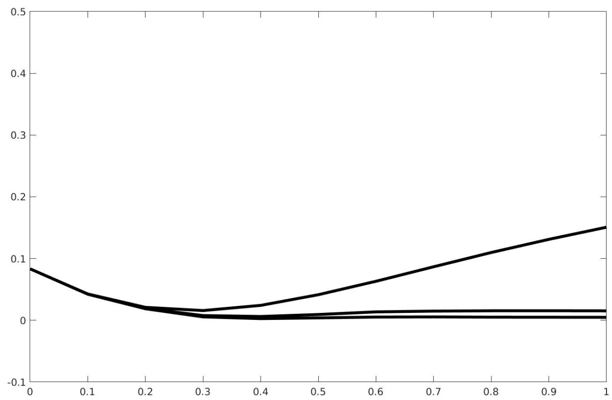

Figure 5.2 – First order moment of the second state (vertical axis) of the occupation measure of the regularized system, as a function of time (horizontal axis), for different values of the regularization parameter (top curve $\lambda = 0.5$, middle curve $\lambda = 0.1$, bottom curve $\lambda = 0.05$)

6

Conclusion and Perspectives

6.1 General Conclusions

This thesis has considered stability analysis and numerical approximation for a class of constrained dynamical systems. In Chapter 1, we discussed the relevance of such systems with some motivating examples and drew connections with different types of set-valued/nonsmooth dynamical systems (complementarity system, projected dynamical system, Moreau's sweeping processes). The mathematical background related to the tools used in developing the results was collected in Chapter 2, where we recalled relevant notions and results from the fields of convex analysis, sum-of-squares representation of polynomials, convex optimization, and the solutions of nonsmooth systems. The first two chapters, therefore, provide a brief introduction and an overview of the topic covered in this thesis. The main results of this dissertation were then organized in three core chapters, whose technical contributions are listed below:

- **In Chapter 3**, we first described the stability notions of our interest and gave the definition of Lyapunov functions with constrained domains for system trajectories. Then we addressed our first main result, that is the stability analysis for a class of complementarity systems using the method of Lyapunov functions. Questions pertaining to the existence of continuously differentiable cone-copositive Lyapunov functions were answered in the affirmative for exponentially stable complementarity systems by constructing a Lyapunov function as a functional of the solution trajectories. Then, we showed the existence of homogeneous Lyapunov function for the case when the vector field is homogeneous, which is useful for the numerical computation.

- **In Chapter 4**, under certain conditions on the vector field in the system dynamics, some refinements of our results in Chapter 3 allowed us to restrict our search for cone-copositive Lyapunov functions within the class of rational functions of homogeneous polynomials. These statements indeed bring tractability to the numerical methods that we have been proposed for computing Lyapunov functions. In particular, two hierarchies of convex optimization problems were obtained using the methods based on discretization and SOS approximation, respectively for computing the desired Lyapunov function. As an illustration of our corresponding two algorithms, we studied some examples which are solved by using Matlab toolboxes.

- **In Chapter 5**, we studied the time evolution of nonsmooth constrained dynamical systems when the initial condition is described by a probability measure. We proposed an approximation technique based on constructing Lipschitz approximations for the original nonsmooth system, and then using the Liouville equation for the approximate Lipschitz dynamics. Numerical methods for computing the approximation of solutions of Liouville equation then allowed us to compute the moments and support of the probability measures associated to the original system using the moment-SOS hierarchy method.

6.2 Perspectives

Building up on the work summarized in previous section, let us conclude this dissertation by indicating certain possible paths for future research which may emerge from the works presented here.

6.2.1 Converse Results

Several immediate questions of interest emerge from our work in Chapter 3. The first one is to extend our results to broader classes of complementarity systems. Systems of the form (3.1) are one particular class of relative degree one systems, but in applications, one sees more complex complementarity systems of the form studied in [146]. In such a wider class of systems, one sees different kinds of constraints on the state trajectories. Moreover, the constraints may vary with time in which case one has to consider the possibility of time-varying Lyapunov functions. It would be interesting to consider converse questions for this broader class of systems.

6.2.2 Computation of Lyapunov Functions

Some extensions at the level of designing algorithms are also of potential interest. In our current treatment, we have considered, in Chapter 4, discretization algorithms in $\mathbb{R}^2$ and $\mathbb{R}^3$ for the computation of Lyapunov functions, where it is relatively straightforward to write algorithms for partition of simplices. It remains to be seen how the algorithms for simplicial partition in higher dimensions perform in computing such functions. It is also interesting to see how the method of discretization can be applied for more generic sets than polyhedral cones. Just like the generalizations that can be carried out for addressing converse results, we can also study the discretization algorithm for computing Lyapunov functions for more general dynamical systems (with possibly different complementarity relations). Moreover, for the SOS method treated in Chapter 4, we used Pólya's theorem on conic sets and Putinar's theorem on compact semi-algebraic sets, for expressing positive polynomials. One may explore the questions of finding appropriate representation of positive polynomials for more general sets, such as unbounded sets which are not necessarily cones.

6.2.3 Connections with Liouville Equation

A first direction of research that appears from Chapter 5 is to seek improvements in the approach adopted in this chapter. It is observed that the proposed Lipschitz approximations are difficult to simulate numerically. In particular, for the illustrated example, we implemented the projection map onto a cone by splitting the Liouville equation in different regions of the state space, where each of them corresponds to the region where the approximating ODE is easily described by elementary smooth functions (compatible with GloptiPoly). One could use some recent work on approximating ODEs with twice differentiable right-hand side [60] to see if the resulting implementation is easier to simulate for a broader class of constraint sets.

Another potential direction of research that comes out from our work presented in Chapter 5 is the possibility of using the proposed tools for optimal control problems. As was done for ODEs [103], it is possible to use the formalism of Liouville equation for optimal control problems. The optimal control for the class of nonsmooth systems studied in this dissertation is a challenging problem, and it has been addressed recently in [46, 59, 151]. The analysis of optimal control problems is a branch of mathematical optimization, which finds its origin in the calculus of variations and has many applications in a wide variety of topics. Hence, it would be interesting to see if the methods proposed in Chapter 5 provide a numerically constructive solution to such challenging problems.

6.3 Non-convex extensions

For the constrained dynamical systems considered in this dissertation, we have limited ourselves to the constraints given by convex sets. One more direction of research that could be explored is the formulation of similar problems for certain classes of non-convex sets. The primary difference that occurs with non-convex sets is that in this case, we are going to loose the maximal monotonicity of the set-valued mapping in the dynamics, but for some special classes of nonconvex, one could find some closely related analysis tools in the literature. One possible example of such nonconvex sets is the so-called *prox-regular* sets; they form a class of non-convex sets for which the normal cone operator is not monotone but there is a parameter that quantifies the extent of nonconvexity in the underlying set. To define prox-regular sets, let us just recall the notion of Fréchet normal cone for (general) closed sets.

Definition 31. (Fréchet normals [115]). For a closed set $\mathcal{S} \subset \mathbb{R}^n$, and $x \in \mathcal{S}$, the vector $w \in \mathbb{R}^n$ is called a Fréchet normal to the set $\mathcal{S}$ at x if, for every $\epsilon > 0$, there exists $\delta > 0$ such that

$$\langle w, x' - x \rangle \leqslant \epsilon |x' - x|, \quad \forall x' \in \mathcal{S},\ |x - x'| < \delta. \tag{6.1}$$

The set of all Fréchet normals at a point $x \in \mathcal{S}$ form a cone denoted by $\mathcal{N}(\mathcal{S}; x)$.

We use this definition for introducing uniformly prox-regular sets as follows:

Definition 32. (Uniformly prox-regular set [132]). A set $\mathcal{S}$ is called uniformly prox-regular with constant $\frac{1}{r}$, or simply r-prox-regular, if for each $x \in \mathcal{S}$, and each $w \in \mathcal{N}(\mathcal{S}; x)$ with $|w| < 1$, it holds that $\mathrm{proj}_{\mathcal{S}}(x+rw) = \{x\}$, that is, x is the unique nearest vector to $x + rw$ in the set $\mathcal{S}$.

It follows from the above definition that $\mathcal{S}$ is an r-prox-regular set, if and only if, for each $x, x' \in \mathcal{S}$, and each $w \in \mathcal{N}(\mathcal{S}; x)$, with $|w| < 1$, we have

$$|rw|^2 = |x + rw - x|^2 < |x + rw - x'|^2 = |x - x'|^2 + 2\langle rw, x - x' \rangle + |rw|^2,$$

or equivalently for each $w \in \mathcal{N}(\mathcal{S}; x)$,

$$\left\langle \frac{w}{|w|}, x - x' \right\rangle \geqslant -\frac{1}{2r}|x - x'|^2, \quad \forall x' \in \mathcal{S}. \tag{6.2}$$

Letting $r \to \infty$ in the inequality (6.2) gives us that w is the normal vector at $x \in \mathcal{S}$ in the classical sense of convex analysis. Thereby we say that the case $r \to \infty$ corresponds to $\mathcal{S}$ being convex. Then the convex sets are a particular case of the r-prox-regular sets with r being arbitrarily large.

With such a characterization, it is possible to study stability of an equilibrium for dynamical systems where the constraints are prox-regular sets [145]. To get an idea of the kind of stability results that one obtains in such a setup, let us consider the following system:

$$\dot{x}(t) \in Ax(t) - \mathcal{N}_{\mathcal{S}}(x(t)) \tag{6.3}$$

where $x(t) \in \mathbb{R}^n$, $A \in \mathbb{R}^{n \times n}$, and there exists a constant $r > 0$ such that $\mathcal{S}$ is a nonempty, closed, and r-prox-regular set. The following theorem gives sufficient conditions for asymptotic stability of system (6.3).

Theorem 6.1. *[145] Consider system (6.3) and suppose that the following inequality is satisfied for some $\theta > 0$:*

$$A + A^\top \leqslant -\theta I. \tag{6.4}$$

For $0 < \beta < 1$, define

$$\mathcal{R}_\rho := \{x \in \mathbb{R}^n \mid \|x\| \leqslant \frac{\beta\theta r}{\|A\|}\}. \tag{6.5}$$

If θ is large enough and $0 \in \mathcal{S}$, then system (6.3) is asymptotically stable and the basin of attraction contains the set $\mathcal{R}_\rho \cap \mathcal{S}$.

As we can observe, the primary difference compared to the stability conditions proposed for convex valued sets is that the asymptotic stability no longer holds globally for non-convex sets. So, we see qualitative differences in the results that one gets by replacing convex sets by prox-regular sets. Moreover, the underlying toolset for analysis also changes as one can no longer use the monotonicity relations. It therefore becomes interesting to revisit some of the questions from earlier chapters, which are summarized below:

- In what context, one can develop converse Lyapunov result considering the fact that the stability holds only locally in general for non-convex sets?

- To what extent we can generalize the key ideas behind our algorithms for computing Lyapunov functions when the domain is non-convex? It is interesting to note that Putinar's representation of positive polynomials does not require convexity, but only compactness. On the other hand, certain arguments used in the discretization algorithm do not necessarily require convexity.

- Lastly, how can we study the time evolution probability measure with nonconvex domains? For the approach we have adopted in our work, the convergence of the solutions of approximating ODEs to the solution construct a sequence of ODEs with Lipschitz continuous right-hand sides which approximate the solution of the nonsmooth system for a fixed initial condition.

A

Preliminaries to Chapter 5

In this appendix, we provide some preliminaries concerning measures, moments, occupation measures and dual spaces to provide the reader some background and references for the work carried out in Chapter 5.

A.1 Measures and Moments

Most of the material in this section has been borrowed from [84]. Let X be a compact subset of the Euclidean space $\mathbb{R}^n$. Let $\mathscr{B}(X)$ denotes the Borel σ-algebra defined as the smallest collection of subsets of X which contains all open sets.

Definition 33. (Signed measure). A signed measure is a function $\mu : \mathscr{B}(X) \to \mathbb{R} \cup \{\infty\}$ such that $\mu(\emptyset) = 0$ and $\mu(\cup_{k\in\mathbb{N}} X_k) = \sum_{k\in\mathbb{N}} \mu(X_k)$ for any pairwise disjoint $X_k \in \mathscr{B}(X)$.

Definition 34. (Positive measure). A positive measure is a signed measure which takes only nonnegative values.

Positive measures on the Borel σ-algebra are often called Borel measures, and positive measures which takes finite values on compact sets are often called Radon measures.

Definition 35. (Probability measure). A probability measure μ on X is a positive measure such that $\mu(X) = 1$.

Let us denote by $\mathscr{M}_+(X)$ the cone of positive measures supported on X, and by $\mathscr{P}(X)$ the set of probability measures supported on X. Geometrically, $\mathscr{P}(X)$ is an affine section of $\mathscr{M}_+(X)$.

Example 15. (Dirac measure). The Dirac measure at x, denoted δ_x, is a probability measure such that $\delta_x(A) = 1$ if $x \in A$, and $\delta_x(A) = 0$ if $x \notin A$.

For a given compact set $X \subset \mathbb{R}^n$, let $\mathscr{M}(X)$ denote the Banach space of signed measures supported on X, so that a measure $\mu \in \mathscr{M}(X)$ can be defined as a function that takes any subset of X and returns a real number. Elements of $\mathscr{M}(X)$ are continuous linear functionals acting on the Banach space of continuous functions $\mathscr{C}(X)$, that is, as elements of the dual space $\mathscr{C}(X)'$.

The action of a measure $\mu \in \mathscr{M}(X)$ on a test function $v \in \mathscr{C}(X)$ can be modeled with the duality pairing

$$\langle v, \mu \rangle := \int_X v(x) d\mu(x).$$

Let us denote by $\mathscr{C}_+(X)$ the cone of positive continuous functions on X, whose dual can be identified to the cone of positive measures on X, i.e. $\mathscr{M}_+(X) = \mathscr{C}_+(X)'$.

Definition 36. (Indicator function). The indicator function of a set A is the function $x \mapsto I_A(x)$ such that $I_A(x) = 1$ when $x \in A$ and $I_A(x) = 0$ when $x \notin A$.

Definition 37. (Monomial). For multi-index $\alpha \in \mathbb{N}^n$, and vectors $x \in \mathbb{R}^n$, a monomial is defined as

$$x^\alpha := \prod_{i=1}^{n} x_i^{\alpha_i}.$$

The degree of a monomial with exponent $\alpha \in \mathbb{N}^n$ is equal to $|\alpha| := \sum_{i=1}^{n} \alpha_i$.

Definition 38. (Moment). Given a measure $\mu \in \mathscr{M}(X)$, the real number

$$y_\alpha := \langle x^\alpha, \mu \rangle = \int_X x^\alpha \mu(dx) \tag{A.1}$$

is called its moment of order $\alpha \in \mathbb{N}^n$.

The sequence $(y_\alpha)_{\alpha \in \mathbb{N}^n}$ is called the sequence of moments of the measure μ, and given $d \in \mathbb{N}$, the truncated sequence $(y_\alpha)_{\alpha \leqslant d}$ is the vector of moments of degree d.

Definition 39. (Representing measure). If y is the sequence of moments of a measure μ, which means that if (A.1) is satisfied for all $\alpha \in \mathbb{N}^n$, we say that μ is a representing measure for y.

In the theory of moments, a fundamental problem concerns the identification of infinite or truncated sequences of moments of some measure. Instead of manipulating a measure, which is a abstract object, we manipulate its moments. In fact, a measure on a compact set is uniquely determined by the infinite sequence of its moments.

Lebesgue's dominated convergence:

In measure theory, Lebesgue's dominated convergence theorem provides sufficient conditions under which almost everywhere convergence of a sequence of functions implies convergence in the L^1 norm.

Theorem A.1. *(Lebesgue's dominated convergence theorem). Let (f_n) be a sequence of measurable functions on a measure space (S, Σ, μ). Suppose that the sequence converges pointwise to a function f and is dominated by some integrable function g in the sense that*

$$|f_n(x)| \leqslant g(x)$$

for all numbers n in the index set of the sequence and all points $x \in S$. Then f is integrable (in the Lebesgue sense) and

$$\lim_{n\to\infty} \int_S |f_n - f| d\mu = 0$$

which also implies

$$\lim_{n\to\infty} \int_S f_n d\mu = \int_S f d\mu.$$

Remark A.2. The statement 'g is integrable' means that measurable function g is Lebesgue integrable; i.e.

$$\int_S |g| d\mu < \infty$$

A.2 Occupation Measures

Definition 40. (Occupation measure). Given an initial condition x_0, the occupation measure of a trajectory $x(t|x_0)$ is defined by

$$\mu(A \times B|x_0) := \int_A I_B(x(t|x_0))dt$$

for all $A \in \mathscr{B}([t_0, T])$ and $B \in \mathscr{B}(X)$.

Let us define

$$\mu(dt, dx) = dt\delta_{x(t)}(dx) \in \mathscr{M}_+([t_0, T] \times X).$$

A geometric interpretation is that μ measures the time spent by the graph of the trajectory $(t, x(t|x_0))$ in a given subset $A \times B$ of $[t_0, T] \times X$. An analytic interpretation is that integration w.r.t. μ is equivalent to time integration along a system trajectory, i.e.

$$\int_{t_0}^{T} v(t, x(t|x_0))dt = \int_{t_0}^{T} \int_X v(t, x)\mu(dt, dx|x_0)$$

for every test function $v \in \mathscr{C}([t_0, T] \times X)$.

Now think of initial condition x_0 as a random variable in X, or more abstractly as a probability measure $\xi_0 \in \mathscr{M}_+(X)$, that is a map from the Borel σ-algebra $\mathscr{B}(X)$ of subset of X to the interval $[0, 1] \subset \mathbb{R}$ such that $\xi_0(X) = 1$.

Definition 41. (Average occupation measure). Given an initial measure ξ_0, the average occupation measure of the flow of trajectories is defined by

$$\mu(A \times B) := \int_X \mu(A \times B|x_0)\xi_0(dx_0)$$

for all $A \in \mathscr{B}([t_0, T])$ and $B \in \mathscr{B}(X)$.

A.3 Basic Concepts in Dual Spaces

The notions in this section appear in [109].

Definition 42. (Bounded linear functional). A linear functional f on a normed space X is bounded if there is a constant M such that

$$|f(x)| \leqslant M\|x\|, \;\; \forall x \in X.$$

Definition 43. (Dual space). Let X be a normed linear vector space. The space of all bounded linear functionals on X is called the normed dual of X and is denoted $X^\star$.

The norm of an element $f \in X^\star$ is

$$\|f\| = \sup_{\|x\| \leqslant 1} |f(x)|.$$

Given a normed space X, its normed dual $X^\star$ simply refers to the dual of X. The value of a linear functional $x^\star \in X^\star$ at the point $x \in X$ is denoted by $x^\star(x)$ or by the symmetric notation $\langle x, x^\star \rangle$.

Theorem A.3. *(Riesz representation for $C[a,b]$ space). Let f be a bounded linear functional on $X = C[a,b]$. Then there is a function v of bounded variation on $[a,b]$ such that for all $x \in X$,*

$$f(x) = \int_a^b x(t)\, dv(t)$$

and such that the norm of f is the total variation of v on $[a,b]$. Conversely, every function of bounded variation on $[a,b]$ defines a bounded linear functional on X in this way.

It should be noted that Theorem A.3 does not claim uniqueness of the function of bounded variation v representing a giving linear functional f.

An important concept that arises naturally upon the introduction of the dual space is the weak convergence.

Definition 44. (Weak convergence). A sequence $\{x_n\}$ in a normed linear vector space X is said to converge weakly to $x \in X$ if for every $x^\star \in X^\star$, we have $\langle x_n, x^\star \rangle \to \langle x, x^\star \rangle$. In this case we write $x_n \to x$ weakly.

We have that if $x_n \to x$ strongly, then $x_n \to x$ weakly.

Definition 45. (Weak* convergence). A sequence $\{x_n^\star\}$ in $X^\star$ is said to converge weak-star (or weak*) to the element $x^\star$ if for every $x \in X$, $\langle x, x_n^\star \rangle \to \langle x, x^\star \rangle$. In this case we write $x_n^\star \to x^\star$ weak*.

Then in $X^\star$, we have three notions of convergence: strong, weak, and weak*. Moreover, strong implies weak, and weak implies weak* convergence. In general, the reverse statements do not hold.

Theorem A.4. *(Alaoglu). Let X be a real normed linear space. The closed unit sphere in $X^\star$ is weak* compact.*

Theorem A.5. *(Bolzano–Weierstrass). Each bounded sequence in $\mathbb{R}^n$ has a convergent subsequence.*

A.4 GloptiPoly

We briefly describe the toolbox GloptiPoly in Matlab, used in Chapter 5.

GloptiPoly [87] is a Matlab toolbox that builds convex linear matrix inequality (LMI) relaxations of the generally non-convex global optimization problem of minimizing a multivariable polynomial function subject to polynomial inequalities, equalities or integer constraints. It produces a series of lower bounds monotonically converging to the global optimum. Numerical experiments illustrate that for most of the problems described and available in the literature, the global optimum is reached at low computational cost.

GloptiPoly is intented to solve, or at least approximate, the Generalized Problem of Moments (GPM), an infinite-dimensional optimization problem which can be viewed as an extension of the classical problem of moments. From a theoretical viewpoint, the GPM has impact in various areas of mathematics such as algebra, functional analysis, probability and statistics, etc. Moreover, the GPM has important applications in many fields such as optimization, control, etc.

B

Liouville Equation for a Nonlinear ODE

B.1 Derivation of the Liouville Equation

Let $\dot{x}(t) = f(t, x(t))$ be a nonlinear ODE where f is a given vector field and $x(t) \in X$ is the state of the system, with $X \subset \mathbb{R}^n$. Let $\mathcal{L} : \mathscr{C}^1([t_0, T] \times X) \to \mathscr{C}([t_0, T] \times X)$ be the linear operator defined by

$$v \mapsto \mathcal{L}v := \frac{\partial v}{\partial t} + \sum_{i=1}^{n} \frac{\partial v}{\partial x_i} f_i = \frac{\partial v}{\partial t} + \mathrm{grad} v.f.$$

For $v \in \mathscr{C}^1([t_0, T] \times X)$, we observe that

$$\begin{aligned} v(T, x(T)) - v(t_0, x(t_0)) &= \int_{t_0}^{T} dv(t, x(t)) = \int_{t_0}^{T} \dot{v}(t, x(t)) dt \\ &= \int_{t_0}^{T} \mathcal{L}v(t, x(t)) dt = \langle \mathcal{L}v, \mu \rangle \end{aligned}$$

where μ is the occupation measure defined in Definition 40, which can be written as

$$\langle v, \mu_T \rangle - \langle v, \mu_0 \rangle = \langle \mathcal{L}v, \mu \rangle \tag{B.1}$$

by defining the initial and terminal occupation measures

$$\mu_o(dt, dx) := \delta_{t_0}(dt)\delta_{x(t_0)}(dx), \;\; \mu_T(dt, dx) := \delta_T(dt)\delta_{x(T)}(dx).$$

The adjoint linear operator $\mathcal{L}' : \mathscr{C}([t_0, T] \times X)' \to \mathscr{C}^1([t_0, T] \times X)'$ is defined by $\langle v, \mathcal{L}'\mu \rangle := \langle \mathcal{L}v, \mu \rangle$ for all $\mu \in \mathscr{M}([t_0, T] \times X)$ and $v \in \mathscr{C}^1([t_0, T] \times X)$. Using integration by parts, we obtain :

$$\mu \mapsto \mathcal{L}'\mu := -\frac{\partial \mu}{\partial t} - \mathrm{div} f \mu,$$

where the derivatives of measures are understood in the weak sense, or in the sense of distributions (i.e., via their action on suitable test functions), and the change of sign comes from the integration by parts formula. For more details, the interested reader is referred to any textbook on functional analysis and partial differential equations, e.g., [73].

Equation (B.1) can be written equivalently as

$$\langle v, \mu_T \rangle - \langle v, \mu_0 \rangle = \langle v, \mathcal{L}'\mu \rangle$$

and since this equation holds for all $v \in \mathscr{C}^1([t_0, T] \times X)$, we get a linear partial differential equation (PDE) linking the nonnegative measures μ_T, μ_0 and μ.

$$\mathcal{L}'\mu = \mu_T - \mu_0$$

which is

$$\frac{\partial \mu}{\partial t} + \operatorname{div} f\mu + \mu_T = \mu_0. \tag{B.2}$$

This linear transport equation is called the continuity equation, or Liouville's equation, or the equation of conservation of mass. This equation is classical in fluid mechanics and statistical physics.

Note that we can disintegrate the occupation measure as follows

$$\mu(dt, dx) = dt\ \xi(dx|t)$$

where $\xi(.|t) \in \mathscr{M}_+(X)$ is the conditional of μ with respect to t. Liouville's equation (B.2) can be equivalently written as a linear PDE satisfied by the probability measure ξ, that is

$$\frac{\partial \xi}{\partial t} + \operatorname{div} f\xi = 0 \tag{B.3}$$

with an initial measure $\xi(.|t = 0) = \xi_0$.

Lemma B.1. *(Cauchy ODE = Liouville PDE). There exists a unique solution to the Liouville PDE (B.2), which is concentrated on the solution of the Cauchy ODE $\dot{x}(t) = f(t, x(t))$.*

B.2 Notions for the Liouville Equation

This section presents some important notions and a theorem related to the Liouville equation which can be seen in [47] (Apprendix, section 5.4).

Definition 46. Let T be a Borel map : $X \to Y$, the push forward (or image measure) of μ through T is the Borel measure, denoted $T\sharp\mu$ defined on Y by

$$T\sharp\mu(B) = \mu(\{x \in X : T(x) \in B\}), \text{ for every Borel subset B of Y.}$$

We remark that $T\sharp\mu$ can equivalently be defined by the change of variables formula:

$$\int_Y \varphi dT\sharp\mu = \int_X \varphi(T(x))d\mu(x), \ \ \forall\varphi \in \mathscr{C}(Y). \tag{B.4}$$

Let f be a smooth vector-field $\mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^d$ such that there is a constant C such that

$$|f(t,x)| \leqslant C(1+|x|), \ \ |f(t,x) - f(t,y)| \leqslant C|x-y|, \ \ \forall (t,x,y).$$

For $x \in \mathbb{R}^d$, let us denote the flow map $t \mapsto X_t(x)$ as the value at time t of the solution of the nonautonomous ODE

$$\dot{y}(s) = f(s, y(s)), \ \ y(0) = x.$$

Which means that X_t satisfies

$$\partial_t X_t(x) = f(t, X_t(x)), \ \ X_0(x) = x, \ \ (t,x) \in \mathbb{R}_+ \times \mathbb{R}^d.$$

Let ξ_0 be a probability measure on $\mathbb{R}^d$, that captures an initial spatial distribution of particles that follow the flow of f. We shall see that $\xi_t := X_t\sharp\xi_0$ is identified by the following PDE, the Liouville equation:

$$\partial_t \xi + \mathrm{div}(\xi f) = 0 \tag{B.5}$$

with the initial condition:

$$\xi|_{t=0} = \xi_0. \tag{B.6}$$

Since none of regularity assumption is made on ξ (ξ could be a Dirac mass and then $X_t\sharp\xi_0$ would remain a Dirac mass for every $t > 0$), we have to interpret the continuity equation in the weak sense i.e. in the sense of distributions. The family of probability measures $t \mapsto \xi_t$ is a measure-valued solution of (B.5)-(B.6) if:

- it is continuous in the sense that, $\forall\phi \in \mathscr{C}_c(\mathbb{R}^d)$, the map

$$M_\phi : t \mapsto \int_{\mathbb{R}^d} \phi d\xi_t \ \text{ is continuous on } \ [0,\infty) \ \text{ and } \ M_\phi(0) = \int_{\mathbb{R}^d} \phi d\xi_0, \tag{B.7}$$

- $\forall T > 0$, $\forall r > 0$ and $\forall\varphi \in \mathscr{C}^1([0,T] \times \mathbb{R}^d)$ such that $\varphi(T,.) = 0$ and $\varphi(t,.)$ is supported by B_r for every $t \in [0,T]$, we have

$$\int_0^T \Big(\int_{\mathbb{R}^d} (\partial_t \varphi(t,x) + f(t,x).\nabla\varphi(t,x)) d\xi_t(x) \Big) dt = -\int_{\mathbb{R}^d} \varphi(0,x) d\xi_0(x). \tag{B.8}$$

The equation (B.8) is the weak formulation of the continuity equation (B.5). The demonstration consists of taking φ a test-function, multiplying

(B.5) by φ and then integrating, we obtain:

$$\begin{aligned}
&\int \varphi(\partial_t \xi + \operatorname{div}(\xi f)) = 0 \\
&\Longrightarrow \int \varphi.\partial_t \xi + \int \varphi.\operatorname{div}(\xi f) = 0 \\
&\Longrightarrow -\int \partial_t \varphi.\xi + \underbrace{\int \varphi(T,.)d\xi_T}_{0} - \int \varphi(0,.)d\xi_0 - \int grad\varphi.\xi.f = 0 \\
&\Longrightarrow \int (\partial_t \varphi + grad\varphi.f)\xi = -\int \varphi(0,.)d\xi_0 \\
&\Longrightarrow \int_0^T \Big(\int_{\mathbb{R}^d} (\partial_t \varphi + f.\nabla\varphi) d\xi \Big) dt = -\int_{\mathbb{R}^d} \varphi(0,.)d\xi_0.
\end{aligned}$$

Theorem B.2. *The measure-valued curve $t \mapsto X_t \sharp \xi_0$ is the unique measure-valued solution of* (B.5)-(B.6).

Proof : It is obvious that $t \mapsto \xi_t := X_t \sharp \xi_0$ satisfies the continuity condition (B.7). Then it remains to prove that $t \mapsto \xi_t := X_t \sharp \xi_0$ satisfies the continuity equation. Let φ be a test-function, then using the definition $\xi_t := X_t \sharp \xi_0$, Fubini's theorem and $\varphi(T,.) = 0$, we have

$$\begin{aligned}
&\int_0^T \Big(\int_{\mathbb{R}^d} (\partial_t \varphi(t,x) + f(t,x).\nabla\varphi(t,x)) d\xi_t(x) \Big) dt \\
&= \int_{\mathbb{R}^d} \Big(\int_0^T (\partial_t \varphi(t, X_t(x)) + f(t, X_t(x)).\nabla\varphi(t, X_t(x))) dt \Big) d\xi_0(x) \\
&= \int_{\mathbb{R}^d} \Big(\int_0^T \frac{d}{dt}(\varphi(t, X_t(x))) dt \Big) d\xi_0(x) \\
&= \int_{\mathbb{R}^d} (\varphi(T, X_T(x)) - \varphi(0, X_0(x))) d\xi_0(x) = -\int_{\mathbb{R}^d} \varphi(0,x) d\xi_0(x).
\end{aligned}$$

So $X_t \sharp \xi_0$ is a measure-valued solution of (B.5)-(B.6).

To prove uniqueness, suppose that $t \mapsto \xi_t$ and $t \mapsto \nu_t$ are two solutions and let $\mu_t = \xi_t - \nu_t$, so for every test-function φ, we have

$$\int_0^T \Big(\int_{\mathbb{R}^d} (\partial_t \varphi(t,x) + f(t,x).\nabla\varphi(t,x)) d\mu_t(x) \Big) dt = 0. \tag{B.9}$$

Let $\psi \in \mathscr{C}_c((0,+\infty) \times \mathbb{R}^d)$ and let us consider the linear transport PDE:

$$\partial_t \varphi + f.\nabla\varphi = \psi \;\; on \;\; (0,T) \times \mathbb{R}^d, \;\; \varphi(T,.) = 0. \tag{B.10}$$

It can be equivalently written as

$$\frac{d}{dt}[\varphi(t, X_t(x))] = \psi(t, X_t(x)), \;\; \varphi(T,.) = 0$$

which can be integrated as

$$\varphi(t, X_t(x)) = -\int_t^T \psi(s, X_s(x)) ds$$

which gives that the unique solution of (B.10) is

$$\varphi(t,x) = -\int_t^T \psi(s, X_s \circ X_t^{-1}(s))ds.$$

This function is compactly supported in space uniformly in time $t \in [0,T]$, we can then use it as test-function in (B.9), which gives

$$\int_0^T \int_{\mathbb{R}^d} \psi(t,x)d\mu_t(x)dt = 0.$$

This implies $\mu_t = 0$ because ψ is arbitrary. So the uniqueness is proved. $\diamondsuit$

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