Event-based control for some PDEs

Florent Koudohode

THÈSE

En vue de l'obtention du

DOCTORAT DE L'UNIVERSITÉ DE TOULOUSE

Délivré par : *l'Université Toulouse 3 Paul Sabatier (UT3 Paul Sabatier)*

Présentée et soutenue le *26/10/2023* **par :**

Mahuklo Florent KOUDOHODE

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JURY

Julie VALEIN	MdC Polytech Nancy	Rapporteure
Romain POSTOYAN	DR CNRS	Rapporteur
Frederic GOUAISBAUT	MdC UT3 Paul Sabatier	Examinateur
Jean-Michel CORON	Pr émérite SU	Examinateur
Antoine GIRARD	DR L2S	Président du Jury
Lucie BAUDOUIN	DR LAAS-CNRS	Directrice de thèse
Sophie TARBOURIECH	DR LAAS-CNRS	Directrice de thèse

École doctorale et spécialité :
EDSYS : Automatique 4200046

Unité de Recherche :
Equipe MAC, LAAS-CNRS (UPR 8001)

Directeur de Thèse :
Lucie BAUDOUIN et Sophie TARBOURIECH

Rapporteurs :
Julie VALEIN et *Romain POSTOYAN*

Je dédie cette thèse à
ma maman Juliette ADO
et à mes parents adoptifs
Christine AVOCETIEN
Vincent AVOCETIEN

Remerciements

Tel un arbre majestueux dont les racines plongent profondément dans la terre bien avant que sa cime ne perce le ciel, cette thèse en automatique est le fruit d'une préparation souterraine bien avant qu'elle ne voie le jour. Dans ce chapitre de remerciements, je rends hommage à ceux qui ont été mes jardiniers visibles comme invisibles, veillant à la concrétisation de ce dernier diplôme universitaire à travers leur soutien, leur encouragement, leur enseignement, leur mentorat, leur disponibilité, leur amour.

En tout premier lieu, je tiens à exprimer ma profonde gratitude envers mes directrices de thèse, Lucie Baudouin et Sophie Tarbouriech, pour leur bienveillance, leur gentillesse, leur patience et leur expertise qui ont été des piliers inestimables tout au long de ces trois années de thèse. La proposition de ce sujet, de même que votre disponibilité constante pour relire nos travaux à plusieurs reprises, ont grandement facilité mon parcours. Les échanges enrichissants, les critiques constructives et les encouragements sincères que vous m'avez offerts ont joué un rôle déterminant dans ma croissance intellectuelle et personnelle. Chaque réunion, chaque dialogue, a constitué une précieuse opportunité d'apprentissage et d'évolution et je ressens une profonde gratitude pour cette expérience formatrice. Merci d'avoir partagé votre expertise avec générosité et d'avoir créé un environnement propice à l'épanouissement intellectuel.

Je souhaite ensuite exprimer ma reconnaissance envers les deux rapporteurs, Julie Valein et Romain Postoyan. Vous avez, en un temps très court, lu, examiné et rapporté cette thèse. Nos échanges et vos questions très pertinentes ont clairement amélioré la qualité de ce travail. La technique de démonstration de stabilité utilisée dans le chapitre sur l'équation de Schrödinger a été plus affinée grâce à un article de Julie sur KdV et des travaux de Romain sur la commande événementielle en dimension finie m'ont permis de vite me familiariser avec le sujet. Je vous en suis très reconnaissant. Mes remerciements vont aussi envers Antoine Girard, Frederic Gouaisbaut et Jean-Michel Coron qui ont accepté d'être examinateurs de cette thèse. Fred et Antoine connaissent bien mon travail puisqu'ils ont été membres de mon comité de suivi individuel depuis la première année de thèse. Merci à vous pour votre disponibilité, votre évaluation approfondie et vos précieux commentaires qui ont contribué à améliorer cette thèse.

En licence, nous étions à un cours d'introduction aux EDPs, quand le professeur Marcos Aboubacar nous informait, à son allure exceptionnelle de vulgarisation des sciences, de la venue l'année suivante du "meilleur au monde" en théorie du contrôle en la personne du professeur Jean-Michel Coron. C'était la première fois que j'entendais parler de cette discipline, mais il en parlait tellement bien que je ne pus m'empêcher de dire à qui voulait l'entendre qu'après ma licence je ferais la théorie du contrôle. Et heureusement, ce rêve devint réalité car j'avais eu la bourse du projet CEA-SMIA financé par la banque mondiale, pour faire le master qui me permit de rencontrer le prof Coron à l'Institut de Mathématiques et de Sciences Physiques au Bénin. Sa façon de manier théorie et applications avec une extraordinaire pédagogie m'a convaincu de mon choix : je me rapprochai de lui pour faire un stage sur la stabilisation en temps fini de KdV en fin de master. C'est le début d'une belle aventure

de recherche qui me conduisit finalement à cette thèse. D'abord, étant à quelques mois de sa retraite, il ne pouvait me prendre en thèse, c'est ainsi qu'il m'aida à venir à Toulouse pour travailler avec Sylvain Everdoza à l'IMT. Avec Sylvain, Alexandre Seuret et Swann Marx, que je remercie très chaleureusement, en tant qu'étudiant visiteur, j'approfondis mon apprentissage du domaine de contrôle puis obtins la bourse du Labex CIMI pour mon second master Research and innovation, parcours P.D.E, Numerical Analysis and Control theory de l'Université Paul Sabatier. Je profite de cette occasion pour remercier tout le personnel et les enseignants de l'IMT et de l'IMSP qui m'ont fourni les outils nécessaires à la réussite de mes études aussi bien au Bénin qu'en France.

La première fois que j'ai entendu parler de la commande événementielle, c'était lors d'un workshop organisé par le labex CIMI à Toulouse : je n'imaginais pas être en train de passer trois ans sur le sujet quand Nicolas Espitia m'en parlait avec passion lors du dîner de ce workshop, vu que c'était son sujet de thèse quelques années plus tôt. Quelques mois plus tard, lorsque je l'avais recontacté pour quelques explications de ses travaux dans le domaine, il a été très disponible et m'avait même proposé de regarder le cas réaction-diffusion avec retard (au dernier chapitre). Merci infiniment Nicolas, pour cette collaboration qui m'a permis d'apprendre sur les petits gains, la stabilité ISS, le backstepping et surtout d'avoir eu l'honneur de travailler avec le professeur Miroslav Krstic qui est d'une finesse intellectuelle et d'une rapidité de réaction incroyables. Merci également de m'avoir parlé du postdoc ERC de Nikolaos Bekiaris-Liberis en Crète que je commence juste après ma thèse.

Je remercie aussi toute l'équipe MAC (et sa sœur de couloir POP) de m'avoir accueilli pendant ces trois années. Nos échanges à la salle café, au restaurant et dans le groupe WhatsApp Laas Gang avec toujours de bons plans me manqueront. Un remerciement tout particulier à mes co-bureaux pendant ma thèse, Thomas, Nicola, Olga, Ghania, Tichique, Joseph. Spécial merci à Flavien, Matteo (les deux), Marianne (pour les biscuits à 18h), et Mathieu pour mon intégration au début, puis à Manon, Yoni, Corbinian, Vit, Nicolas, Thiago, Carla, Ramon, Constantinos, Adrien, Camille, Alberto, Jared, Alexey, Saroj, Mauro, Larbi, Falak, Sara, Suha, Giovanni, Anubhab etc pour les dîners culturels chez Nicola, Corbi, Mathieu ou Saroj, les sorties, campings, randonnées, montagnes et surf. Merci aussi à tout le personnel de l'école doctorale Edsys, du LAAS (Sysadmin, Magasin, service gestion, bibliothèque) pour la bienveillance, la résolution rapide des difficultés d'ordi, de départ en mission, de reception de colis, de livres etc.

Merci à mes camarades Danhane, Michaelis, Ismaïla, Cyprien, Branda, Zongo, Émile, Jean-Jacques, Ousmane, Marcel, Romziath, Rocard, Clotilde, Boris, Kenneth, Larissa, Hans, Ariel, Helmut, Harlem, Sorel, Christelle etc on a étudié ensemble et ou élaboré des projets de développement des sciences (en témoignent nos séminaires de dimanche soir et création de l'Association Sciences pour l'Afrique durant nos thèses) et pour moi ce sont de meilleurs moments d'entraîde, de belles discussions et d'amitié. Je ne saurais ne pas remercier des personnes avec qui j'ai partagé des colocations, des moments de sortie, de divertissement et d'apprentissage : Lionel, Benjamin, Marie-Laure, Sèdami, Alexis, Lisa, Jean, Micha, Solène, Esmeralda, Rosalinda, Bignon, Mélinda, Emmanuella, Gloria, Oswald, Voltère. Merci beaucoup pour votre amitié, compréhension et tous ces beaux moments passés.

Je voudrais enfin exprimer ma profonde gratitude à ma mère Juliette ADO et à mes parents adoptifs Vincent et Christine AVOCETIEN. Quand très tôt mon père quitta ce monde, je suis devenu votre dernier garçon, vous m'avez pris en charge, éduqué, donné une passion à la lecture et aux sciences avec les nombreux livres de la bibliothèque de la maison. Ça y est, votre fils est docteur, soyez en très fiers car c'est surtout grâce à vous et je vous en suis éternellement reconnaissant. Merci à mes frères et sœurs Mouna, Viako, Merline, Zaché, André, Joachim, Clarisse, Martine et à mes neveux et nièces Ola, Sessi, Alicia, Stella, Sènan, Dègnon, Blessings, Tèni, Mahouklo pour votre soutien indéfectible, votre encouragement, vos appels visio et prières qui pour moi ont été d'une grande aide et d'inspiration.

Résumé

Au cours des dernières décennies, les stratégies de commande événementielle ont attiré une attention considérable en raison de leur application à divers domaines tels que les systèmes de contrôle en réseau, les processus industriels, la robotique, les systèmes biologiques, la transmission de données et le traitement du signal, etc. En commande événementielle, le contrôle n'est mis à jour que si nécessaire, ce qui favorise une utilisation optimisée des ressources et une économie d'énergie. Tous ces aspects sont cruciaux pour minimiser les ressources de communication et de calcul. Cela a conduit à une riche littérature sur la commande événementielle, en particulier pour des systèmes modélisés par des équations différentielles ordinaires. Dans cette thèse, notre objectif est de contribuer à la commande basée-événement d'équations aux dérivées partielles (EDPs), qui modélisent élégamment de nombreuses dynamiques physiques, biologiques ou sociales et pour lesquelles il existe peu de modèles de lois d'échantillonage. À ce titre, nous nous concentrerons sur la construction de commande événementielle pour trois EDPs spécifiques : les équations des ondes, de Schrödinger et de réaction-diffusion.

Tout d'abord, nous montrons comment maintenir la stabilité exponentielle de l'équation des ondes soumise à un terme source d'amortissement distribué et variable dans le temps lorsqu'il est soumis à une loi d'échantillonage, induisant un amortissement constant sur chaque intervalle d'échantillonnage. Après avoir vérifié l'existence et la régularité suffisante des solutions du système en boucle fermée où la loi d'échantillonage impose les mises à jour, nous établissons une condition suffisante basée sur des inégalités matricielles pour assurer la stabilité exponentielle globale de l'état. La preuve s'appuie sur une fonctionnelle de Lyapunov adéquate. Un autre point important est la démonstration de l'absence du phenomène Zeno, correspondant à l'accumulation d'instants de mises à jour de la loi de commande. Ces résultats s'étendent également à une équation des ondes excitée.

Ensuite, nous proposons une loi d'échantillonage statique (dépendant de l'état) pour l'équation de Schrödinger linéaire soumise à un terme source d'amortissement localisé. Nous prouvons qu'avec cette loi le phénomène Zeno ne se produit pas et que la stabilité exponentielle globale est conservée grâce à des estimations d'énergie qui utilisent une inégalité d'observabilité bien connue. De plus, nous utilisons une approche similaire pour proposer une loi d'échantillonage dynamique afin d'enrichir la loi d'échantillonage statique.

Enfin, nous considérons une équation de réaction-diffusion en une dimension d'espace avec un retard en entrée et soumise à une loi d'échantillonage. Nous traitons le retard comme une équation de transport, transformant le problème en un système de contrôle en cascade EDP-EDP puis nous effectuons une émulation sur le contrôle construit par backstepping. Nous démontrons qu'entre deux instants d'échantillonage, il existe un temps minimum d'inter-exécution qui garantit le caractère bien-posé et la stabilité exponentielle du système en boucle fermée. Notre analyse de stabilité s'appuie sur la propriété de stabilité ISS pour les EDP et utilise des arguments de petit gain.

Mots clés : Commande événementielle, stabilisation, observabilité, théorie de Lyapunov, méthode de backstepping, systèmes à retard, équation des ondes, équa-

tion de Schrödinger, équation de diffusion de réaction.

Abstract

Over the past few decades, event-triggered control strategies have garnered significant attention due to their potential to deal with diverse domains such as networked control systems, industrial processes, robotics, biological systems, data transmission, and signal processing etc. In event-based control paradigm, the control is not executed unless it is required, leading to optimized resource utilization, and energy conservation. All of these aspects are crucial when minimizing communication and computation resources. This has led to a substantial body of literature on event-based control in particular for systems described by ordinary differential equations. In this thesis, our objective is to contribute to the event-based control for the field of partial differential equations (PDEs), that elegantly model many physical, biological or social dynamics and for which there exist few designs of event-based control strategies. As such, we will focus on event-based control for three specific PDEs: wave equation, Schrödinger equation, and reaction-diffusion equations.

First, we prove how to maintain the exponential stability of the wave equation under a distributed and time varying damping source term when it is subjected to an event-triggering mechanism, inducing a constant damping on each sampling interval. After checking existence and sufficient regularity of the solutions of the closed loop-system where the event-triggering law imposes the updates, we establish a sufficient matrix inequality type condition for ensuring the global exponential stability of the state. The proof relies on an adequate Lyapunov functional, and besides, an important point is to prove the absence of Zeno behavior by preventing accumulation points in the sequence of updates. These results extend to an excited wave equation as well.

Secondly, we introduce the design of a state-dependent (static) event-triggering mechanism for the linear Schrödinger equation and consider both localized and constant damping source terms. We prove the avoidance of Zeno behavior and the global exponential stability is established through energy estimates that leverage some well-known observability inequality. Furthermore, we use a similar approach to provide a dynamic event-triggering law in order to enrich the static one.

Finally, we propose an event-triggered boundary control strategy for stabilizing a 1-D reaction-diffusion equation with input delay. We treat the delay as a transport equation, transforming the problem into a cascade of PDE-PDE controlled systems and then we perform emulation on the backstepping control. We demonstrate that under the proposed event-triggered boundary control, there exists a minimum dwell-time between two triggering times that guarantees the well-posedness and exponential stability of the closed-loop system. Our stability analysis relies on Input-to-State stability theory for PDEs and employs small-gain arguments.

Keywords: Event-triggered control, Stabilization, Observability, Lyapunov theory, Backstepping method, time-delay systems, wave equation, schrödinger equation, reaction diffusion equation.

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Notation and acronyms

Usual sets

- $\mathbb{N}$: set of natural integers, $\mathbb{N}^\star := \mathbb{N} \setminus \{0\}$: set of positive integers,
- $\mathbb{R}$: set of real numbers, $\mathbb{R}_+ := \{x \in \mathbb{R} : x \geq 0\}$: set of nonnegative real numbers,
- $\mathbb{C}$: set of complex numbers, $\mathbf{i}$: complex number s.t. $\mathbf{i}^2 = -1$, $\mathcal{R}e(z)$: real part of $z \in \mathbb{C}$, $\mathcal{I}m(z)$: imaginary part of $z \in \mathbb{C}$,

Linear algebra

- $\mathbb{R}^{n\times m}$: space of matrices with n rows and m columns with coefficients in $\mathbb{R}$,
- $m_{i,j}$: for $\mathrm{M} \in \mathbb{R}^{n\times m}$, refers to the coefficient on the i^{th} row and j^{th} column,
- $\mathrm{M}^\top$: transposition of a matrix M,
- $\mathbb{S}^n$: space of real symmetric matrices with n rows ($\mathrm{M}^\top = \mathrm{M}$),
- $I \in \mathbb{R}^{n\times n}$: is the identity matrix,
- $\mathbb{S}^n_+$: cone of symmetric positive semi-definite matrices, $\mathrm{M} \in \mathbb{S}^n_+ \Leftrightarrow \mathrm{M} \succeq 0$,
- $\mathbb{S}^n_{++}$: open cone of symmetric positive definite matrices, $\mathrm{M} \in \mathbb{S}^n_{++} \Leftrightarrow \mathrm{M} \succ 0$,
- $\mathrm{M} \preceq 0 \Leftrightarrow -\mathrm{M} \succeq 0$, $\mathrm{M} \prec 0 \Leftrightarrow -\mathrm{M} \succ 0$ and $\mathrm{M} \preceq \mathrm{N} \Leftrightarrow \mathrm{N} - \mathrm{M} \succeq 0$.
- The symbol $\star$ stands for symmetric blocks in symetric partitioned matrix.

Differential analysis

Let $\mathbf{\Omega} \subset \mathbb{R}^n$ be an open or compact set, $N \in \mathbb{N}^\star$.

- $\dot{\mathbf{x}} := \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t}$: derivative of the vector function $t \mapsto \mathbf{x}(t)$,
- ∂_x: partial differentiation operator with respect to the variable x,
- $\partial^k_{x_{i_1},\ldots,x_{i_k}} := \partial_{x_{i_1}} \cdots \partial_{x_{i_k}}$: k^{th} partial differentiation operator w.r.t. $x_{i_1}, \ldots, x_{i_k}$,
- $\partial_i u = u_{x_i} = \dfrac{\partial u}{\partial x_i}$ is the partial derivative of u with respect to the variable x_i
- $\nabla u = (\partial_{x_1} u, \ldots \partial_{x_N} u)$ and $\Delta u = \displaystyle\sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2}$.
- Given a compact set $\Omega, C^k(\Omega)$ denotes the space of functions $k-$times continuously differentiable on Ω (where k is an integer such that $k \geq 0$). Note moreover, that
$$C^\infty(\Omega) = \cap_{k\geq 0} C^k(\Omega).$$

- Given an open set Ω, $L^2(\Omega) = \left\{u : \Omega \to \mathbb{R}; \int_\Omega |u(x)|^2 dx < +\infty\right\}$ endowed with the norm:
$$\|u\|^2 = \langle u, u\rangle = \int_\Omega |u(x)|^2 dx$$
- $L^\infty(\Omega)$ denotes the space of essentially bounded function
- $H^1(\Omega) = \left\{u \in L^2(\Omega); \nabla u \in L^2(\Omega)^n\right\}$ endowed with the norm
$$\|u\|^2_{H^1(\Omega)} = \|u\|^2 + \|\nabla u\|^2 \text{ if } \Omega \subset \mathbb{R}^n$$
- $H^1_0(\Omega)$ is the closure in $H^1(\Omega)$ of $\mathcal{D}(\Omega) = C^\infty_c(\Omega)$, the Fréchet space of C^∞ function $\Omega \to \mathbb{R}$(or $\mathbb{C}$) compactly supported in Ω. It is also defined as the set of $u \in H^1(\Omega)$ such that $u = 0$ in the boundary $\partial\Omega$ of Ω
- $H^2(\Omega) = \left\{z \in L^2(\Omega), \nabla z \in L^2(\Omega)^n, \partial_{x_j}\partial_{x_i} z \in L^2(\Omega) \quad \forall i, j \in \{1, 2, 3, \ldots, n\}\right\}$, which is the set of functions z such that $\int_\Omega (|z|^2 + |\nabla z|^2 + |\Delta z|^2)$ is finite.
- The dual space of a Sobolev space H is denoted H'.
- For integrable functions $f : (x, t) \mapsto f(x, t)$ one will often write $\int_\Omega f(t)$ instead of $\int_\Omega f(x, t)dx$ in the sake of simplicity.
- By $C^0(A; \Omega)$, we denote the class of continuous functions on A, which take values in $\Omega \subseteq \mathbb{R}$.
- By $C^k(A; \Omega)$, where $k \geq 1$ is an integer, we denote the class of functions on A, which take values in Ω and have continuous derivatives of order k. $L^2(0, 1)$ denotes the equivalence class of Lebesgue measurable functions $f : [0, 1] \to \mathbb{R}$ such that $\|f\| = \left(\int_0^1 |f(x)|^2 dx\right)^{1/2} < \infty$.
- For an interval $J \subseteq \mathbb{R}_+$, the space $C^0(J; L^2(0, 1))$ is the space of continuous mappings $J \ni t \mapsto u(t, \cdot) \in L^2(0, 1)$. $H^2(0, 1)$ denotes the Sobolev space of functions $f \in L^2(0, 1)$ with square integrable (weak) first and second-order derivatives $f'(\cdot), f''(\cdot) \in L^2(0, 1)$.
- A function $f : J \to \mathbb{R}$ is called right continuous on an interval $J \subseteq \mathbb{R}_+$, if for every $s \in J$ and $\varepsilon > 0$ there exists $\delta(\varepsilon, s) > 0$ such that for all $\tau \in J$ with $s \leq \tau < s + \delta(\varepsilon, s)$ it holds that $|f(\tau) - f(s)| < \varepsilon$.
- A right continuous function $f : J \to \mathbb{R}$ is piecewise C^1 on J (and we denote it as $\mathcal{C}^1_{rpw}(J, \mathbb{R})$) if for every compact $K \subseteq J$ there exists a finite set $B \subset J \cap K$ such that f is C^1 on $(J \cap K)\backslash B$ and all meaningful limits $\lim_{h\to 0^+}(\dot{f}(s + h)), \lim_{h\to 0^+}(\dot{f}(s - h)), \lim_{h\to 0^+}(f(s + h)), \lim_{h\to 0^+}(f(s - h))$ exist for all $s \in J$ and are finite.
- The sup-norm is defined by $\|f\|_\infty = \max_{x\in J}(|f(x)|)$ for an interval $J \subseteq \mathbb{R}_+$.
- $I_m(\cdot)$, $J_m(\cdot)$ with $m \in \mathbb{Z}$, denote the modified Bessel and nonmodified Bessel functions of the first kind.

Abbreviations

ODE	Ordinary differential equation.
PDEs	Partial differential equations.
ETM	Event-Triggering Mechanism.
ETC	Event-Triggered Control.
s.t	such that.
i.e	that is.
resp.	respectively.

1 Introduction

"Wise men speak because they have something to say, fools because they have to say something "- Plato 370 BC

This thesis explores event-based control strategies, which have gained attention for their potential applications in various domains. The research focuses on extending event-based control to partial differential equations (PDEs), specifically addressing the wave equation, Schrödinger equation, and reaction-diffusion equations, where limited designs of event-based control strategies exist, aiming at improving system performance and resource utilization. In this introductive chapter, we will discuss the development of an efficient event-triggering mechanism, illustrating its general form with reference examples from both PDE and ODE context, and highlighting our contributions to this field.

1.1 On event-based control

1.1.1 What is event-triggered control?

In many modern control applications, controllers are implemented using digital platforms. In this digital setup, the control task involves periodically measuring the plant's outputs, calculating new actuator signals, and applying them as represented in Figure 1.1. This periodic execution allows for the analysis and design of the closed-loop system using established theories for sampled-data systems. For example, in networked control systems sensors and/or actuators usually send their data at specific time intervals determined by a clock, a concept known as time-triggered (or periodic) control [46, 47].

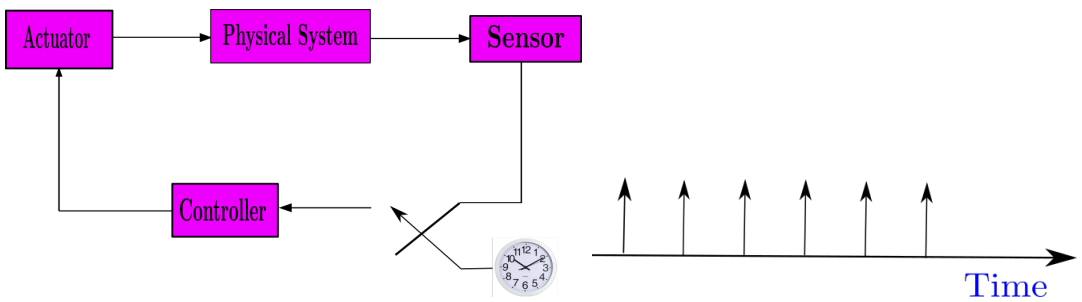

Figure 1.1: Time-triggered control

Paraphrasing Plato, time-triggered control *speaks because it has to say something,* not because *it has something to say* leading to inefficient usage of resources. Specifically, performing the control task when the system is operating optimally and no disturbances are present can be seen as wasteful in terms of computational resources. Moreover, if the measured outputs and actuator signals need to be transmitted over a shared (potentially wireless) network, this approach can lead to unnecessary network utilization or increased power consumption for wireless radios. Event-based control offers a promising alternative to this conventional periodic approach [88, 47, 115]. Indeed, event-based control is a computer control strategy that only updates the control value when the system needs attention while preserving stability and performance. It is a cutting-edge approach to managing systems, that operates on an asynchronous and event-triggered basis.

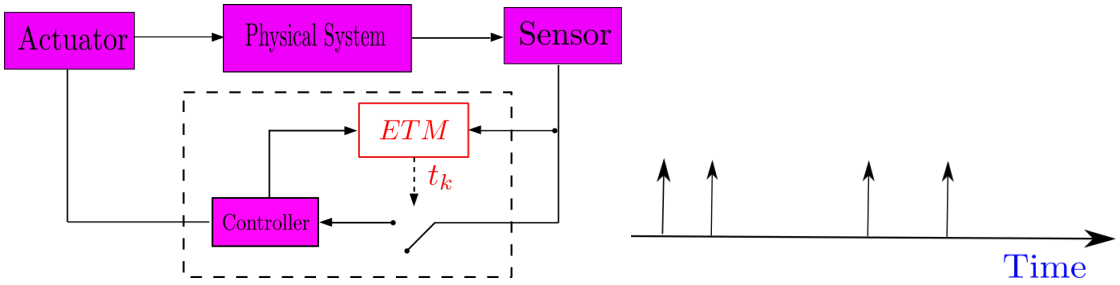

Figure 1.2: Control system subject to event-triggering mechanism (ETM).

Instead of continuously measuring and updating control actions, in event-based control systems, the sensor readings are only taken when a relevant event is detected or when there is a specific change in the system (Figure 1.2). These events can be predefined thresholds, sensor measurements reaching a certain value, or other triggers depending on the system's requirements. This paradigm is reshaping engineering, robotics [84, 25], and various other fields, providing substantial benefits in terms of the efficient usage of computational and communication resources such as power usage, bandwidth, etc (see, e.g., [111, 46, 76]). Consider for instance a robotic arm in a manufacturing setting. With event-based control, the arm does not constantly recalibrate its position; it waits until a specific event, like a change in the object's position, is detected [84, 25]. This allows the robot to allocate its resources more effectively, reducing wear and tear, and speeding up tasks.

While event-based control offers many advantages, its implementation can be complex and requires careful consideration of event detection, system modeling and controller design. A fundamental issue when dealing with event-triggered controllers is to avoid any situation where the mechanism could induce infinitely many updates of the control in a bounded time interval, corresponding to the occurence of a Zeno phenomenon [42, Chapter 2],[41].

1.1.2 Zeno phenomenon

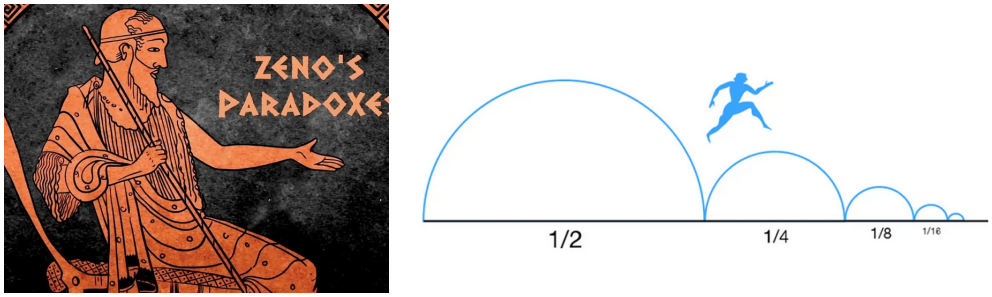

Figure 1.3: Zeno Paradox (©Buzzco Associates, inc; TED Ed, YouTube)

The Zeno's paradox[1] is a philosophical paradox that raises questions about motion and continuity. One of the most famous versions, known as the *Dichotomy Paradox*, involves a runner (let us name him Lyapunov) trying to reach a destination as illustrated in Figure 1.3. The paradox suggests that before Lyapunov can reach the destination, he must first reach the halfway point (1/2). However, before reaching the halfway point, he must cover half of that distance (1/4), and so on, leading to an infinite number of smaller distances that need to be covered before reaching the destination. The paradox highlights the idea that if we keep dividing distances into smaller and smaller segments, it becomes unclear how motion and change are possible in a finite amount of time.

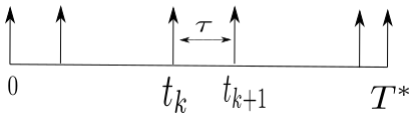

Figure 1.4: Strong Non-Zenoness

In the event-triggered control context, this paradox corresponds to the occurrence of infinitely many updates (or transmission) of the control in a bounded time interval $[0, T^*]$. As an illustration, for some update instant t_k in the finite interval $[0, T^*]$ one has $\lim_{k \to +\infty} t_k = T < T^*$. This phenomenon is problematic since it clearly induces inefficient usage of computational and communication resources. Therefore, in the design of event-triggering mechanism, one must avoid this behavior. To do so, there exists many solutions among which one has:

- **Dwell-time approach:** the event-triggering law includes a triggering condition under which it is possible to obtain a dwell-time (also called inter-execution time): there exists $\tau > 0$ such that for all $k \in \mathbb{N}$, $t_{k+1} - t_k \geq \tau$ as illustrated in Figure 1.4. Remark that in the periodic framework, the triggering instants are given by $t_k = kh$ where $h > 0$ is the period and then the dwell-time is $\tau = h$. But one should notice that having a dwell-time only brings a sufficient condition. That is nevertheless pratically desirable, preventing the updates to accumulate at infinity.

[1] `https://socratesjourney.org/zenos-paradoxes`

- **Accumulation point avoidance:** This approach provides an event-triggering rule under which no accumulation point of the sequence $(t_k)_{k\in\mathbb{N}}$ is possible. In other words, this approach ensures that if there is an infinite number of updates or transmissions, the upper time bound T^* is infinite, preventing the system from becoming trapped in a state of perpetual updates and ensuring efficient resource utilization.

A natural question that may arise around this subject when thinking about event-triggered control is how can we design the event-triggering mechanism while preserving stability properties of the control system and avoiding the Zeno phenomenon?

1.1.3 General form of event-triggering mechanism

One of the first alternative to periodic sampling is *variable sampling* defined as follows:

$$t_{k+1} - t_k = T_k \leq h, \quad \forall k = 0, 1, 2, \cdots, \tag{1.1}$$

where T_k and h are positives numbers. This sampling rule is for example considered in [37] on sampled-data state-feedback stabilization of linear system with a piecewise-continuous delay control input and in [100] for energy control of a pendulum. Note that one recovers the periodic law if the variable sampling T_k is constant and equal to some positive constant h.

Let us state here that events could be initiated once a function G can be designed in such a way that [111, 112, 113, 88, 29]

$$\begin{cases} t_0 = 0 \\ t_{k+1} = \inf\left\{t \geq t_k, G(z(t), e_k(t)) \geq 0\right\} \end{cases} \tag{1.2}$$

where z is the state of control system and $e_k(t)$ denotes the error when sampling. This sampling error could be the deviation between the last sampled state (or the parameter of the control system which is sampled like the velocity as we will see in Chapter 2 or the reaction coefficient [56]) and the current one.

Hence, different kinds of function G are now discussed.

1. **Absolute threshold triggering mechanism**

   The so-called absolute threshold criterion consists of updating the control only when the norm of the deviation error e_k reaches a threshold $\eta > 0$, which can be carefully chosen by the user. Thus the function G is explicitly given by

   $$G(z(t), e_k(t)) = \|e_k(t)\|^2 - \eta. \tag{1.3}$$

   This triggering law is introduced in [1] for significant decreases in central processing unit (CPU) utilization with only minimal control performance deterioration; or in [46] in order to reduce the total costs of a mailing system without losing the tight synchronization between the conveyor belt and the sheet-feeders. It is also applied for the event-driven control for the diffusion process using mobile sensor and actuator in [52].

2. **Relative threshold or static triggering mechanism**

   Another important principle in event-based control involves in adjusting the threshold applied to the norm of $e_k(t)$ based on the energy of state z. The concept behind this is that when $\|z(t)\|$ (that is the energy of the system at time t) is significant, a larger sampling-induced error can be accepted without compromising the stability of the closed-loop system. The static event-triggering rule can be characterized by

$$G(z(t), e_k(t)) = \|e_k(t)\|^2 - \delta\|z(t)\|^2 \tag{1.4}$$

   where $\delta > 0$ is a design parameter that may has to be chosen appropriately. This concept is known as a relative threshold and was first considered for nonlinear finite dimensional control system in [111] where the norm of the deviation $e_k(t)$ and the norm of the state $z(t)$ are replaced by $\mathcal{K}_\infty$[2] functions under the hypothesis that the controller to be implemented ensures that the closed-loop system achieves Input-to-State Stability (ISS) with respect to the measurement errors. This event-triggering mechanism is also considered in [38] for for semilinear time-fractional diffusion systems with distributed feedback; in [66] (see also Chapter 2) for the wave equation and in [77] for nonlinear coupled reaction–diffusion system with finite-time synchronization control.

3. **Combination of absolute and relative threshold**

   In practice, it is common to combine the two previous mechanisms, as this could reduces the number of updates and facilitates the proof of strong non-zenoness. Hence, by defining the function G as follows:

$$G(z(t), e_k(t)) = \|e_k(t)\|^2 - \delta\|z(t)\|^2 - \eta \tag{1.5}$$

   or with the corresponding norm in the state space or $\mathcal{K}_\infty$ fonctions, one obtains the triggering mechanism which consists in combining the absolute and relative threshold triggering mechanisms. We refer to [46, Section V] or to [24] where the authors presented event-triggered control using dynamical output-based controllers for linear time-invariant system and established the proof of the existence of minimum inter-event time preventing the Zeno behavior. We also refer to [28, 5] for 1-dimensional linear hyperbolic systems of conservation laws and multidimensional wave equation with in domain damping. We get here the qualities of both absolute and relative threshold. The Zeno behavior should be easily avoided and a Lyapunov approach should be possible to get at least pratical stability.

4. **Dynamic triggering mechanism**

   The dynamic triggering mechanism is obtained by adding an internal dynamical variable as introduced in [41] in order to enrich the previous event-triggering laws. Hence, the dynamic triggering mechanism could be generated

[2] A continuous function $\alpha : [0, a) \to \mathbb{R}_0^+$ is said to be of class $\mathcal{K}$ if it is strictly increasing and $\alpha(0) = 0$. If $a = \infty$ and $\alpha(r) \to \infty$ as $r \to \infty$, it is said to be of class $\mathcal{K}_\infty$.

with a function G defined by:

$$G(z(t), e_k(t)) = \|e_k(t)\|^2 - \delta\|z(t)\|^2 + \frac{1}{\eta}m(t) \tag{1.6}$$

where

$$\dot{m}(t) = -\eta m(t) + f(z(t), e_k(t)).$$

f is an appropriate function of the state z and the deviation e_k and other parameters of the control system, η and γ are design parameters. Remark that we can obtain the static event-triggering mechanism by considering the limite case of the dynamic one. Dynamic event-triggering mechanism is proposed in [23] for state-feedback systems with simultaneously guarantee of a finite $\mathcal{L}_p$-gain from disturbance to output and a strictly positive lower bound on the inter-event times (implying Zeno-freeness). It is also considered in [128, 50] to solve the consensus problem for multi-agent systems and in the survey [39] on a unified event-triggered control and estimation framework. The time-delay approach has been developed for networked systems with state multiplicative noise in [130] and in [29, 30] respectively for linear systems with conservation laws for 2×2 hyperbolic system using the backstepping approach. In [65], we design a dynamic event-triggering law for a linear Schrodinger equation with in-domain damping.

5. **Event-trigger with *dead-time zone* or time-regularized event-triggered control**

   Consider one of the previous event-triggering mechanisms defined by (1.2) and one of function G in (1.3),(1.4),(1.5),(1.6), but under the assumption that following the transmission of the measurement, the sensor remains inactive for a waiting period of $T_d > 0$ seconds. The time-regularized event-triggering rule is therefore characterized by:

$$\begin{cases} t_0 = 0 \\ t_{k+1} = \inf\Big\{t \geq t_k + T_d, \quad G(z(t), e_k(t)) \geq 0\Big\}. \end{cases} \tag{1.7}$$

   The purpose of introducing a dead-time zone is evidently to prevent the occurence of the Zeno phenomenon, T_d being an obvious dwell time. Dead-time zone event-triggering law was studied in, e.g. [112] for the design of a decentralized dynamic controller for the Linear Time-Invariant (LTI) system; in [113] for event-triggered output feedback controllers for LTI systems over sensor controller-actuator networks and in [100] for sampled-data energy control of a pendulum. We additionally mention [102] for insights into the semilinear diffusion PDE and [101] where they obtain a dead-time zone event-triggering law by switching a periodic sampling and a static event-trigger mechanisms.

   Note that one danger when dealing with PDEs and using this event-triggering law concerns the well-posedness of the system. The imposed dwell-time may exeed a possible limit time T of existence of solution.

1.1.4 Emulation & co-design approaches

Let us recall that two approaches can be considered in the event-triggered control framework: *emulation* and *co-design* approaches. The strategy where the controller is a priori given and the event-triggering mechanism has to be designed corresponds to the emulation approach: see for example [88, 33]. The co-design approach refers to the situation where the design of both the controller and the event-triggering mechanism have to be performed, simultaneously: see for instance [105, 47] and references therein.

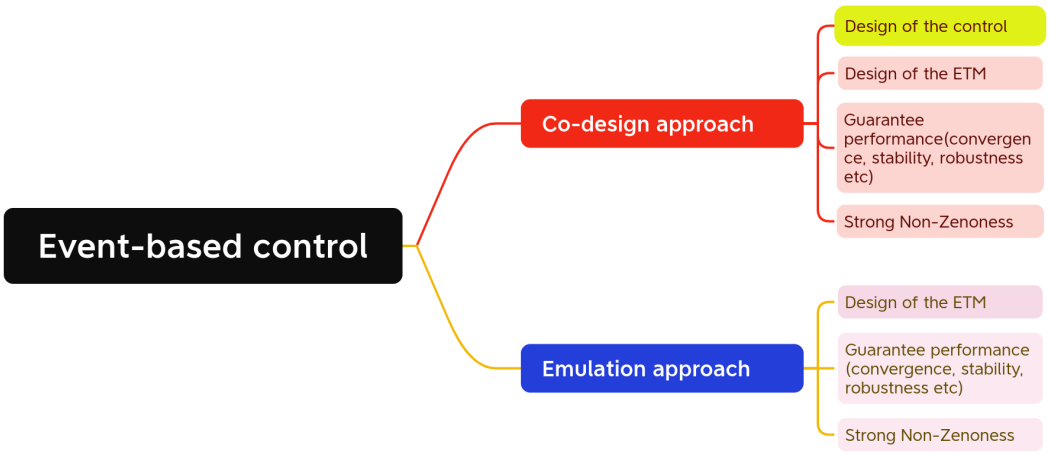

Figure 1.5: Emulation & Co-design approaches

1.1.5 Further literature on event-triggered control

Event-based control (ETC) is well studied for classical finite dimensional systems but has been investigated only recently for infinite dimensional system e.g. described by partial differential equations. For instance, in the context of finite dimensional system, we refer to the seminal works [1, 3] or the most recent ones [111, 46, 41] (for linear systems), [88, 49, 119] (for nonlinear systems), [114] (with dynamic quantization), [85] (with high gain approach) and [99] (for the stabilization of event-triggered control systems affected by measurement noise). ETC for PDEs has gained a lot of interest during the last few years, and several contributions have since then been proposed for wide classes of PDEs. In [127, 31] and [33, 17, 5] event-based control strategies were considered for parabolic and hyperbolic PDE.

For hyperbolic PDEs, [28] proposes an output feedback event-triggered boundary controller for 1-D linear hyperbolic systems of conservation laws through Lyapunov techniques. Using the backstepping approach, [30] and [26, 17] introduce dynamic triggering conditions to the event-triggered boundary controllers for the stabilization of coupled 2×2 linear hyperbolic systems by full-state feedback and output feedback, respectively. The methodology is further employed and advanced in [123]

and [125], the latter proposing an event-triggered adaptive control for coupled hyperbolic PDEs. The results on event-triggered control using backstepping-based methods have been applied to load-moving cable systems [122] and traffic flow control on connected roads [27].

For parabolic PDEs, [102] proposes a decentralized event-triggered control to reduce the number of transmitted measurements, while [63] builds on modal decomposition and comes up with sampled-data and observer-based event triggered boundary control for 1-D reaction-diffusion systems in the presence of time-varying input delays. Such a contribution includes a novel switching-based dynamic triggering condition depending on the finite modes of the estimated state and a suitable time regularization, allowing the avoidance of the Zeno phenomenon. On the other hand, using ISS properties for PDEs and small gain arguments, [32] proposes a backstepping-based full-state feedback ETC strategy for a 1-D reaction-diffusion system with constant parameters and Dirichlet boundary actuation. In addition, [94] proposes an observer-based event-triggered backstepping boundary control in the case of Robin boundary actuation. The ETC strategy includes a dynamic triggering condition under which it is possible to obtain a dwell-time, thus avoiding the Zeno phenomenon. Moreover, [124] extends the results of [60] and [94] and proposes a novel adaptive event-triggered boundary control for a parabolic PDE-ODE system with uncertain parameters, whereas [92] goes further with applications to the Stefan problem. Event-triggered control strategies for other classes of PDEs (including abstract infinite-dimensional systems [121, 120]) are reported in [55] for nonlinear Korteweg–de Vries (KdV) under averaged measurements.

In this thesis we are concerned with the emulation approach in an infinite dimensional context, building new bricks to complete the above overview. More precisely, leveraging on a continuous-in-time control, that ensures the exponential stability of some PDEs (wave equation, Schrödinger equation and reaction diffusion equation) by means of a classical viscous damping term or backstepping controller, we propose a systematic approach for the design of static and dynamic event-triggering rules allowing to guarantee the well-posedness and the exponential stability of the new event-triggered closed-loop control system. Furthermore, the avoidance of Zeno phenomenon is addressed thanks to the accumulation point avoidance and the existence of dwell-time approaches.

1.2 Contributions and structure of this thesis

The remainder of this manuscript falls into three other chapters, devoted to three different PDEs. The first two chapters concern the multidimensional wave and Schrödinger equations with event-triggered in-domain dampings. The last one is about a 1-dimensional reaction diffusion PDE with input delay under event-triggered backstepping control.

1.2.1 Wave equation

Using an adequate Lyapunov functional, related to the energy of the system, we propose a sufficient matrix inequality condition to carry out the global exponen-

tial stability of the wave equation under a static event-triggering mechanism that updates a damping source term. The damping is distributed in the whole space but sampled in time. The results are presented in Chapter 2 by adapting our paper [66] and can be viewed as complementary to those developed in [116] and [5]. Indeed, [116] deals with a semi-linear wave equation in one dimensional space dimension, while we consider here a linear multi-dimensional wave equation. The event-triggering rule we present here is simpler and different from the one in [5], getting rid of any spatial constraints on the stability result and bringing a completely new and interesting proof around the avoidance of the Zeno phenomenon based on the notion of accumulation point. Furthermore, we extend the result to the anti-damped wave equation.

1.2.2 Schrödinger equation

We consider for the first time the design of an event-triggering mechanism for the multi-dimensional Schrödinger equation subject to a local in domain damping control. We design first a static and then a dynamic triggering law for the damping. The following results are tackled: the existence of solution to the closed-loop event-triggered control system; the avoidance of the Zeno behavior due to the absence of any accumulation point of the sequence of time instants and the exponential stability based on energy estimate exploiting a well-known observability inequality. These results are presented in our papers [67] and [65] and are detailed in Chapter 3.

1.2.3 Reaction-diffusion equation with input delay

We consider the problem of event-triggered stabilization of a 1-D reaction-diffusion PDE system with input delay. The approach relies on reformulating the delay problem as an actuated transport PDE, which cascades into the reaction-diffusion PDE, and on the emulation of backstepping control originally introduced in [68] extending the results of [32] to the case of delayed input. We then propose a static triggering condition that establishes the time instants at which the control value needs to be updated. It is shown that under the proposed event-triggered boundary control, there exists a minimal dwell-time (independent of the initial conditions) between two triggering times which allows to guarantee the avoidance of the Zeno Phenomenon and the well-posedness of the closed-loop system. The choice of the space norms (L^2-norm for the reaction-diffusion PDE and supremum-norm for the hyperbolic PDE) is crucial in the design of the triggering policy and for the stability analysis, which is based on Input-to-State Stability and small-gain arguments. These results are presented in Chapter 4.

1.3 List of publications

1.3.1 Peer-reviewed international journals

[66] F. Koudohode, L. Baudouin, S. Tarbouriech. Event-based control of the linear wave equation, Automatica, vol. 146, 110 627, 2022

1.3.2 Peer-reviewed international conferences

[67] F. Koudohode, L. Baudouin, S. Tarbouriech. Event-based control of a damped linear Schrödinger equation, European Control Conference (ECC 2022), pp. 2099-2104, London (UK), July 2022

[65] F. Koudohode, L. Baudouin, S. Tarbouriech. Dynamic event-based control of a damped linear Schrödinger equation (Joint IFAC Conference: SSSC – TDS – LPVS, Montreal, Canada, 2022), IFAC PaperOnline, pp. 120-125

1.3.3 Submitted to peer-reviewed international journals

- F. Koudohode, N. Espitia, M. Krstic Event-triggered. Boundary Control of an unstable reaction diffusion PDE with input delay (Provisionally accepted in SCL).

1.3.4 Outgoing works

- F. Koudohode, L. Baudouin, S. Tarbouriech. Stability of the Schrödinger equation with saturating distributed input, in progress.
- F. Koudohode, L. Baudouin, S. Tarbouriech. Event-based control of the schrödinger equation subject to cone-bounded nonlinearity, in progress.

2

Event-based control of the wave equation

The general problem in this chapter is the study of a multi-dimensional wave equation under in-domain event-triggered control. We will first recall and detail modeling, well-posedness and exponential stability results of a continuous-in-space damped wave equation before setting and studying the event-triggered control context. This chapter is based on the journal article [66].

2.1 Model description

Let Ω be an open bounded domain in $\mathbb{R}^n$, with smooth boundary $\partial\Omega$. We consider the following multi-dimensional controlled wave equation

$$\begin{cases} \partial_t^2 z(x,t) - \Delta z(x,t) = f(x,t) & \forall\ (x,t) \in \Omega \times \mathbb{R}^+, \\ z(x,t) = 0 & \forall\ (x,t) \in \partial\Omega \times \mathbb{R}^+, \\ z(x,0) = z_0(x) & \forall\ x \in \Omega, \\ \partial_t z(x,0) = z_1(x) & \forall\ x \in \Omega. \end{cases} \tag{2.1}$$

where $(z, \partial_t z)$ denotes the state, f is the source term control inputs and (z_0, z_1) is the initial data. The wave PDE system arises in fluid dynamics, acoustics and electromagnetic and models the evolution and the propagation of wave's amplitude (water waves, sound waves, seismic waves or light waves) [64].

In this chapter we consider the feedback control input $f(x,t) = -\alpha\partial_t z(x,t)$, with $\alpha > 0$ the damping coefficient. Let us present rapidly here three different settings to illustrate possible applications.

• In a one-dimensional media, it is called the vibrating cord or string equation [11, Section 10.3], [74] and can model the dynamics of an elastic slope vibrating around its rest position and submitted to external friction forces [78, Section 4.3], tunable through the coefficient α.

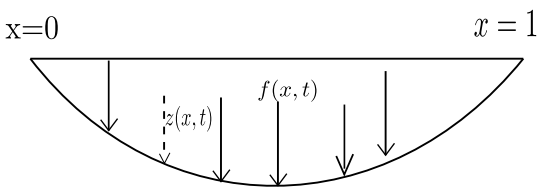

Figure 2.1: Vibration of an elastic slope subject to external force

• In two space dimensions, it can be a model to study the vibration of a stretched membrane like the skin of a drum. The control task consists in the reduction of the vibrations [51], whereas the control input is assumed to be distributed over the whole membrane's surface. Actually, the motivation for incorporating internal material damping in the wave equation arises first from the fact that inherent (and small) material damping is always present in real materials. The task of the control is then to do better than this natural distributed damping, for instance thinking about using smart materials to compose the membrane: see, for example, [96], [97].

• System (2.1) can also model the process of oil drilling (depicted on Figure 2.2). In fact, by considering $z = \begin{pmatrix} \xi \\ \phi \end{pmatrix}$ with the axial ξ and torsional (angular) ϕ vibrations in the drill string. That is described in [40] by two one-dimensional wave equations:

$$\partial_t^2 \xi(x,t) - c_\xi^2 \partial_x^2 \xi(x,t) = -\alpha_\xi \partial_t \xi(x,t),$$
$$\partial_t^2 \phi(x,t) - c_\phi^2 \partial_x^2 \phi(x,t) = -\alpha_\phi \partial_t \phi(x,t),$$

where $c_\xi = \sqrt{\frac{E}{\rho}}$ is the axial velocity, ρ is the pipe mass density, E its Young's modulus, $c_\phi = \sqrt{\frac{G}{\rho}}$ is the propagation speed of the angle, G is the shear modulus and α_ξ, α_ϕ are the internal damping and the axial distributed damping respectively .

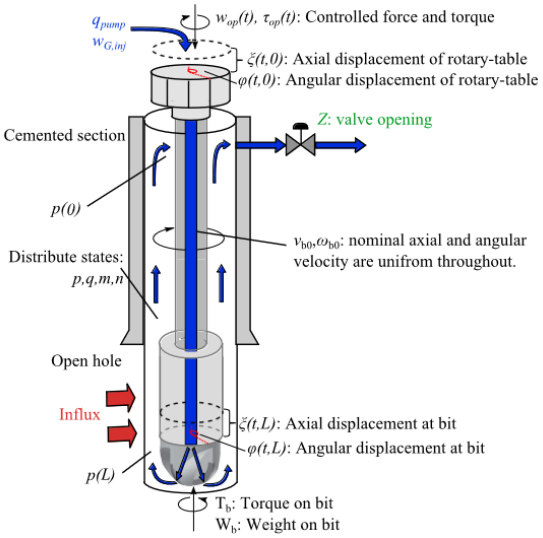

Figure 2.2: Well schematic of oil drilling originally taken from [21].

These applications examples are solely proposed to illustrate the practical utility of the mathematical models that we are interested in. In the particular situation of system (2.1), we have here the case where the external force $f = -\alpha \partial_t z$ is chosen as the control, making the feedback law only based on the speed $\partial_t z$ of the deformation, which is a part of the complete state's system $(z, \partial_t z)$ and could be viewed as the linear case of the ones studied (for $n = 1$) in [89], [14].

2.2 Existing results

When dealing with PDEs, it is crucial to guarantee their well-posedness since if a PDE is not well-posed [53, 54], it may not have a unique solution, or its solution may be very sensitive to small perturbations of source initial or boundary data. In such cases, the PDE may not have any physical or meaningful interpretation, or it may be impossible to use numerical methods to compute approximate solutions.

Let us start by the proof of existence and uniqueness of weak and strong solution to the problem. We refer to the book by Evans [34, Section 7.2] for the definition of weak solution to hyperbolic system and to the paper [72] (Theorems 2.1 and 2.2) for the definition of strong and weak solution to the wave equation. In a nutshell, the notion of strong solution corresponds to the continuity of the solution with respect to the space variable x and is related to stronger hypotheses on the initial data. Let us recall the following classical definition.

Definition 2.1: Weak solution to the wave equation

A weak solution by *transposition* of the system (2.1) is a function

$$z \in C^0([0,T]; H_0^1(\Omega)) \cap C^1([0,T]; L^2(\Omega)) \tag{2.2}$$

such that for all $\varphi \in C^2(\Omega \times [0,T])$ verifying

$$\varphi(x,T) = \partial_t \varphi(x,T) = 0 \qquad \forall x \in \Omega \quad \text{and} \tag{2.3}$$

$$\varphi(x,t) = 0 \quad \forall (x,t) \in \partial\Omega \times [0,T]. \tag{2.4}$$

one has:

$$\int_0^T\int_\Omega \nabla\varphi(x,t)\cdot\nabla z(x,t)dxdt - \int_\Omega \varphi(x,0)z_1(x)dx + \int_\Omega \partial_t\varphi(x,0)z_0(x)dx = \int_0^T\int_\Omega \partial_t\varphi(x,t)z(x,t)dxdt. \tag{2.5}$$

To obtain (2.5) we multiply the first line of (2.1) by φ and integrate on $\Omega \times [0,T]$, performing integrations by parts and using the Green formula (Lemma A.3 in Appendix) and (2.3).

Let us denote the full state of the system (2.1) by

$$Z = (z, \partial_t z)^\top \in \mathbb{R}^2.$$

Then

$$\dot{Z} = \begin{pmatrix} \partial_t z \\ \partial_t^2 z \end{pmatrix} = \begin{pmatrix} \partial_t z \\ \Delta z - \alpha \partial_t z \end{pmatrix}.$$

Therefore, system (2.1) can be rewritten as

$$\begin{cases} \dot{Z} & = \mathcal{A}Z \\ Z(0) & = (z_0, z_1)^\top \end{cases} \tag{2.6}$$

where the operator $\mathcal{A}$ is defined by $\mathcal{A} = \begin{pmatrix} 0 & I \\ \Delta & -\alpha \end{pmatrix}$ with domain

$$\mathcal{D}(\mathcal{A}) = (H^2(\Omega) \cap H_0^1(\Omega)) \times H_0^1(\Omega).$$

Let us denote by $\mathcal{H}$ the Hilbert space

$$\mathcal{H} = H_0^1(\Omega) \times L^2(\Omega)$$

equipped with the usual inner product

$$\left\langle \begin{pmatrix} u \\ v \end{pmatrix}, \begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix} \right\rangle = \int_\Omega \nabla u(x) \nabla \tilde{u}(x) dx + \int_\Omega v(x)\tilde{v}(x) dx$$

and the norm

$$\left\| \begin{pmatrix} u \\ v \end{pmatrix} \right\| = \int_\Omega |\nabla u(x)|^2 dx + \int_\Omega |v(x)|^2 dx.$$

Definition 2.2: Strong solution to the wave equation

We say that z is a strong solution of the system (2.1) if

$$(z, \partial_t z)^\top \in C^0([0,T]; \mathcal{D}(\mathcal{A})) \tag{2.7}$$

and the first equation of 2.1 holds in $C([0,T]; L^2(\Omega))$.

The problem of existence and the uniqueness of the wave equation is well-documented in the litterature. One has the following theorem.

Theorem 2.3: Well-posedness of the damped wave equation [13]

- For $T > 0$ and for any initial conditions $(z_0, z_1) \in \mathcal{H}$, there exists a unique weak solution to (2.1) satisfying

$$z \in C([0,T]; H_0^1(\Omega)) \cap C^1([0,T]; L^2(\Omega)).$$

- Moreover, for any initial conditions $(z_0, z_1) \in \mathcal{D}(\mathcal{A})$, there exists a unique strong solution to (2.11) satisfying

$$z \in C([0,T); H^2(\Omega) \cap H_0^1(\Omega)) \cap C^1([0,T); L^2(\Omega)). \tag{2.8}$$

The proof of Theorem 2.3 is done in [13, Theorem 2.3 and Theorem 3.4], and in [74, 75] using the semigroup theory.

The control and the stability analysis of the wave equation (2.1) have been widely studied in the literature. For instance the multiplier method used by [13] and [74], a micro-local analysis approach by [73] and a backstepping method by [108] are used to characterize the stability and prove some controllability and stabilization results of this equation. For intance in [13], it is proved that system (2.1) is exponentially stable as stated in the following theorem.

Theorem 2.4: Exponential stability [13]

For all initial state $(z_0, z_1) \in \mathcal{H} = H_0^1(\Omega) \times L^2(\Omega)$, the solution to the system (2.1) has its energy decaying exponentially: there exist $K, \beta > 0$ such that

$$E(t) \leq KE(0)e^{-\beta t}, \tag{2.9}$$

where the energy E is defined as the sum of the kinetic and potential energies by

$$E(t) = \frac{1}{2}\left(\|\partial_t z(t)\|^2 + \|\nabla z(t)\|^2\right). \tag{2.10}$$

The proof is based on multiplier technique (see [74]) and consists of multiplying the equation appropriately to bring out specific expressions such as energy, then integrating by part in time and space to make the desired estimates.

2.3 ETC for the damped wave equation

In this section, using an adequate Lyapunov functional, related to the energy of the system, an exponential stability condition for the closed-loop system under event-triggered control is formulated as a linear matrix inequality to satisfy. The feasibility of such an inequality is proven to be always guaranteed.

2.3.1 Problem formulation

We are interested by an event-triggering implementation of the control term $f(x,t) = -\alpha\partial_t z(x,t)$, so that the control signal applied to the plant is updated only at certain instants $\{t_k\}_{k\in\mathbb{N}}$, defined by a mechanism. We assume that the control action is held constant between two successive events. Moreover, differently from classical periodic sampling techniques, the inter-sampling time $t_{k+1} - t_k$ is not assumed to be constant. Hence the system under study can be illustrated by the block diagram (Figure 2.3) and described by the following equations :

$$\begin{cases} \partial_t^2 z(x,t) - \Delta z(x,t) = -\alpha\partial_t z(x,t_k), & \text{for } (x,t) \in \Omega \times [t_k, t_{k+1}), k \in \mathbb{N} \\ z(x,t) = 0, & \text{for } (x,t) \in \partial\Omega \times \mathbb{R}^+, \\ z(x,0) = z_0, \partial_t z(x,0) = z_1, & \text{for } x \in \Omega. \end{cases} \tag{2.11}$$

This closed-loop system can then be described in Figure 2.3:

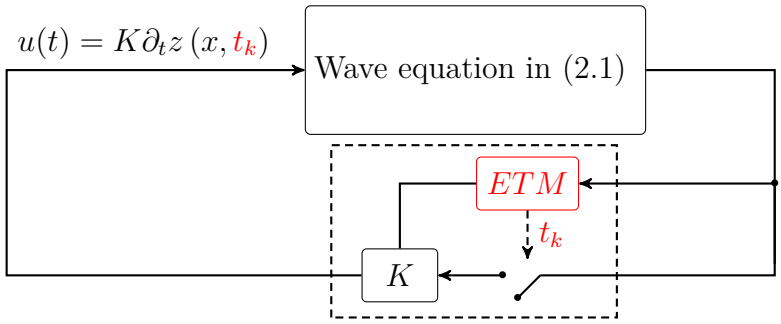

Figure 2.3: Block diagram Event-Triggering Mechanism (ETM) for wave equation.

Then, the problem we want to adress consists in design the red box in Figure 2.3, that is to design the event-triggering mechanism while preserving the closed-loop stability.

Note that $t_k, k \in \mathbb{N}$ are the triggering instants that satisfy

$$0 = t_0 < t_1 < \cdots < t_k < t_{k+1} < \cdots .$$

Hence, the problem we intend to solve can be summarized as:

Problem 1: Control objective

Design a triggering mechanism in order to guarantee:

1. the well-posedness of the closed-loop system (2.11),
2. the avoidance of Zeno behavior,
3. the exponential stability of the system (2.11).

To address Problem 1, as a stepping stone, we exploit and expand the results about the continuous-in-time version of system (2.11), correponding to system (2.1).

2.3.2 Definition of the event-triggering mechanism

Let us introduce the error deviation from the speed at the last triggering instant, for all $x \in \Omega$ and $t \in [t_k, t_{k+1})$:

$$e_k(x,t) = \partial_t z(x,t) - \partial_t z(x,t_k). \tag{2.12}$$

In this chapter, we will consider a static event-triggering mechanism. That corresponds to case presented in the introductive Chapter 1 where we choose the function G defined in (1.4) as:

$$G(z(t), e_k(t)) = \|e_k(t)\|^2 - 2\gamma E(t) = \|e_k(t)\|^2 - \gamma\|\partial_t z(t)\|^2 - \gamma\|\nabla z(t)\|^2.$$

The idea consists in measuring the deviation of the wave's speed between the last sampled state and the current one and authorizing it to be in a 2γ proportion of the current energy. In other words, between two instants t_k and t_{k+1}, it holds

$$\|e_k(t)\|^2 \leq 2\gamma E(t), \tag{2.13}$$

and as soon as this becomes false, an update is generated. More precisely, the event-triggering rule can be characterized as:

$$\begin{cases} t_0 = 0 \\ t_{k+1} = \inf\Big\{t \geq t_k, \|e_k(t)\|^2 > 2\gamma E(t)\Big\}. \end{cases} \tag{2.14}$$

where $\gamma > 0$ is a design parameter that has to be chosen appropriately.

Remark 2.1 *The monitoring of the deviation by the energy will be some thing good regarding our Lyapunov study. But it is not clear that the Zeno behavior can be avoided. For instance, in order to facilitate this Zeno phenomenon, the choice of the combination of the absolute threshold triggering mechanism and the static event-triggering mechanism (see also* (1.5)*) is considered for the wave equation with in-domain damping in [5] where the function G is defined as follows:*

$$G(z(t), e_k(t)) = \|e_k(t)\|^2 - 2\gamma E(t) - E(0)e^{-\theta t}$$

so that the event-triggering algorithm is defined by

$$\begin{cases} t_0 = 0 \\ t_{k+1} = \inf\Big\{t \geq t_k, \|e_k(t)\|^2 > 2\gamma E(t) + E(0)e^{-\theta t}\Big\}. \end{cases} \tag{2.15}$$

A comparative study of these two triggering rules will be addressed in the numerical simulation Section 2.3.6.

Using (2.12), the closed-loop system under consideration can be written as follows:

$$\begin{cases} \partial_t^2 z - \Delta z = -\alpha\partial_t z + \alpha e_k, & \text{in } \Omega \times [t_k, t_{k+1}), \forall k \in \mathbb{N} \\ z = 0, & \text{on } \partial\Omega \times \mathbb{R}^+, \\ z(\cdot, 0) = z_0, \partial_t z(\cdot, 0) = z_1, & \text{in } \Omega. \end{cases} \tag{2.16}$$

In the sequel we separate the study of the well-posedness of system (2.16), from the guarantee of the avoidance of Zeno behavior and the exponential stability of the closed-loop system.

2.3.3 Well-posedness

Let us begin by defining the maximal time T^* under which the system (2.11), or equivalently (2.16) subjected to the event-triggering law (2.14) has a solution:

$$\begin{cases} T^* = +\infty & \text{if } (t_k) \text{ is a finite sequence,} \\ T^* = \limsup\limits_{k \to +\infty} t_k & \text{if not.} \end{cases} \tag{2.17}$$

Leveraging on some regularity of the solutions to the wave equation we prove the following theorem.

Theorem 2.5: Well-posedness of the event-triggered control system

Let Ω be an open bounded domain of class C^2. For any initial condition $(z_0, z_1) \in H^2(\Omega) \cap H_0^1(\Omega) \times H_0^1(\Omega)$, there exists a unique strong solution to (2.11) under the event-triggering mechanism (2.14), satisfying

$$z \in C([0, T^*); H^2(\Omega) \cap H_0^1(\Omega)) \cap C^1([0, T^*); H_0^1(\Omega)). \tag{2.18}$$

Proof : First of all, we show by induction the well-posedness on every sampled interval $[t_k, t_{k+1}]$. From the definition (2.17) of T^*, this will allow to obtain a unique solution in the class (2.18).

• **Initialization.** On the first time interval $[0, t_1]$, system (2.11) reads as a basic wave equation with initial data $(z_0, z_1) \in H^2(\Omega) \cap H_0^1(\Omega) \times H_0^1(\Omega)$ and source term $f(x,t) = -\alpha z_1(x,t)$:

$$\begin{cases} \partial_t^2 z - \Delta z = -\alpha z_1, & \text{in } \Omega \times (0, t_1), \\ z = 0, & \text{on } \partial\Omega \times (0, t_1), \\ z(\cdot, 0) = z_0, \partial_t z(\cdot, 0) = z_1, & \text{in } \Omega. \end{cases}$$

Since $z_1 \in H_0^1(\Omega)$, then $f \in L^1([0, t_1]; H_0^1(\Omega))$. Thus from Theorem 2.3, it follows that there exists a unique solution satisfying

$$z \in C([0, t_1]; H^2 \cap H_0^1(\Omega)) \cap C^1([0, t_1]; H_0^1(\Omega)).$$

• **Heredity.** Let $k \geq 0$ be fixed and assume that

$$z \in C([t_k, t_{k+1}]; H^2(\Omega) \cap H_0^1(\Omega)) \cap C^1([t_k, t_{k+1}]; H_0^1(\Omega)).$$

Consider now the closed-loop system (2.11) over the next time interval $[t_{k+1}, t_{k+2}]$:

$$\begin{cases} \partial_t^2 z - \Delta z = -\alpha z_{2k+3}, & \text{in } \Omega \times (t_{k+1}, t_{k+2}), \\ z = 0, & \text{on } \partial\Omega \times (t_{k+1}, t_{k+2}), \\ (z(\cdot, t_{k+1}), \partial_t z(\cdot, t_{k+1})) = & (z_{2k+2}, z_{2k+3}) \text{ in } \Omega, \end{cases}$$

where we have denoted by z_{2k+2} and z_{2k+3} the position and velocity function values of the wave at t_{k+1} given by the previous system over $[t_k, t_{k+1}]$. This is again a wave equation with source term $f(x,t) = -\alpha z_{2k+3}$ which belongs to $L^1([t_{k+1}, t_{k+2}]; L^2(\Omega))$ since we assumed $z \in C^1([t_k, t_{k+1}]; H_0^1(\Omega))$ and $\partial_t z(t_{k+1}) = z_{2k+3}$. Therefore, applying again Theorem 2.3 we conclude to the existence and the uniqueness of the solution z in the same functional spaces on next time interval $[t_{k+1}, t_{k+2}]$.

By induction, this regularity holds for any $k \in \mathbb{N}$. Therefore, from the extension by continuity at the update instants t_k, one can conclude that system (2.11), or equivalently system (2.16), has a unique solution in the class (2.18). ♢

The fact that Theorem 2.5 holds means that we solved item 1 of Problem 1.

2.3.4 Avoidance of Zeno behavior

In this section, we address the second item of Problem 1, namely we prove that we avoid Zeno behavior, where the closed-loop system would generate an infinite number of updates in a finite time. Before proving that this phenomenon cannot occur, let us show that the natural energy (2.10) of the closed-loop system has a useful property stated as follows.

Lemma 2.6: Boundedness of the energy

Let α be the damping coefficient and γ the design parameter for the triggering law (2.14). For all $t \in [0, T^*)$ the energy E (defined by (2.10)) of the system (2.16) under the event-triggering mechanism (2.14) verifies:

$$E(0)e^{-2\alpha(1+\sqrt{\gamma})t} \leq E(t) \leq E(0)e^{2\alpha(1+\sqrt{\gamma})t}. \tag{2.19}$$

Remark 2.2 *In this lemma, one should notice that:*

1. *The inequality $E(t) \geq E(0)e^{-2\alpha(1+\sqrt{\gamma})t}$ is crucial in the proof of the avoidance of Zeno behavior as we will see in Theorem 2.8.*

2. *The inequality $E(t) \leq E(0)e^{2\alpha(1+\sqrt{\gamma})t}$ is classically used in the energy/Galerkin's method to prove the well-posedness of the wave system.*

Proof : Let us first calculate the time-derivative of the energy $E(t)$. From now on, the mute variable x is ghosted in order to ease the reading. From (2.10) one can write

$$\dot{E}(t) = \int_\Omega \partial_t^2 z(t)\partial_t z(t) + \int_\Omega \nabla\partial_t z(t) \cdot \nabla z(t).$$

Since (2.16) brings $\partial_t^2 z(t) = \Delta z(t) - \alpha\partial_t z(t) + \alpha e_k(t)$ in Ω for all $t \geq 0$, we get, using Green's formula (see Lemma A.3):

$$\begin{aligned}\dot{E}(t) &= \int_\Omega \partial_t z(t)\Delta z(t) - \alpha\int_\Omega |\partial_t z(t)|^2 + \alpha\int_\Omega \partial_t z(t)e_k(t) + \int_\Omega \nabla\partial_t z(t)\cdot\nabla z(t)\\ &= -\int_\Omega \nabla\partial_t z(t)\cdot\nabla z(t) + \int_{\partial\Omega}(\partial_t z \nabla z\cdot\nu)\,ds - \alpha\int_\Omega |\partial_t z(t)|^2\\ &\quad + \alpha\int_\Omega \partial_t z(t)e_k(t) + \int_\Omega \nabla\partial_t z(t)\cdot\nabla z(t).\end{aligned}$$

Since $z = 0$ on $\partial\Omega \times \mathbb{R}_+$ in (2.16) we get $\partial_t z(t)$ on $\partial\Omega \times \mathbb{R}_+$ and we conclude

$$\dot{E}(t) = -\alpha\int_\Omega |\partial_t z(t)|^2 + \alpha\int_\Omega \partial_t z(t)e_k(t) \tag{2.20}$$

so that one gets

$$|\dot{E}(t)| \leq \alpha\|\partial_t z(t)\|^2 + \alpha\|e_k(t)\|\|\partial_t z(t)\|. \tag{2.21}$$

From the definition (2.17) of T^*, since $t \in [0, T^*)$, either there exists k such that $t \in [t_k, t_{k+1})$ if the sequence $(t_k)_{k\geq 0}$ is not finite, or t may be greater than the

last t_k and the definition of (2.14) allows to call $t_{k+1} = T^*$. From (2.13) and $\|\partial_t z(t)\|^2 \leq 2E(t)$ we get:

$$\begin{aligned} &|\dot{E}(t)| \leq 2\alpha E(t) + \alpha\sqrt{2\gamma E(t)}\sqrt{2E(t)} \text{ or equivalently} \\ &|\dot{E}(t)| \leq 2CE(t) \quad \text{with } C = \alpha(1+\sqrt{\gamma}). \end{aligned} \tag{2.22}$$

It follows that $-2CE(t) \leq \dot{E}(t) \leq 2CE(t)$.

Gronwall's Lemma applied on $[t_k, t]$ (Lemma A in Appendix) to both inequalities gives

$$E(t_k)e^{-2C(t-t_k)} \leq E(t) \leq E(t_k)e^{2C(t-t_k)}. \tag{2.23}$$

Then taking $t = t_{k+1}$, it becomes:

$$E(t_k)e^{-2C(t_{k+1}-t_k)} \leq E(t_{k+1}) \leq E(t_k)e^{2C(t_{k+1}-t_k)}.$$

Inferring what it gives for $E(t_k)$, one can deduce

$$E(t_{k-1})e^{-2C(t_{k+1}-t_{k-1})} \leq E(t_{k+1}) \leq E(t_{k-1})e^{2C(t_{k+1}-t_{k-1})}$$

and since $t_0 = 0$, by induction we get:

$$E(0)e^{-2Ct_{k+1}} \leq E(t_{k+1}) \leq E(0)e^{2Ct_{k+1}}.$$

Then inequality (2.23) yields:

$$E(0)e^{-2Ct_k}e^{-2C(t-t_k)} \leq E(t) \leq E(0)e^{2Ct_k}e^{2C(t-t_k)},$$

showing that (2.19) holds for all $t \in [0, T^*)$. ♢

We will also need the following lemma which bound the term $\|\Delta z(t)\|$.

Lemma 2.7: Intermediate result

For any $(z_0, z_1) \in H^2(\Omega) \cap H_0^1(\Omega) \times H_0^1(\Omega)$, the closed-loop system (2.16) with (2.14) has a unique solution satisfying $z \in C([0, T^*); H^2(\Omega) \cap H_0^1(\Omega))$, then there exists a constant $C_\Delta > 0$ such that $\forall t \in [0, T^*)$

$$\|\Delta z(t)\| \leq \|\Delta z\|_{L^\infty(0,T^*;L^2(\Omega))} \leq C_\Delta, \tag{2.24}$$

where C_Δ depends on $\|z_0\|_{H^2(\Omega)}$ and $\|z_1\|_{H_0^1(\Omega)}$.

Proof : Combining Lemma 2.6 and Theorem 2.1, for all solution to the system

$$\begin{cases} \partial_t^2 z - \Delta z = -\alpha \partial_t z(t_k), \\ z(0) = z_0, \partial_t z(0) = z_1, \end{cases} \tag{2.25}$$

we obtain

$$\|\nabla z(t)\|^2 + \|\partial_t z(t)\|^2 \leq K\left(\|z_0\|^2 + \|\partial_t z_1\|^2\right) \tag{2.26}$$

with $K = e^{2\alpha(1+\sqrt{\gamma})T^*}$.

Let us set $w = \partial_t z$ so that w verifies

$$\begin{cases} \partial_t^2 w - \Delta w = 0 \\ w(0) = z_1, \partial_t w(0) = \Delta z_0 - \alpha z_1. \end{cases} \tag{2.27}$$

Therefore, since $\Delta z_0 \in L^2(\Omega)$, $z_1 \in H_0^1(\Omega)$ we have:

$$\begin{aligned} \|\partial_t v(t)\|^2 &\leq K \left(\|z_1\|_{H_0^1}^2 + \|\partial_t \Delta z_0 - \alpha z_1\|^2 \right) \\ &\leq K \left(\|z_0\|_{H^2}^2 + \|z_1\|_{H_0^1}^2 \right) \end{aligned}$$

This leads to

$$\begin{aligned} \|\Delta z(t)\|^2 &\leq K \left(\|\partial_t^2 z(t)\|^2 + \|\partial_t z(t_k)\|^2 \right) \\ &= K \left(\|\partial_t w(t)\|^2 + \|\partial_t z(t_k)\|^2 \right) \\ &\leq K \left(\|z_0\|_{H^2 \cap H_0^1}^2 + \|z_1\|_{H_0^1}^2 \right) = C_\Delta. \end{aligned}$$

♢

We can now state the following result concerning Zeno behavior. The idea is to consider the maximal time T^* under which we proved that the system (2.11) subjected to the event-triggering law (2.14) has a solution. From the definition of T^* in (2.17), one can verify that if $T^* < +\infty$, then T^* is an accumulation point of the sequence $(t_k)_{k\geq 0}$ and a Zeno behavior occurs. Thus, avoiding Zeno phenomenon is a consequence of proving that $T^* = +\infty$.

Theorem 2.8: Zeno free

There is no Zeno phenomenon for the system (2.16) under the event-triggering mechanism (2.14). Equivalently, the maximal time (of existence of solution) defined by (2.17) is actually $T^* = +\infty$.

Proof : By taking inspiration from the reasoning in [111, 41] the proof is based on the study of the function φ defined on $[t_k, t_{k+1})$ by

$$\varphi : t \mapsto \varphi(t) = \frac{\|e_k(t)\|^2}{2\gamma E(t)}. \tag{2.28}$$

The function φ is nonnegative and satisfies $\varphi(t_k) = 0$ for any k since $e_k(t_k) = 0$. Moreover, it jumps from $\lim_{t \to t_{k+1}} \varphi(t) = 1$ to $\varphi(t_{k+1}) = 0$ when a triggering event occurs, according to the law (2.14). Let us estimate the time-derivative of φ:

$$\dot{\varphi}(t) = \frac{\int_\Omega \dot{e}_k(t) e_k(t)}{\gamma E(t)} - \frac{\dot{E}(t) \|e_k(t)\|^2}{2\gamma \left(E(t)\right)^2}. \tag{2.29}$$

On the one hand, from we have (2.12) $\dot{e}_k(x,t) = \partial_t^2 z(x,t)$, using (2.16) and the Cauchy Schwarz's inequality, for all $t \in [t_k, t_{k+1})$ we have

$$\begin{aligned} \int_\Omega \dot{e}_k(t) e_k(t) &= \int_\Omega \Delta z(t) e_k(t) - \alpha \int_\Omega \partial_t z(t) e_k(t) + \alpha \|e_k(t)\|^2, \\ &\leq \|e_k(t)\| \|\Delta z(t)\| + \alpha \|e_k(t)\| \|\partial_t z(t)\| + \alpha \|e_k(t)\|^2. \end{aligned}$$

Then using $\|\partial_t z(t)\|^2 \leq 2E(t)$, (2.13) and Lemma 2.7 it follows

$$\frac{\int_\Omega \dot{e}_k(t)e_k(t)}{\gamma E(t)} \leq \frac{C_\Delta\sqrt{2\gamma E(t)}}{\gamma E(t)} + \frac{\alpha\sqrt{2\gamma E(t)}\sqrt{2E(t)}}{\gamma E(t)} + 2\alpha\varphi(t),$$

leading to

$$\frac{\int_\Omega \dot{e}_k(t)e_k(t)}{\gamma E(t)} \leq \frac{C_\Delta\sqrt{2}}{\sqrt{\gamma E(t)}} + \frac{2\alpha}{\sqrt{\gamma}} + 2\alpha\varphi(t). \tag{2.30}$$

On the other hand, from (2.22), one has $-\dot{E}(t) \leq 2\alpha(1+\sqrt{\gamma})E(t)$ so that

$$\frac{-\dot{E}(t)\|e_k(t)\|^2}{2\gamma\left(E(t)\right)^2} \leq 2\alpha(1+\sqrt{\gamma})\varphi(t). \tag{2.31}$$

Gathering (2.30) and (2.31) we obtain:

$$\dot{\varphi}(t) \leq \frac{C_\Delta\sqrt{2}}{\sqrt{\gamma E(t)}} + \frac{2\alpha}{\sqrt{\gamma}} + 2\alpha(2+\sqrt{\gamma})\varphi(t).$$

Let us now recall that from the event-triggering law (2.14), an event occurs if $\varphi(t) > 1$, and as long as $\varphi(t) \leq 1$, no update event is triggered. Hence it follows that for all $t \in [t_k, t_{k+1})$,

$$\dot{\varphi}(t) \leq A + \frac{B}{\sqrt{E(t)}} \tag{2.32}$$

with $A = \dfrac{2\alpha}{\sqrt{\gamma}} + 2\alpha(2+\sqrt{\gamma})$ and $B = C_\Delta\sqrt{\dfrac{2}{\gamma}}$.

From Lemma 2.6, T^* still being the maximal time under which the closed-loop event-triggered control system has a solution, one has, for all $t \in [0, T^*)$,

$$E(t) \geq E(0)e^{-2Ct} \geq E(0)e^{-2CT^*},$$

and (2.32) becomes $\dot{\varphi}(t) \leq A + \dfrac{Be^{CT^*}}{\sqrt{E(0)}}$.

Then $\forall k \in \mathbb{N}$, integrating on (t_k, t_{k+1}) and using that $\varphi(t_k) = 0$ and $\varphi(t_{k+1}) = 1$ we obtain :

$$1 \leq \left[A + \frac{Be^{CT^*}}{\sqrt{E(0)}}\right](t_{k+1} - t_k). \tag{2.33}$$

Finally, let $t_k \to T^*$ as $k \to +\infty$ in (2.33), then we get a contradiction if $T^* \neq +\infty$. Therefore, we need to consider $T^* = +\infty$, leading to the absence of any accumulation points. Thereby, the avoidance of Zeno behavior is guaranteed. ♢

Remark 2.3 *It is quite usual, in the finite dimensional framework, to prove that there is no Zeno phenomenon in a hybrid system or an event-triggered control loop using the fact that a minimal dwell time between two triggering instants can be characterized and independant of the time window. However, one should know that it is only a sufficient condition, and not a necessary one according to the strict Zeno phenomenon definition. Here, we follow another route by using the strict mathematical definition that corresponds to the absence of accumulation points in the time sequence of updates. Note that the inequality* (2.33) *gives a dwell time* $\tau^* = \frac{1}{A+\frac{Be^{CT^*}}{\sqrt{E(0)}}}$ *which depends unfortunetly on the time* T^* *and the initial condition. Nevertheless, we should say that from a pratical point of view, such a dwell time is not satisfying.*

The proof of Theorem 2.8 means that we solved item 2 of Problem 1. It also means that the result of Theorem 2.5 consequently holds for $T^* = +\infty$ as stated in the following corallary.

Corollary 2.9: Well-podeness

Let Ω be an open bounded domain of class C^2. For any initial conditions $(z_0, z_1) \in H^2(\Omega) \cap H_0^1(\Omega) \times H_0^1(\Omega)$, there exists a unique strong solution to (2.11) under the event-triggering mechanism (2.14), satisfying

$$z \in C([0,+\infty); H^2(\Omega) \cap H_0^1(\Omega)) \cap C^1([0,+\infty); H_0^1(\Omega)). \tag{2.34}$$

2.3.5 Exponential stability

In this section we address item 3 of Problem 1, that is, we propose sufficient conditions in order to ensure the exponential stability of system (2.11)-(2.14) or equilalently (2.16)-(2.14). Let be a tuning parameter $\varepsilon > 0$ and define the following Lyapunov functional candidate:

$$\begin{aligned} V(t) &:= E(t) + \frac{\alpha\varepsilon}{2}\int_\Omega |z(t)|^2 + \varepsilon\int_\Omega z(t)\partial_t z(t). \\ &= \int_\Omega |\partial_t z(t)|^2 dx + \int_\Omega |\nabla z(t)|^2 dx + \frac{\alpha\varepsilon}{2}\int_\Omega |z(t)|^2 + \varepsilon\int_\Omega z(t)\partial_t z(t). \end{aligned} \tag{2.35}$$

The following result can be stated.

Theorem 2.10: Stability of the event-triggered of the wave equation

Given the damping parameter $\alpha > 0$, assume there exist positive scalars $\gamma, \lambda_1, \lambda_2, \delta$ and $\varepsilon < 1/C_\Omega$ such that the following matrix inequality holds:

$$\Phi := \begin{pmatrix} -\lambda_1 + \alpha\varepsilon\delta & \delta\varepsilon & \frac{\alpha\varepsilon}{2} & 0 \\ \star & \phi_{22} & \frac{\alpha}{2} & 0 \\ \star & \star & -\lambda_2 & 0 \\ \star & \star & \star & \phi_{44} \end{pmatrix} \prec 0, \tag{2.36}$$

with $\phi_{22} = \varepsilon - \alpha + \delta + \lambda_2\gamma$, $\phi_{44} = \delta - \varepsilon + \lambda_1 C_\Omega^2 + \lambda_2\gamma$, C_Ω the constant in the Poincaré inequality. Then, for any initial condition

$$(z_0, z_1) \in H^2(\Omega) \cap H_0^1(\Omega) \times H_0^1(\Omega),$$

the closed-loop system (2.11) or (2.16) under the event-triggering mechanism (2.14) tuned by γ is exponentially stable with decay rate δ. In other words, there exists $K > 0$ such that

$$E(t) \leq KE(0)e^{-2\delta t} \quad \forall t > 0. \tag{2.37}$$

Furthermore, if the above matrix inequality holds with $\delta = 0$, then the closed-loop system is exponentially stable with a small enough decay rate.

Proof : The relationship between $V(t)$ and $E(t)$ is described thanks to Cauchy-Schwarz and Poincaré's inequalities. On the one hand,

$$\left|\varepsilon \int_\Omega z(t)\partial_t z(t)\right| \leq \varepsilon\|z(t)\|\|\partial_t z(t)\| \leq \varepsilon C_\Omega\|\nabla z(t)\|\|\partial_t z(t)\| \leq \varepsilon C_\Omega E(t).$$

By choosing $\varepsilon < 1/C_\Omega$ its follows

$$V(t) \geq (1 - \varepsilon C_\Omega)E(t).$$

On the other hand, the same tools bring

$$V(t) \leq (1 + \varepsilon C_\Omega)E(t) + \frac{\alpha\varepsilon C_\Omega^2}{2}\|\nabla z(t)\|^2 \leq \left(1 + \varepsilon C_\Omega + \varepsilon\alpha C_\Omega^2\right)E(t).$$

Hence we have

$$(1 - \varepsilon C_\Omega)E(t) \leq V(t) \leq (1 + \varepsilon C_\Omega + \varepsilon\alpha C_\Omega^2)E(t). \tag{2.38}$$

The cornerstone of the proof is now to ensure that there exists $\delta > 0$ such that:

$$\dot{V}(t) + 2\delta V(t) \leq 0, \qquad \forall t \geq 0$$

under the assumptions of the Theorem. Thus let us start by computing the time-derivative of V along the trajectories of (2.11):

$$\dot{V}(t) = \dot{E}(t) + \alpha\varepsilon \int_\Omega z(t)\partial_t z(t) + \varepsilon \int_\Omega |\partial_t z(t)|^2 + \varepsilon \int_\Omega z(t)\partial_t^2 z(t). \tag{2.39}$$

We already know $\dot{E}(t)$ from (2.20). Since z satisfies (2.16), and using the Green formula, we also have

$$\begin{aligned}\varepsilon\int_\Omega z(t)\partial_t^2 z(t) &= \varepsilon\int_\Omega z(t)\Delta z(t) - \alpha\varepsilon\int_\Omega z(t)\partial_t z(t)dx + \alpha\varepsilon\int_\Omega z(t)e_k(t)\\ &= -\varepsilon\int_\Omega |\nabla z(t)|^2 - \alpha\varepsilon\int_\Omega z(t)\partial_t z(t) + \alpha\varepsilon\int_\Omega z(t)e_k(t).\end{aligned}$$

Therefore

$$\begin{aligned}\dot{V}(t) &= -\alpha\int_\Omega |\partial_t z(t)|^2 + \alpha\int_\Omega \partial_t z(t)e_k(t) + \alpha\varepsilon\int_\Omega z(t)\partial_t z(t) + \varepsilon\int_\Omega |\partial_t z(t)|^2\\ &\quad - \varepsilon\int_\Omega |\nabla z(t)|^2 - \alpha\varepsilon\int_\Omega z(t)\partial_t z(t) + \alpha\varepsilon\int_\Omega z(t)e_k(t)\\ &= (\varepsilon-\alpha)\int_\Omega |\partial_t z(t)|^2 - \varepsilon\int_\Omega |\nabla z(t)|^2 + \alpha\int_\Omega \partial_t z(t)e_k(t) + \alpha\varepsilon\int_\Omega z(t)e_k(t),\end{aligned}$$

and we obtain:

$$\begin{aligned}\dot{V}(t) + 2\delta V(t) &= \alpha\varepsilon\delta\int_\Omega |z(t)|^2 + (\delta-\varepsilon)\int_\Omega |\nabla z(t)|^2 + \alpha\varepsilon\int_\Omega z(t)e_k(t) + \alpha\int_\Omega \partial_t z(t)e_k(t)\\ &\quad + (\varepsilon-\alpha+\delta)\int_\Omega |\partial_t z(t)|^2 + 2\delta\varepsilon\int_\Omega z(t)\partial_t z(t).\end{aligned}$$

Therefore

$$\dot{V}(t) + 2\delta V(t) = \int_\Omega \psi^\top(x,t)M_1\psi(x,t)dx, \tag{2.40}$$

with $\psi = \begin{pmatrix} z & \partial_t z & e_k & \nabla z\end{pmatrix}^\top$ and a symmetric matrix

$$M_1 = \begin{pmatrix} \alpha\varepsilon\delta & \delta\varepsilon & \frac{\alpha\varepsilon}{2} & 0\\ \star & \varepsilon-\alpha+\delta & \frac{\alpha}{2} & 0\\ \star & \star & 0 & 0\\ \star & \star & \star & \delta-\varepsilon\end{pmatrix}.$$

The goal is to find some conditions under which we want to satisfy $\dot{V}(t)+2\delta V(t) \leq 0$ or equivalently $\int_\Omega \psi^\top(t)M_1\psi(t) \leq 0$ subject to some mathematical constraints.

The first constraint comes from the Poincaré's inequality $\|z(t)\|^2 \leq C_\Omega^2\|\nabla z(t)\|^2$ (see Lemma A in Appendix) and it is equivalent to

$$\int_\Omega \psi^\top(t)M_2\psi(t) \geq 0, \text{ with } M_2 = \mathrm{diag}(-1,0,0,C_\Omega^2).$$

The second constraint comes from the event-triggering law that imposes (2.14) reading $\|e_k(t)\|^2 \leq 2\gamma E(t)$, $\forall t \in [t_k, t_{k+1})$, i.e., while no triggering event occurs. This last inequality can be written $\|e_k(t)\|^2 \leq \gamma\left(\|\partial_t z(t)\|^2 + \|\nabla z(t)\|^2\right)$ or equivalently

$$\int_\Omega \psi(t)^\top M_3\psi(t) \geq 0, \text{ with } M_3 = \mathrm{diag}(0,\gamma,-1,\gamma).$$

Using the S-procedure [10, Section 2.6.3], one therefore wants to satisfy the following:

$$\dot{V}(t) + 2\delta V(t) + \lambda_1\int_\Omega \psi(t)^\top M_2\psi(t) + \lambda_2\int_\Omega \psi(t)^\top M_3\psi(t) \leq 0 \tag{2.41}$$

for any two positive scalars λ_1 and λ_2.

From (2.40), inequality (2.41) reads :

$$\int_\Omega \psi^\top(x,t)(M_1+\lambda_1 M_2+\lambda_2 M_3)\psi(x,t)dx \leq 0. \tag{2.42}$$

Hence, by defining Φ as $\Phi = M_1+\lambda_1 M_2+\lambda_2 M_3$, the satisfaction of relation (2.36) means that relation (2.42) and (2.41) are also satisfied, and therefore one obtains

$$\dot{V}(t)+2\delta V(t)\leq 0, \quad \forall t\in\mathbb{R}^+.$$

That corresponds to have $V(t)\leq e^{-2\delta t}V(0)$. Taking (2.38) into account, it follows that

$$E(t)\leq \frac{1+\varepsilon C_\Omega+\varepsilon\alpha C_\Omega^2}{1-\varepsilon C_\Omega}E(0)e^{-2\delta t}.$$

The proof of Theorem 2.10 is complete. ♢

Let us now provide some insights on the matrix inequality (2.36). First, we can use a change of variable $\bar{\gamma}=\lambda_2\gamma$ and search both λ_2 and $\bar{\gamma}$ as decision variables of

$$\Phi := \begin{pmatrix} -\lambda_1+\alpha\varepsilon\delta & \delta\varepsilon & \frac{\alpha\varepsilon}{2} & 0 \\ \star & \varepsilon-\alpha+\delta+\bar{\gamma} & \frac{\alpha}{2} & 0 \\ \star & \star & -\lambda_2 & 0 \\ \star & \star & \star & \delta-\varepsilon+\lambda_1 C_\Omega^2+\bar{\gamma} \end{pmatrix} \prec 0. \tag{2.43}$$

This is a sufficient condition allowing to ensure the exponential stability of the closed loop. In the following proposition we show that there always exists a solution $(\lambda_1,\lambda_2,\bar{\gamma},\delta)$ such that (2.43) is satisfied.

Proposition 2.11: Feasibility of the matrix inequality

Given $\alpha > 0$, condition (2.36) of Theorem 2.10, or equivalently condition (2.43), enjoys the following properties

(i) Given $\delta = 0$, condition (2.43) is always feasible;

(ii) There always exists a strictly positive scalar δ such that (2.43) is feasible.

Proof : Let us denote by Φ_0 the matrix corresponding to Φ in the case $\delta = 0$:

$$\Phi_0 := \begin{pmatrix} -\lambda_1 & 0 & \alpha\varepsilon/2 & 0 \\ 0 & \varepsilon-\alpha+\bar{\gamma} & \alpha/2 & 0 \\ \alpha\varepsilon/2 & \alpha/2 & -\lambda_2 & 0 \\ 0 & 0 & 0 & -\varepsilon+\lambda_1 C_\Omega^2+\bar{\gamma} \end{pmatrix}.$$

Condition (2.43) reads $\Phi_0 \prec 0$ and then implies:

$$\begin{aligned} &\lambda_1>0,\ \lambda_2>0 \\ &-\varepsilon+\alpha-\bar{\gamma}>0 \Longleftrightarrow \varepsilon<\alpha-\bar{\gamma} \\ &\varepsilon-\lambda_1 C_\Omega^2-\bar{\gamma}>0 \Longleftrightarrow \varepsilon>\lambda_1 C_\Omega^2+\bar{\gamma}. \end{aligned} \tag{2.44}$$

Now, consider λ_1, ε and $\bar{\gamma}$, satisfying (2.44) and the extra constraint $\varepsilon < 1/C_\Omega$ in assumptions of Theorem 2.10, which always exist. Then one gets

$$\begin{pmatrix} -\lambda_1 & 0 \\ 0 & \varepsilon - \alpha + \bar{\gamma} \end{pmatrix} \prec 0.$$

By using the Schur complement (Lemma A.6) on the first 3×3 block one obtains:

$$-\lambda_2 + \frac{\alpha^2}{4} \begin{pmatrix} \varepsilon & 1 \end{pmatrix} \begin{pmatrix} \lambda_1^{-1} & 0 \\ 0 & -(\varepsilon - \alpha + \bar{\gamma})^{-1} \end{pmatrix} \begin{pmatrix} \varepsilon \\ 1 \end{pmatrix} < 0 \tag{2.45}$$

Therefore, since the quadratic term in the left-hand side of (2.45) is positive, there always exists a positive value for λ_2 such that (2.45) holds.

In summary there exists a solution such that $\Phi_0 \prec 0$ holds. Furthermore, the tuning parameter γ is easily recovered from $\bar{\gamma}$ and λ_2.

Consider now $\delta \neq 0$, then one can write Φ as follows:

$$\Phi = \Phi_0 + \delta \begin{pmatrix} \alpha\varepsilon & \varepsilon & 0 & 0 \\ \varepsilon & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{2.46}$$

Since there exist $\varepsilon, \bar{\gamma}, \lambda_1, \lambda_2$ such that $\Phi_0 \prec 0$, it follows that there always exists a small enough $\delta > 0$ such that $\Phi \prec 0$. The proof of Proposition 2.11 is complete.

♢

2.3.6 Numerical simulation

We illustrate the efficiency of the event-triggering law proposed in this section by considering the example of a one-dimensional wave equation (2.1) on $\Omega = (0, \pi)$. We consider the damping coefficient $\alpha = 1$ and the initial conditions

$$z_0(x) = \sin(x) \text{ and } z_1(x) = \sin(2x) \tag{2.47}$$

that are consistent with the homogeneous boundary data of (2.1).

We aim at comparing the continuous-in-time version of the closed-loop system versus the event-triggered closed-loop version. In other words we compare the behavior of system (2.1) with the one of system (2.11) under the event-triggering rule (2.14).

The design parameter γ in the event-triggering rule (2.14) plays a key role in the exponential stability of system (2.11) under the mechanism (2.14). The choice of γ influences the number of updates imposed by (2.14): the smaller γ, the more frequent the updates. A feasible solution to condition (2.36) in Theorem 2.10 is $\lambda_1 = 0.1$, $\lambda_2 = 1$, $\gamma = 0.3$, $\delta = 0.25$ and $\varepsilon = 0.6$.

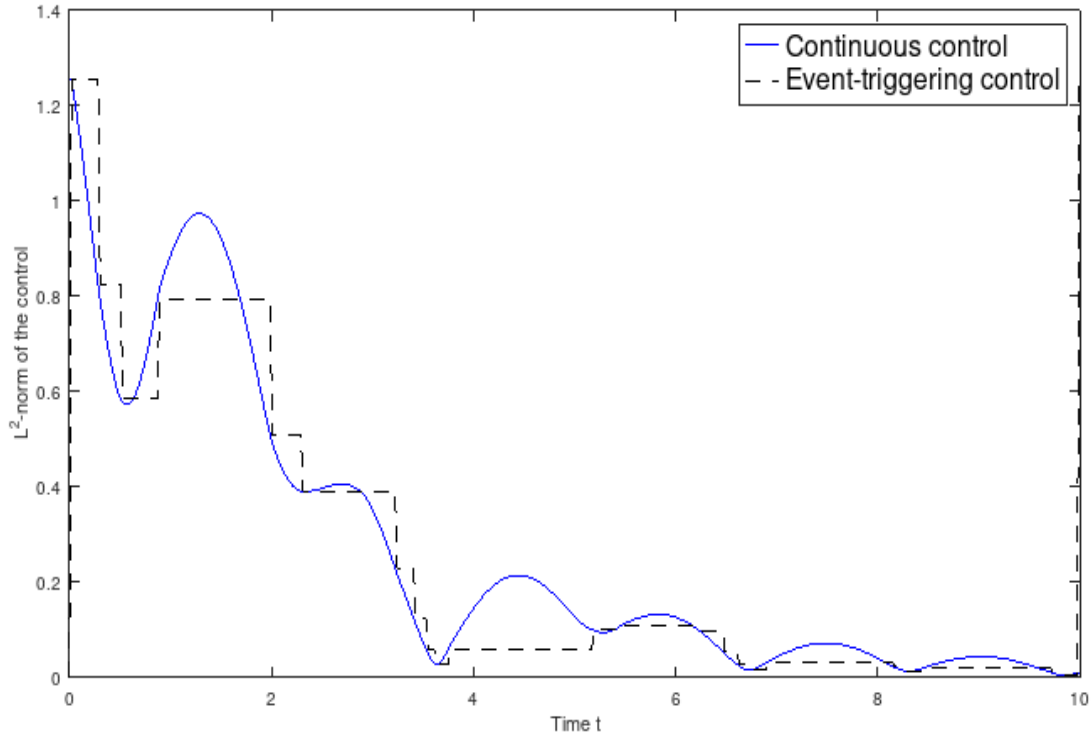

Figure 2.4: Evolution of the L^2-norm of the continuous-in-time controller in blue and the event-triggered controller (in black dashed line) with damping coefficient $\alpha = 1$.

To begin with, let us present Figure 2.4 that depicts the evolution of the magnitude of the controller $\|f(t)\|_{L^2(0,\pi)}$ in the continuous-in-time and the event-triggering frameworks. We notice that the update times are not regular and there is a large variation in the magnitude of the continuous-in-time controller allowing to conclude that the event-triggered control approach is energy efficient.

Now in order to better understand how the sampling acts on the exponential stability result, we present in Figure 2.5 the repartition of the updates instants, and the evolution of the natural energy $E(t)$ of the closed-loop system (2.11), for $\alpha = 1$, in the following cases:

- Under the continuous-in-time control (blue line)
- Under the event-triggered controller (black dotted line) with t_k given by the event-triggering rule (2.14).
- Under the fixed initial time controller $f(t) = -z_1$ (green line).
- With the controller $f(x,t) = -\partial_t z(x, k\tau)$ (in red) build with periodic sampling under period $\tau = 0.5$ so that the number of updates is the same as the one observed during the time $T = 3$ when following (2.14).
- Under the event-triggered controller with t_k given by the event-triggering rule of [5](see Remark 2.1) that has an additional term $E(0)e^{-\theta t}$ (black line) with $\theta = 2.5$.

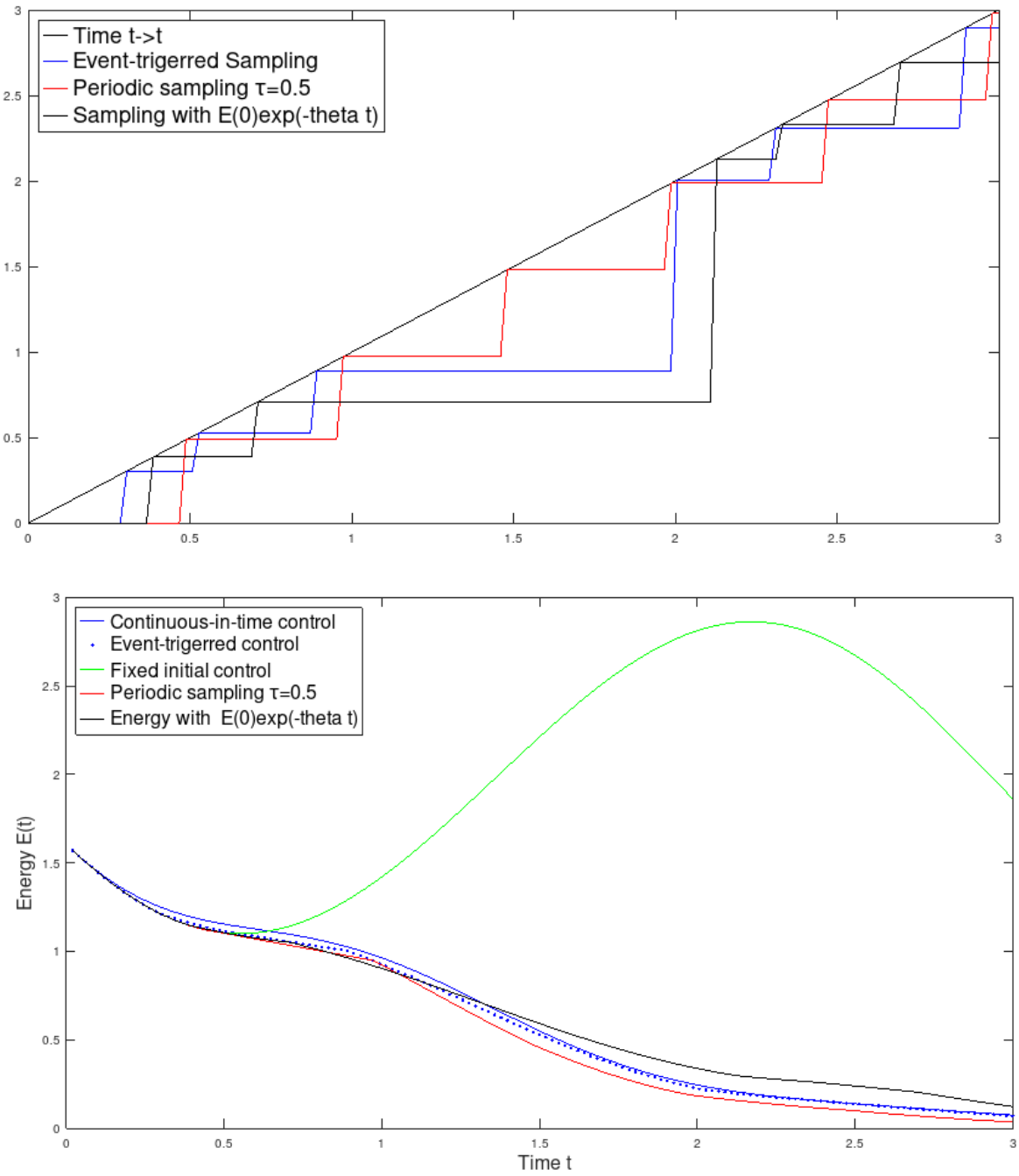

Figure 2.5: Top: Illustration of the event-triggering mechanisms (2.14) and from [5] for $\gamma = 0.3$ and $\theta = 2.5$ and periodic sampling of period $\tau = 0.5$. Bottom: Evolution of the energy $E(t)$ defined in (2.10)

First, we remark that when the controller is fixed as $f(t) = -z_1$, the energy evolves as the sinusoidal z_1. This corresponds to the first sampling period where the control is only based on the initial velocity. Second, Figure 2.5 also shows that the evolution of energy of the event-triggered control system is similar to the one of the continuous–in-time controlled system and to the one under ad-hoc periodic sampling.

Nevertheless, it is important to recall that to the best of our knowledge a proof of the exponential decay rate of the energy in the case of periodic sampling does not exist. However, as it can be seen, a good choice of the period τ seems to lead to the exponential decay rate of the corresponding energy. More precisely, using a trial and error method, one can find that this particular system becomes unstable under periodic sampling when $\tau > 1.2$.

Let us illustrate the inter-execution time. We chose the same tuning parameter $\gamma = 0.3$ for the event-triggering mechanism (2.15) and our event-triggering mechanism (2.14) and take $\theta = 2.5$ in (2.14). Both choices allow the feasibility of the mandatory matrix inequalities to obtain the exponential stability of the corresponding closed-loop systems.

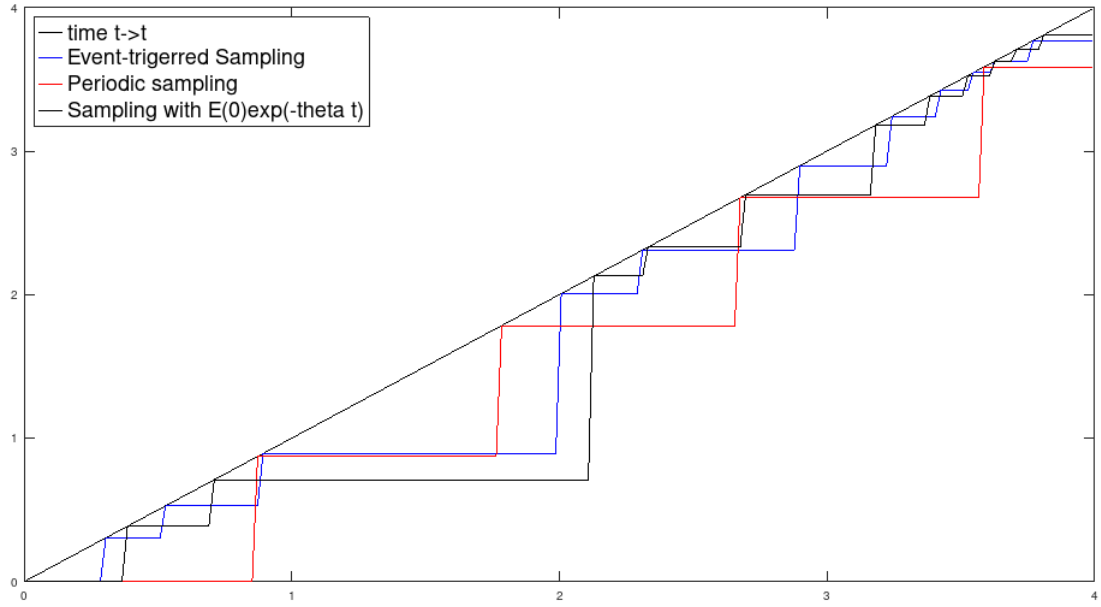

Figure 2.6: Illustration of the event-triggering mechanisms (2.13) and (2.15) for $\gamma = 0.3$ and $\theta = 2.5$. Periodic sampling of period $\tau = 0.9$.

Figure 2.6 allows to observe the number and the repartition of the time updates generated by several cases : the event-triggering mechanism (2.14), the one from (2.15) with $\theta = 2.5$ and a periodic sampling one. The parameter γ is chosen to allow the exponential stability. The parameter θ has the specificity to be an upper bound of the best possible decay rate δ and can be chosen large if we aim at a very efficient exponential decay rate, to the cost of more frequent updates or small, if the number of updates should be minimized. Finally, the period τ of the periodic sampling was chosen through a trial-and-error method just to find a value that brings stability, since we do not have any proof of stability in the periodic case.

Finally, Figure 2.7 illustrates the evolution of the energy of the corresponding closed-loop systems. Note that the fixed initial control graph corresponds to consider

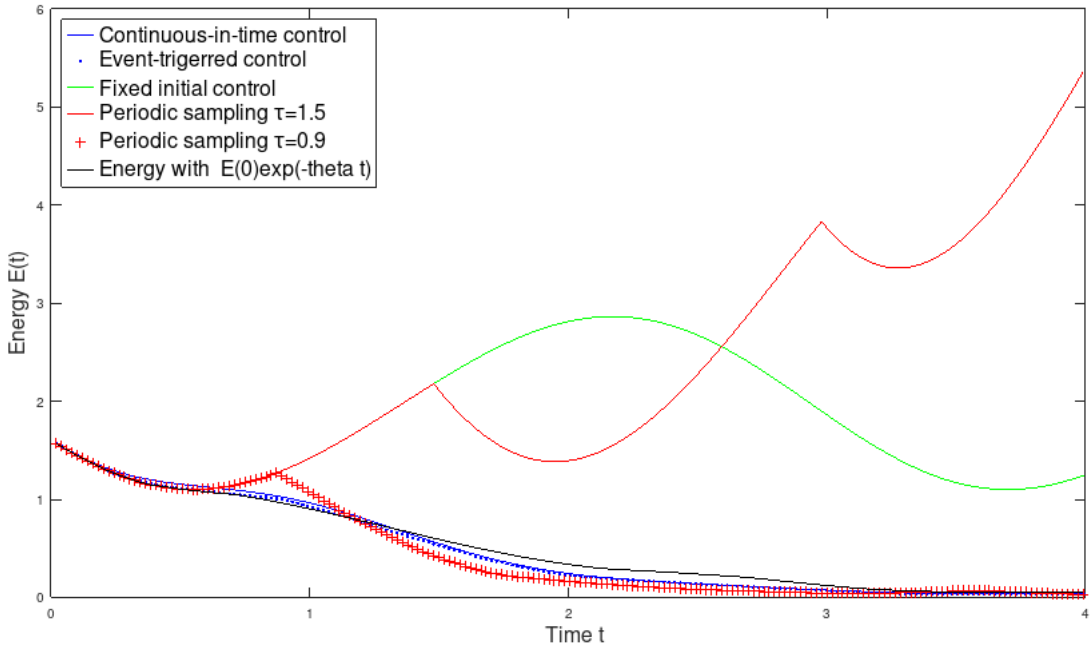

Figure 2.7: Evolution of the closed-loop energy $E(t)$ computed in various cases, with parameters $\gamma = 0.3$, $\theta = 2.5$, initial velocity $z_1(x) = sin(2x)$, and $\tau = 0.9$ or 1.5.

$u(x,t) = -\alpha z_1(x)$ and that we took an initial velocity condition $z_1(x) = sin(2x)$. A trial-and-error method was used to find that for a period $\tau < 1.2$ we obtain exponential stability and otherwise, there is no stability, as depicted in red in Figure 2.7 (with $\tau = 0.9$ and $\tau = 1.5$).

2.4 ETC for the anti-damped wave equation

Remark that if we consider $\alpha = 0$ in system (2.1), corresponding to the undamped open-loop, the system will be stable. It is however possible to consider an unstable open-loop system, by adding in the right hand side of (2.1) a source term as, for example, $+b\partial_t z(x,t)$ with $b > 0$, leading to the modified system:

$$\begin{cases} \partial_t^2 z - \Delta z = -\alpha \partial_t z(t_k) + b\partial_t z(x,t), & \text{in } \Omega \times [t_k, t_{k+1}), \\ & k \in \mathbb{N} \\ z = 0, & \text{on } \partial\Omega \times \mathbb{R}^+, \\ z(\cdot,0) = z_0, \partial_t z(\cdot,0) = z_1, & \text{in } \Omega. \end{cases} \tag{2.48}$$

where the event-triggering mechanism is defined by the static rule:

$$\begin{cases} t_0 = 0 \\ t_{k+1} = \inf \left\{ t \geq t_k, \|e_k(t)\|^2 > 2\gamma E(t) \right\} \end{cases} \tag{2.49}$$

with the energy of the system also given by

$$E(t) = \frac{1}{2} \left(\|\partial_t z(t)\|^2 + \|\nabla z(t)\|^2 \right). \tag{2.50}$$

In that case, without control (i.e., $\alpha = 0$), the energy of the system satisfies

$$\dot{E}(t) = b \int_\Omega |\partial_t z(t)|^2,$$

meaning that if $b = 0$, the energy will remain constant over time, and if $b > 0$, then the system undergoes an excitation source term and the energy can only grow. Meanwhile, with a continuous-in-time control $u(x,t) = -\alpha \partial_t z(x,t)$ if one chooses α such that $\alpha > b$ then one can prove exponential decrease of the energy.

2.4.1 Well-posedness and Avoidance of Zeno behavior

Theorem 2.12: Well-posedness

Let Ω be an open bounded domain of class C^2. For any initial conditions $(z_0, z_1) \in H^2(\Omega) \cap H_0^1(\Omega) \times H_0^1(\Omega)$, there exists a unique strong solution to (2.11) under the event-triggering mechanism (2.14), satisfying

$$z \in C([0,T^*); H^2(\Omega) \cap H_0^1(\Omega)) \cap C^1([0,T^*); H_0^1(\Omega)). \tag{2.51}$$

Proof : The proof is constructed by induction and is similar to the one that has been presented in Theorem 2.5 for the damped wave equation since the nature of the operator is the same. ♢

As we have recalled in this chapter, in event-triggered control events must be triggered in such a way that the closed-loop system does not generate an infinite number of updates in a finite time which is known as the Zeno phemenon.

To do so, we rewrite the system (2.48) in order to consider the deviation error e_k :

$$\begin{cases} \partial_t^2 z(x,t) - \Delta z(x,t) = (b-\alpha)\partial_t z(x,t) + \alpha e_k(t), & \forall (x,t) \in \Omega \times (0,T^*), \\ z(x,t) = 0, & \forall (x,t) \in \partial\Omega \times \mathbb{R}^+, \\ z(x,0) = z_0(x), \quad \partial_t z(x,0) = z_1(x), & \forall x \in \Omega, \end{cases} \tag{2.52}$$

with

$$e_k(x,t) = \partial_t z(x,t) - \partial_t z(x,t_k). \tag{2.53}$$

As for the static case the energy E is bounded as follows.

Lemma 2.13:

Under the event-triggering mechanism (2.49) there exists a constant $C_b > 0$ such that for all $t \in [0, T^*)$:

$$E(0)e^{-2C_b t} \leq E(t) \leq E(0)e^{2C_b t}. \tag{2.54}$$

Proof : Using (2.48) and the Green formula (Lemma A.3), the time derivative of E is reduced to

$$\dot{E}(t) = (b-\alpha)\int_\Omega |\partial_t z(t)|^2 + \alpha \int_\Omega \partial_t z(t) e_k(t) \tag{2.55}$$

so that one gets

$$|\dot{E}(t)| \leq (b+\alpha)\|\partial_t z(t)\|^2 + \alpha \|e_k(t)\| \|\partial_t z(t)\|. \tag{2.56}$$

From the definition (2.17) of T^*, since $t \in [0, T^*)$, either there exists k such that $t \in [t_k, t_{k+1})$ if the sequence $(t_k)_{k\geq 0}$ is not finite, or t may be greater than the last t_k and the definition of (2.49) allows to call $t_{k+1} = T^*$. From (2.14) and $\|\partial_t z(t)\|^2 \leq 2E(t)$ we get:

$$\begin{aligned} &|\dot{E}(t)| \leq 2(b+\alpha)E(t) + \alpha\sqrt{2\gamma E(t)}\sqrt{2E(t)} \text{ or equivalently} \\ &|\dot{E}(t)| \leq 2C_b E(t) \quad \text{ with } C_b = b + \alpha(1+\sqrt{\gamma}). \end{aligned} \tag{2.57}$$

It follows that $-2C_b E(t) \leq \dot{E}(t) \leq 2C_b E(t)$.

The same reasoning as for the proof of Lemma 2.6 allows to conclude.

♢

We can now state the following result concerning Zeno behavior.

Theorem 2.14:

There is no Zeno phenomenon for the system (2.48) under the event-triggering mechanism (2.14). Equivalently, the maximal time of existence of solution defined by (2.17) is actually $T^* = +\infty$.

Proof : The proof is based on the study of the function φ defined on $[t_k, t_{k+1})$ by

$$\varphi : t \mapsto \varphi(t) = \frac{\|e_k(t)\|^2}{2\gamma E(t)}. \tag{2.58}$$

The function φ is nonnegative and satisfies $\varphi(t_k) = 0$ for any k and jumps from $\lim_{t \to t_{k+1}} \varphi(t) = 1$ to $\varphi(t_{k+1}) = 0$ when a triggering event occurs, according to the law (2.49). Let us estimate the time-derivative of φ:

$$\dot{\varphi}(t) = \frac{\int_\Omega \dot{e}_k(t) e_k(t)}{\gamma E(t)} - \frac{\dot{E}(t)\|e_k(t)\|^2}{2\gamma \left(E(t)\right)^2}. \tag{2.59}$$

On the one hand, from (2.53), (2.52) and the Cauchy Schwarz's inequality, for all $t \in [t_k, t_{k+1})$, we have

$$\begin{aligned}\int_\Omega \dot{e}_k(t) e_k(t) &= \int_\Omega \Delta z(t) e_k(t) + (b-\alpha) \int_\Omega \partial_t z(t) e_k(t) + \alpha \|e_k(t)\|^2, \\ &\leq \|e_k(t)\| \|\Delta z(t)\| + (b-\alpha)\|e_k(t)\|\|\partial_t z(t)\| + \alpha\|e_k(t)\|^2.\end{aligned}$$

Since there exists a constant $C_\Delta > 0$ such that $\forall t \in [0, T^*)$

$$\|\Delta z(t)\| \leq \|\Delta z\|_{L^\infty(0,T^*;L^2(\Omega))} \leq C_\Delta, \tag{2.60}$$

where C_Δ depends on $\|z_0\|_{H^2(\Omega)}$ and $\|z_1\|_{H_0^1(\Omega)}$, and $\|\partial_t z(t)\|^2 \leq 2E(t)$ it follows

$$\frac{\int_\Omega \dot{e}_k(t) e_k(t)}{\gamma E(t)} \leq \frac{C_1\sqrt{2}}{\sqrt{\gamma E(t)}} + \frac{2(b-\alpha)}{\sqrt{\gamma}} + 2\alpha\varphi(t). \tag{2.61}$$

Besides, from (2.57), one has $-\dot{E}(t) \leq 2\alpha(1+\sqrt{\gamma})E(t)$ so that

$$\frac{-\dot{E}(t)\|e_k(t)\|^2}{2\gamma\left(E(t)\right)^2} \leq 2[b + \alpha(1+\sqrt{\gamma})]\varphi(t). \tag{2.62}$$

Gathering (2.61) and (2.62) we obtain:

$$\dot{\varphi}(t) \leq \frac{C_1\sqrt{2}}{\sqrt{\gamma E(t)}} + \frac{2(b-\alpha)}{\sqrt{\gamma}} + 2[b + \alpha(1+\sqrt{\gamma})]\varphi(t).$$

Using the same arguments as for the proof of Theorem 2.8, we obtain :

$$1 \leq \left[A + \frac{Be^{CT^*}}{\sqrt{E(0)}}\right](t_{k+1} - t_k) \tag{2.63}$$

with $A = \dfrac{2(b-\alpha)}{\sqrt{\gamma}} + 2b + 2\alpha(2+\sqrt{\gamma})$ and $B = C_1\sqrt{\dfrac{2}{\gamma}}$.

Let $t_k \to T^*$ as $k \to +\infty$ in (2.63), then we get a contradiction if $T^* \neq +\infty$. Therefore, we need to consider $T^* = +\infty$, leading to the absence of any accumulation points. Thereby, the avoidance of Zeno behavior is guaranteed. $\diamondsuit$

2.4.2 Exponential stability

The only necessity to preserve the stability result is to have b smaller than the damping coefficient α. More specifically, to ensure that LMI (2.36) (in Theorem 2.10) of the current anti-damping problem is feasible, one only needs $b < \alpha$ and one can mimic the proof of Theorem 2.10 in the following corresponding Theorem.

Theorem 2.15: Stability of the event-triggered of the wave equation

Given the damping parameter $\alpha > b$, assume there exist positive scalars $\gamma, \lambda_1, \lambda_2, \delta$ and $\varepsilon < 1/C_\Omega$ such that the following matrix inequality holds: $\Phi_{b,\delta} \prec 0$ where

$$\Phi_{b,\delta} := \begin{pmatrix} -\lambda_1 + (\alpha - b)\varepsilon\delta & \delta\varepsilon & \frac{\alpha\varepsilon}{2} & 0 \\ \star & \varepsilon + b - \alpha + \delta + \lambda_2\gamma & \frac{\alpha}{2} & 0 \\ \star & \star & -\lambda_2 & 0 \\ \star & \star & \star & \delta - \varepsilon + \lambda_1 C_\Omega^2 + \lambda_2\gamma \end{pmatrix} \tag{2.64}$$

and C_Ω is the constant in the Poincaré inequality (Appendix A.2). Then, for any initial condition

$$(z_0, z_1) \in H^2(\Omega) \cap H_0^1(\Omega) \times H_0^1(\Omega),$$

the closed-loop system (2.48) under the event-triggering mechanism (2.49) tuned by γ is exponentially stable with decay rate δ. In other words, there exists $K > 0$ such that

$$E(t) \leq KE(0)e^{-2\delta t} \quad \forall t > 0. \tag{2.65}$$

Furthermore, if the above matrix inequality holds with $\delta = 0$, then the closed-loop system is exponentially stable with a small enough decay rate.

Proof : Assuming that $\alpha - b > 0$ the proof of this theorem is the same as the one 2.10, relying on the following Lyapunov function:

$$V(t) := E(t) + \frac{(\alpha - b)\varepsilon}{2} \int_\Omega |z(t)|^2 + \varepsilon \int_\Omega z(t)\partial_t z(t).$$

Using $\psi = \begin{pmatrix} z & \partial_t z & e_k & \nabla z \end{pmatrix}^\top$, it leads, with the same reasoning, to

$$\int_\Omega \psi^\top(x,t)\Phi_{b,\delta}\psi(x,t)dx \leq 0, \tag{2.66}$$

where $\Phi_{b,\delta}$ is defined by (2.64).

From there, one can conclude to the feasibility of this LMI condition in the same way as for the case $b = 0$

♢

2.5 Conclusion and Perspectives

In the present chapter, the exponential stability of the damped (and anti-damped) linear wave equation under an event-triggering mechanism was proved under some specific conditions. A sufficient matrix inequality-based condition for the exponential stability of the system was formulated. The avoidance of the Zeno behavior through the absence of accumulation points in the updates sequence was proved.

This work opens the door for future investigations, as for example:

- *Over (under)damping effect*

In [16] considering the damped wave equation (2.1), the decay rate

$$\omega(\alpha) = \inf\{\delta; \exists C(\delta) > 0 \text{ s.t } E(t) \leq CE(0)e^{-2\delta t}\}$$

is proved to be equal to the spectral abscissa $\mu(\alpha)$ of the operator

$$\mathcal{A} : \mu(\alpha) = \sup\{\mathcal{R}e(\lambda) : \lambda \in \sigma(\mathcal{A})\} = -\alpha + \mathcal{R}e(\sqrt{\alpha^2 - \pi^2}).$$

This allows to characterize the over(under)damping effect when α is less (greater) than π.

The Lyapunov functional V (2.35) has the advantage to allow us to bypass this over(under)damping effect thanks to the term $\alpha\varepsilon$. But it also has the drawback to prevent us from seeing that when α grows, even if we expect δ to grow as well, it doesn't. Indeed, from (2.46) (Proposition 2.11) one has $\Phi = \Phi_0 + \delta \begin{pmatrix} \alpha\varepsilon & \varepsilon & 0 & 0 \\ \varepsilon & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ with $\Phi_0 \prec 0$, but α grows implies that we must have δ small to get $\Phi \prec 0$.

A solution to deal with this effect is to consider all the state $(z, \partial_t z)$ in the control term. The numerical implementation of the new control could be considered.

- *Event-triggering mechanism for Aeroelastic system: model of the flow-induced vibration.*

The model of the flow-induced vibration is given by a wave equation with anti-damping term throughout the 1-D domain $(0, \frac{1}{c})$:

$$\begin{cases} \partial_t^2 z(x,t) - \partial_x^2 z(x,t) = 2\lambda \partial_t z(x,t) - \beta z(x,t) \\ z(0,t) = 0 \\ z(\frac{1}{c}, t) = U(t) \end{cases}$$

where λ and β are function of free-stream density of the fluid ρ, velocity U and the Mach number M. In [106], the following control is proposed in other to

$$U(t) = \int_0^{\frac{1}{c}} r(\tfrac{1}{c}, y)v(y,t)dy + \frac{1}{h(\frac{1}{c})}\left[\int_0^{\frac{1}{c}} k(\tfrac{1}{c}, y)\omega(y,t)dt + \int_0^{\frac{1}{c}} s(\tfrac{1}{c}, y)\partial_t\omega(y,t)dy\right]$$

where $v(y,t)$ and $\omega(y,t)$ are used to defined the direct and inverse backstepping transformations. The kernels r, k and s are well known thanks to a three-stage backstepping transformation.

For $U(t) = U(t_k), t \in [t_k, t_{k+1})$, what kind of event-triggered law the sequence (t_k) should follow in order to maintain this exponential decay result and avoid Zeno behavior?

- *Event-based dynamic boundary control of the wave PDE*

Event-based control can be considered for other kinds of boundary control in 1-D setting, as the wave PDE compensation with Neumann Actuation:

$$\begin{cases} \dot{X}(t) = AX(t) + Bz(0,t) \\ \partial_t^2 z(x,t) = \partial_x^2 z(x,t) \\ \partial_x z(0,t) = 0 \\ \partial_x z(D,t) = U(t) \end{cases}$$

or with Dirichlet actuation

$$\begin{cases} \dot{X}(t) = AX(t) + B\partial_x z(0,t) \\ \partial_t^2 z(x,t) = \partial_x^2 z(x,t) \\ z(0,t) = 0 \\ z(1,t) = U(t) \end{cases}$$

where $(X, z, \partial_t z)$ is the state ans U is the control corresponding to a force on the string's boundary. Backstepping technique is used in [69] to design the control U and the state is proven to be exponentially stable in the sence of the norm:

$$\left(|X(t)|^2 + u(0,t)^2 + \int_0^D u_x(x,t)^2 dx + \int_0^D u_t(x,t)^2 dx \right)^{1/2}$$

It would be relevant to study these system when they are subjected to event-triggered control, $U(t)$ becoming $U(t_k)$ $\forall t \in [t_k, t_{k+1})$.

- *Event-triggered boundary control of the wave equation*

Consider system

$$\begin{cases} \partial_t^2 z(x,t) - \Delta z(x,t) = 0 & \forall\ (x,t) \in \Omega \times (0,T), \\ z(x,t) = 0 & \forall\ (x,t) \in \Gamma_0 \times (0,T), \\ \partial_\nu z(x,t) = -\alpha(x)\partial_t z(x,t) & \forall\ (x,t) \in \Gamma_1 \times (0,T), \\ z(x,0) = z_0(x) & \forall\ x \in \Omega, \\ \partial_t z(x,0) = z_1(x) & \forall\ x \in \Omega, \end{cases}$$

where the damping parameter $\alpha \in L^\infty(\Omega)$ satisfies

$$\alpha(x) = (x - x_0) \cdot \nu(x) \geq \alpha_0 > 0, \quad \forall x \in \Gamma_1.$$

with Γ_1 a suitable part of the boundary $\partial\Omega$. For $x_0 \notin \mathbb{R}^n \setminus \bar{\Omega}$, we can choose

$$\Gamma_1 = \{x \in \partial\Omega, (x - x_0) \cdot \nu(x) > 0\}$$

and if Γ_1 and $\Gamma_0 = \partial\Omega \setminus \Gamma_1$ are such that $\Gamma_0 \cap \Gamma_1 = \varnothing$.

Note that one of the difficulty in the well-posedness of the system (2.5), is the fact that the initial data (z_0, z_1) have to satisfy the compatibility conditions with the boundary data:

$$\begin{aligned} z_0(x) &= 0 \quad \text{for all } x \in \Gamma_0 \\ \partial_\nu z_0(x) &= \alpha(x) z_1 \quad \text{for all } x \in \Gamma_1. \end{aligned}$$

We refer to [117] where this problem is well adressed. Moreover, the origin of the closed-loop system is proved to be globally exponentially stable in [117, 13].

The event-triggerred control system in this case could reads:

$$\begin{cases} \partial_t^2 z(x,t) - \Delta z(x,t) = 0 & \forall\ (x,t) \in \Omega \times (0,T), \\ z(x,t) = 0 & \forall\ (x,t) \in \Gamma_0 \times (0,T), \\ \partial_\nu z(x,t) = -\alpha(x) \partial_t z(x,t_k) & \forall\ (x,t) \in \Gamma_1 \times [t_k, t_{k+1}), \\ z(x,0) = z_0(x) & \forall\ x \in \Omega, \\ \partial_t z(x,0) = z_1(x) & \forall\ x \in \Omega. \end{cases} \tag{2.67}$$

with the event-triggering rule which can be defined as follows: $t_0 = 0$,

$$t_{k+1} = \inf \left\{ t \geq t_k, \|\partial_t z(x,t) - \partial_t z(x,t_k)\|^2_{L^2(\Gamma_1)} - \gamma E(t) - \nu_0 \geq 0 \right\}, \tag{2.68}$$

where γ, ν_0 are design parameters. In the event-triggering framework, one will need to carefully adress the following items.

1. Since the control is defined on the boundary, it becomes essential to rigorously establish the well-posedness of the system (2.67). Particularly, when dealing with the ETM, it is important to note that the compatibility conditions of the solution on the boundaries Γ_0 and Γ_1 at time t_k are no longer preserved.

2. For the avoidance of Zeno behavior, the difficulty is that the operator associated to the system is unbounded therefore Lemma 2.6 is no longer valid. This led us to add the ν_0 parameter in the event-triggering law (2.68) (which is the event-triggering mechanism (1.5) introduced in the Introductive Chapter 1).

3. With the event-triggering law (2.68) one may expect exponential convergence to an attractor because of ν_0.

3

Event-based control of the linear Schödinger system

3.1 Introduction

3.1.1 Some ingredients on the Schrödinger equation

The Schrödinger equation, for $x \in \Omega \subset \mathbb{R}^N$ and $t \geq 0$

$$i\partial_t z(x,t) + \Delta z(x,t) = f(x,t),$$

most known in quantum theory, arises for instance in nonlinear optics for laser beam propagation [2] or in cold atom physics to describe Bose Einstein condensation. Its solution z describes the shape of the probability wave function that governs the motion of quantum particles, and the equation specifies how these waves are altered by external influences f [110]. A partial list of concrete applications of the linear Schrödinger equation includes:

- Particle in a Box: The linear Schrödinger equation appears in the study of a particle confined within a box potential [20, 48]. This scenario is used to model quantum systems such as electrons in a one-dimensional semiconductor or atoms trapped in an optical lattice and is important in the design of devices such as quantum wells and quantum dots.

- Quantum Tunneling: A phenomenon where a particle can pass through a potential barrier that classically it would not have enough energy to overcome is called quantum tunneling. The linear Schrödinger equation allows us to calculate the probability of tunneling for a particle encountering a potential barrier. This application has practical implications in various fields, such as scanning tunneling microscopy, where quantum tunneling is exploited to image and manipulate individual atoms on surfaces [43].

- Bose-Einstein Condensates: The linear Schrödinger equation is used to study the behavior of ultra-cold quantum gases, particularly in the context of Bose-Einstein condensates (BECs) [129]. BECs are a state of matter that occurs at extremely low temperatures, where a large number of bosonic particles occupy the same quantum state.

In addition to all these applications, let us mention, even if it is not the subject of our studies, the nonlinear Schrödinger equation (NLSE), which finds applications in various fields of physics, including quantum mechanics, nonlinear optics, and fluid dynamics [110, 9]. It is typically written as:

$$i\hbar\frac{\partial\psi(x,t)}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\psi(x,t)}{\partial x^2} + V(x)\psi(x,t) + g|\psi(x,t)|^2\psi(x,t)$$

where ψ represents the complex wave function, t is time, x is space, $\hbar$ is the reduced Planck constant, m is the particle mass, $V(x)$ is the potential, and g is a nonlinear coefficient. The NLSE describes phenomena such as solitons, which are localized wave packets that maintain their shape during propagation, and self-focusing or self-defocusing effects due to nonlinearities. Applications of the NLSE include modeling optical fibers, where it describes the propagation of intense laser pulses, as well as Bose-Einstein condensates and superfluids. It is also used in the study of quantum turbulence, nonlinear wave phenomena, and other nonlinear systems where the interactions between particles or waves lead to rich and complex dynamics.

These applications demonstrate how the Schrödinger equation provides a powerful mathematical tool for describing and analyzing various quantum phenomena and systems. This justifies the extensive study of this equation in many aspects, including the well-posedness [12], the exact controllability [79], the observability [87], and the stabilization or stability analysis by multiplier techniques and constructing energy functionals [80] or by backstepping approach via the boundary actuation and measurements [70].

Furthermore, there exists a huge litterature on the control of the Schrödinger equation and one may distinguish several kinds of Schrödinger equation

- The finite dimensional Schrödinger equation

$$\frac{d}{dt}z(t) = -\frac{i}{\hbar}[H_0 - \varepsilon(t)\cdot\mu]z(t).$$

  where $z(t) \in \mathbb{R}^N$ is unitary evolution operator of the system at time t, and H_0 is the free Hamiltonian, μ is the dipole operator, and $\varepsilon(t)$ is the control function at time t. This equation fully describes the coherent quantum dynamics of molecular systems in interaction with electric laser fields in the dipole approximation or with spin systems have a magnetic field that varies with time. We refer to [15] where the authors present a perspective of progress about theoretical insights, technological improvements and adaptive feedback control in the laboratory.

- Another form of finite dimensional Schrödinger equation is

$$\frac{d}{dt}\varphi = (H_0 + u(t)H_1)\varphi, \qquad \varphi(0) = \varphi^0, |\varphi^0|_N = 1$$

  where $\varphi \in \mathbb{R}^N$, H_0 is the free Hamiltonian and H_1 the interaction Hamiltonian. H_0 and H_1 are Hermitian matrices so that the state of the system verifies the

conservation of probability: $|\varphi(t)|_N = 1 \quad \forall t \geq 0$ and therefore it evolves on the unit sphere of $\mathbb{C}^N$: $S = \{\mathbf{x} \in \mathbb{C}^N; |\mathbf{x}|_N = 1\}$. The control problem of this last equation is considered in [6] (for LaSalle invariance principle's based convergence analysis),[7] (for global practical stabilization of the eigenstates by explicit feedback laws) and [82] (for an overview of some properties of a quantum harmonic oscillator).

- When we consider the free Hamiltonian $H_0 = -\Delta + V$ one obtain the Schrödinger PDE for which one has an extensive study in many aspects, including the well-posedness [12], the exact controllability [79], the observability [87], and the stabilization or stability analysis by multiplier techniques and constructing energy functionals [80] or by backstepping approach via the boundary actuation and measurements [70].

3.1.2 Problem description

Let us consider $\Omega \subset \mathbb{R}^N$, an open bounded domain with smooth boundary $\partial\Omega$. For any $x_0 \in \mathbb{R}^N$, the set

$$\Gamma_0 = \{x \in \partial\Omega, (x - x_0) \cdot \nu(x) > 0\} \tag{3.1}$$

where $\nu(x)$ denotes the unit outward normal vector to Ω at $x \in \partial\Omega$ and $\cdot$ the scalar product in $\mathbb{R}^N$. This set is shown in Figure 3.1, originally taking from [117].

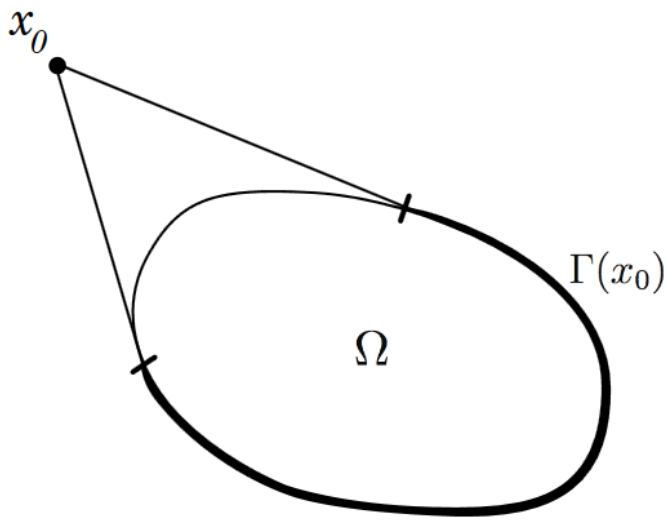

Figure 3.1: The set Γ_0 is an open part of the boundary $\partial\Omega$.

Let us consider the following closed-loop control system with source term in-domain $f(x,t) = -i\alpha(x)z(x,t)$:

$$\begin{cases} i\partial_t z(x,t) + \Delta z(x,t) = f(x,t) & (x,t) \in \Omega \times \mathbb{R}^+, \\ z(x,t) = 0 & (x,t) \in \partial\Omega \times \mathbb{R}^+ \\ z(x,0) = z_0(x) & x \in \Omega. \end{cases} \tag{3.2}$$

In system (3.2), the damping coefficient $\alpha \in L^\infty(\Omega;\mathbb{R})$ satisfies that there exist α_0 and $\alpha_1 \in \mathbb{R}+$ such that

$$\begin{cases} 0 < \alpha \leq \alpha_1 = \|\alpha\|_{L^\infty(\Omega)} & \text{a.e. in } \Omega \\ \exists \alpha_0 > 0 : \alpha \geq \alpha_0 & \text{a.e. in } \omega \subset \Omega, \end{cases} \tag{3.3}$$

where $\omega \subset \Omega$ is a neighborhood of $\overline{\Gamma}_0$ in Ω. This means that the damping will not necessarily act on the whole domain Ω, but at least over a geometrically constrained sub domain ω of Ω.

As in Chapter 2, dealing with the wave equation, a similar event-triggering mechanisms can be considered for the damped Schrödinger equation.

3.2 Results in continuous-in-time framework

Before working on system (3.2), let us recall basic results about the classical Schrödinger equation

$$\begin{cases} i\partial_t w(x,t) + \Delta w(x,t) = f(x,t) & (x,t) \in \Omega \times (0,\tau), \\ w(x,t) = 0 & (x,t) \in \partial\Omega \times (0,\tau) \\ w(x,0) = w_0(x) & x \in \Omega. \end{cases} \tag{3.4}$$

For instance, it has been proved in [12, 80] the following theorems.

Theorem 3.1: Classical energy estimate

For any initial condition $w_0 \in L^2(\Omega)$ and source date $f \in L^1(0,\tau,L^2(\Omega))$, there exists a unique weak solution to (3.4) $w \in C^0(0,\tau;L^2(\Omega)) \cap C^1(0,\tau;(H^2(\Omega) \cap H_0^1(\Omega))')$ such that

$$\|w\|_{L^\infty(0,\tau;L^2(\Omega))} \leq \|w_0\| + \|f\|_{L^1(0,\tau;L^2(\Omega))}. \tag{3.5}$$

Proof : To demonstrate formally that the solution to the equation (3.10) satisfies the inequality (3.5), we will use a classical energy method. The key to this method is to multiply the original equation by an appropriate function, integrate over space-time, and apply Hölder's and Young's inequalities to obtain the desired estimate. Firstly, we multiply equation (3.4) by $\overline{w}$, where $\overline{w}$ is the complex conjugate of w, and integrate over Ω:

$$\int_\Omega \overline{w}(t)(i\partial_t w(t) + \Delta w(t)) = \int_\Omega \overline{w}(t)\, f(t).$$

Integration by parts, leads to

$$i\int_\Omega \overline{w}\partial_t w\, dx = \int_\Omega \overline{\nabla w} \cdot \nabla w\, dx + \int_\Omega \overline{w}\, f\, dx.$$

Taking the imaginary part of this equation and using $\mathcal{I}m(iZ) = -\mathcal{R}e(Z)$, $\forall Z \in \mathbb{C}$, we find:

$$-\int_\Omega \mathcal{R}e(\overline{w}(t)\partial_t w(t)) = \int_\Omega \mathcal{I}m(\overline{w}(t) f(t)).$$

Since

$$\frac{d}{dt}\|w(t)\|^2 = 2\mathcal{R}e\left(\int_\Omega \overline{w}(t)\partial_t w(t)\right)$$

then we obtain thanks to the Cauchy-Schwarz's inequality:

$$\begin{aligned}\frac{d}{dt}\left(\|w(t)\|^2\right) &= -2\int_\Omega \mathrm{Im}(\overline{w}(t)f(t)) \\ &\leq 2\|w(t)\|\|f(t)\| \\ \frac{\frac{d}{dt}\|w(t)\|^2}{2\|w(t)\|} = \frac{d}{dt}\|w(t)\| &\leq \|f(t)\|.\end{aligned}$$

Henceforth, for all $t \in (0,\tau)$, by integrating on $(0,t)$ we obtain

$$\|w(t)\| \leq \|w(0)\| + \int_0^\tau \|f(s)\|\,ds = \|w_0\| + \|f\|_{L^1(0,\tau,L^2(\Omega))}.$$

♢

Moreover, the following observability inequality is investigated in [87].

Lemma 3.2: Observability inequality[87]

Let $\tau > 0$ be given and $\omega \subset \Omega$ be the neighborhood of $\overline{\Gamma}_0 \subset \partial\Omega$ with γ_0 defined in (3.1). There exists $C_{obs} > 0$ such that the solution to the system 3.4 with $f = 0$ satisfies

$$\|w(0)\|^2 \leq C_{obs}\int_0^\tau \int_\omega |w(x,t)|^2 dxdt. \tag{3.6}$$

The proof of this lemma is very important in the proof of local (and exact) controllability results. We refer to [87, 79] where the multiplier technique together with the Holmgren's Uniqueness Theorem (or an interpolation inequality) are used to obtain this inequality.

Now, back to Equation (3.2), the well-posedness and exponential stability are already documented in the literature [12, 87, 79, 80].

Theorem 3.3: Well-posedness ([12])

For any initial conditions $z_0 \in L^2(\Omega)$, there exists a unique weak solution to (3.2) satisfying

$$z \in C^0(\mathbb{R}^+; L^2(\Omega)) \cap C^1([0,T]; (H^2(\Omega)\cap H_0^1(\Omega))'). \tag{3.7}$$

Moreover, for any initial data $z_0 \in H^2(\Omega)\cap H_0^1(\Omega)$, the unique strong solution to (3.2) satisfies

$$z \in C^0(\mathbb{R}^+; H^2(\Omega)\cap H_0^1(\Omega)) \cap C^1(\mathbb{R}^+; L^2(\Omega)). \tag{3.8}$$

As in chapter 1,Theorem 3.2 is proven by using the Hilde-Yossida Theorem.

Theorem 3.4: Stability [80]

Let $\Omega \subset \mathbb{R}^N$ be an open bounded domain with boundary of class C^3. Let $x_0 \in \mathbb{R}^N$ and $\omega \subset \Omega$ a neighborhood of $\bar{\Gamma}_0$ in Ω. Assume that $\alpha \in L^\infty(\Omega)$ satisfies (3.3). For any initial condition in $L^2(\Omega)$, there exist $C_{cont} > 0$ and $\delta > 0$ such that the weak solution z to (3.2) verifies for all $t > 0$

$$E(t) := \frac{1}{2}\|z(t)\|^2 \leq C_{cont} E(0) e^{-2\delta t}. \tag{3.9}$$

Idea of the proof of Theorem 3.4: We give here a sketch of the proof of Theorem 3.4 because it will pave the way for the more intricate setting with the event-triggering mechanism later. Indeed, the proof relies on some energy estimate which can be obtained thanks to the observability inequality (3.6) and the classical estimate (3.5) on the solution to the Schrödinger equation.

Let us set the solution $z = y + \phi$ to the system (3.2) as the sum of two variables $y = y(x,t)$ and $\phi = \phi(x,t)$ which satisfy

$$\begin{cases} i\partial_t y + \Delta y = -i\alpha z & \text{in } \mathbb{R}^+, \\ y = 0 & \text{on } \partial\Omega \times \mathbb{R}^+, \\ y(\cdot, 0) = 0 & \text{in } \Omega, \end{cases} \tag{3.10}$$

and

$$\begin{cases} i\partial_t \phi + \Delta \phi = 0 & \text{in } \Omega \times \mathbb{R}^+, \\ \phi = 0 & \text{on } \partial\Omega \times \mathbb{R}^+, \\ \phi(\cdot, 0) = z_0 & \text{in } \Omega. \end{cases} \tag{3.11}$$

As proved, for example, in [79, 87] and recalled in Lemma 3.2, the solution to system (3.11) with $w \equiv \phi$ satisfies the observability inequality: for all $\omega \subset \Omega$ neighborhood of $\overline{\Gamma}_0 \subset \partial\Omega$ and $\tau > 0$, here exists $C_{obs} > 0$ such that the solution to (3.11) satisfies

$$\|z_0\|^2 = \|\phi(0)\|^2 \leq C_{obs} \int_0^\tau \int_\omega |\phi(x,t)|^2 dx dt. \tag{3.12}$$

Moreover, the classical estimate (3.5) holds for $w \equiv y$ and $w_0 \equiv 0$: Any solution y to the system (3.10) satisfies:

$$\|y\|_{L^\infty(0,\tau;L^2(\Omega))} \leq \|\alpha z\|_{L^1(0,\tau;L^2(\Omega))}. \tag{3.13}$$

We will also need to prove that the following energy estimate holds.

Lemma 3.5: Energy estimate [80]

There exist[a] a time $\tau > 0$ and a constant $C_\tau > 0$ such that

$$E(\tau) \leq C_\tau \int_0^\tau \int_\Omega \alpha(x) |z(x,t)|^2 \, dx \, dt \tag{3.14}$$

for every solution to (3.10) with initial data $z_0 \in L^2(\Omega)$ and α satisfying (3.3)

[a] In fact, we will prove that (3.14) holds for any $\tau > 0$ and some constant $C_\tau = C(\tau) > 0$.

Remark 3.1 *This critical Lemma is the cornerstone of the proof in [80] since it bounds the energy E at time $\tau > 0$ by the integral of something which can vanish outside $\omega \subset\subset \Omega$.*

As we will be employing the concept outlined in the proof of Lemma 3.5 within the context of event-triggered control, let us proceed to furnish the proof for this lemma.

Proof : The time-derivative of $E(t)$ along the trajectories of system (3.2) is given by

$$\begin{aligned}\dot{E}(t) &= \mathcal{R}e\left(\int_\Omega \bar{z}(t)\partial_t z(t)\right)\\ &= \mathcal{R}e\int_\Omega \left(i\bar{z}(t)\Delta z(t) - i\alpha(x)|z(t)|^2\right).\end{aligned}$$

By the Green's formula (Lemma A.3 in Appendix) with $z = 0$ on $\partial\Omega$, and since α takes its values in $\mathbb{R}$,

$$\dot{E}(t) = -\int_\Omega \alpha(x)|z(t)|^2 \leq 0. \tag{3.15}$$

Using this non-increasing character of the energy $E(\cdot)$, we get

$$E(\tau) \leq E(0) = \frac{1}{2}\|z_0\|^2. \tag{3.16}$$

Recalling that $z = y + \phi$, combining (3.16) and the observability inequality (3.12) and using the condition on the damping coefficient α (3.3) and the fact that for any $a, b \in \mathbb{R},\ |a-b|^2 \leq 2(a^2+b^2)$, we obtain

$$\begin{aligned}E(\tau) &\leq \frac{C_{obs}}{2}\int_0^\tau\int_\omega |\phi(x,t)|^2\,dx\,dt,\\ &\leq C_{obs}\int_0^\tau\int_\omega |z(t)|^2\,dt + C_{obs}\int_0^\tau\int_\omega |y(t)|^2\,dt,\\ &\leq \frac{C_{obs}}{\alpha_0}\int_0^\tau\int_\Omega \alpha(x)|z(t)|^2\,dt + C_{obs}\int_0^\tau\int_\Omega |y(t)|^2\,dt,\\ &\leq \frac{C_{obs}}{\alpha_0}\int_0^\tau\int_\Omega \alpha(x)|z|^2\,dt + C_{obs}\|y(t)\|^2_{L^\infty(0,\tau;L^2(\Omega))}.\end{aligned}$$

Using the classical estimate (3.13), the Cauchy-Schwarz's inequality and (3.3), we have

$$\begin{aligned}\|y\|^2_{L^\infty(0,\tau;L^2(\Omega))} &\leq \|\alpha z\|^2_{L^1(0,\tau;L^2(\Omega))}\\ &\leq \left(\int_0^\tau\left(\int_\Omega |\alpha(x)z(x,t)|^2\,dx\right)^{1/2}dt\right)^2,\\ &\leq \|\alpha\|_\infty\tau\int_0^\tau\int_\Omega \alpha(x)|z(x,t)|^2\,dxdt,\\ &\leq \alpha_1\tau\int_0^\tau\int_\Omega \alpha(x)|z(x,t)|^2\,dxdt.\end{aligned}$$

Therefore,

$$\begin{aligned} E(\tau) &\leq \frac{C_{obs}}{\alpha_0}\int_0^\tau\int_\Omega \alpha(x)|z(x,t)|^2\,dx\,dt + \alpha_1\tau C_{obs}\int_0^\tau\int_\Omega |\alpha(x)|z(x,t)|^2 dx\,dt \\ &\leq C_\tau\int_0^\tau\int_\Omega \alpha(x)|z(x,t)|^2\,dx\,dt, \end{aligned}$$

with $C_\tau = (1/\alpha_0 + \alpha_1\tau)\,C_{obs}$. ♢

The proof of Theorem 3.4 comes from Lemma 3.5, the identity (3.15) and the invariance by translation property of the Schrödinger equation.

Indeed, integrating (3.15) on $[0,\tau]$, we obtain:

$$E(\tau) - E(0) \leq -\int_0^\tau\int_\Omega \alpha(x)|z(x,t)|^2 dxdt. \tag{3.17}$$

We can rewrite (3.14) as follows

$$-\int_0^\tau\int_\Omega \alpha(x)|z(t)|^2 dxdt \leq -\frac{1}{C_\tau}E(\tau),$$

Combining this last inequality with (3.17), we get

$$E(\tau) \leq aE(0)$$

with

$$a = \frac{C_\tau}{1+C_\tau}.$$

Next, we use the fact that the linear Schrödinger equation is invariant by translation in time, and this argument applies on the interval $[(n-1)\tau, n\tau]$, for $n = 1, 2, \ldots,$ yields:

$$E(n\tau) \leq aE((n-1)\tau) \leq \cdots \leq a^n E(0) = e^{-n\tau\delta}E(0),$$

where we set $a^n = \exp\left(-n\tau\frac{1}{\tau}\ln\left(\frac{1}{a}\right)\right)$ and $\delta = \frac{1}{\tau}\ln\left(\frac{1}{a}\right) = \frac{1}{\tau}\ln\left(1+\frac{1}{C_\tau}\right) > 0$.

It is therefore possible to have τ very small.

Now, for every positive time t, there exists $n \in \mathbb{N}^*$ such that $(n-1)\tau < t \leq n\tau$. Using (3.16) and integration on $[(n-1)\tau, t]$ we have:

$$\begin{aligned} E(t) &\leq E((n-1)\tau) \\ &\leq e^{-(n-1)\tau\delta}E(0) \\ &\leq e^{-n\tau\delta}e^{\tau\delta}E(0). \end{aligned}$$

Since $e^{-n\tau\delta} \leq e^{-\delta t}$ for $t \leq n\tau$, and $e^{\tau\delta} = 1/a$, we get

$$E(t) \leq \frac{1}{a}e^{-\delta t}E(0)$$

so that (3.9) holds for $C_{cont} = \frac{1}{a} = 1 + \frac{1}{C_\tau}$ and $\delta = \frac{1}{\tau}\ln\left(1+\frac{1}{C_\tau}\right)$.

3.3 Static event-based control for the damped Schrödinger equation

In this section, based on our paper [67], considering a possibly locally damped Schrödinger equation, we design an event-triggering update mechanism for the damping, aiming at maintaining the exponential stability of the closed-loop system.

3.3.1 Definition of the static event-triggering mechanism

We are interested by the implementation of the control term $u = -i\alpha z$, so that the control signal applied to the plant is updated only at certain instants $\{t_k\}_{k\in\mathbb{N}}$, defined by an event-triggering law. We assume that the control action is held constant between two successive updates. Furthermore, differently from classical periodic sampling techniques, the inter-sampling time $t_{k+1}-t_k$ is not assumed to be constant. The closed-loop system can then be described for all $t \in [t_k, t_{k+1})$ as follows[1]:

$$\begin{cases} i\partial_t z + \Delta z = -i\alpha z(t_k), & \text{in } \Omega \times [t_k, t_{k+1}), k \in \mathbb{N} \\ z = 0, & \text{on } \partial\Omega \times \mathbb{R}^+, \\ z(\cdot, 0) = z_0 & \text{in } \Omega \end{cases} \tag{3.18}$$

where $0 = t_0 < t_1 < \cdots < t_k < t_{k+1}$.

Therefore, we can summarize the problem we intend to solve as the one of designing a simple triggering condition in order to guarantee i) the well-posedness of the closed-loop system (3.18), ii) the avoidance of any Zeno behavior and iii) the exponential stability of the closed loop.

In order to expand the event-triggering strategy developed in the context of finite-dimensional systems as for example in [111, 88, 41], let us introduce the following deviation between the last sampled state and the current one $\forall x \in \Omega$ and $t \in [t_k, t_{k+1})$:

$$e_k(x,t) = z(x,t) - z(x,t_k). \tag{3.19}$$

In the sequel, we use the shortcut notation $e_k(t)$ or e_k. Therefore, we can characterize the event-triggering law as:

$$t_{k+1} = \inf\left\{t \geq t_k \text{ such that } \|e_k(t)\|^2 > \gamma\|z(t)\|^2\right\} \tag{3.20}$$

where $\gamma > 0$ is a design parameter. In other words, as soon as the deviation term gets larger than a $2\gamma-$proportion of the energy since $E(t) = \frac{1}{2}\|z(t)\|^2$, an update event is generated. This yields:

$$\|e_k(t)\|^2 \leq \gamma\|z(t)\|^2, \qquad t \in [t_k, t_{k+1}). \tag{3.21}$$

This triggering law corresponds to the static event-triggering mechanism defined in (1.4). In the following we split the study into three steps.

[1]The dependence in x and t is omitted to simplify.

3.3.2 Well-posedness of the closed-loop system

As in the previous chapter, the maximal time T^* under which the system (3.18) subjected to the event-triggering law (3.20) has a solution is defined by

$$\begin{cases} T^* = +\infty & \text{if } (t_k) \text{ is a finite sequence,} \\ T^* = \limsup_{k\to+\infty} t_k & \text{if not.} \end{cases} \tag{3.22}$$

The absence of Zeno behavior will actually be stemming from the proof that $T^* = +\infty$ since no accumulation point of the sequence $(t_k)_{k\geq 0}$ will therefore be possible.

Leveraging on some regularity of the classical solutions to the Schrödinger equation we state the following first result:

Theorem 3.6: Well-posedness

Let Ω be an open bounded domain of class C^2. For any initial conditions $z_0 \in H^2(\Omega)\cap H_0^1(\Omega)$, there exists a unique strong solution to (3.18) under the event-triggering mechanism (3.20), satisfying

$$z \in C^0([0,T^*);H^2(\Omega)\cap H_0^1(\Omega))\cap C^1([0,T^*);H_0^1(\Omega)). \tag{3.23}$$

Proof : The well-posedness on every sampled interval $[t_k, t_{k+1}]$ is proven by induction.

• **Initialization.** On the first time interval $[0,t_1]$, the control system (3.18) reads simply

$$\begin{cases} i\partial_t z + \Delta z = -i\alpha(x)z_0, & \text{in } \Omega\times(0,t_1), \\ z = 0 & \text{on } \partial\Omega\times(0,t_1), \\ z(0) = z_0, & \text{in } \Omega. \end{cases} \tag{3.24}$$

This is a Schrödinger equation with initial data $z_0 \in H^2(\Omega)\times H_0^1(\Omega)$ compatible with Dirichlet boundary data and source term $f(x,t) = -i\alpha(x)z_0(x)$. Since $z_0 \in H_0^1(\Omega)$ and $\alpha \in L^\infty(\Omega)$,

$$f \in L^1(0,t_1;H^2(\Omega)\times H_0^1(\Omega)).$$

Then, Theorem 3.8 allows to deduce that there exists a unique solution satisfying

$$z \in C([0,t_1];H^2(\Omega)\cap H_0^1(\Omega))\cap C^1([0,t_1];H_0^1(\Omega)).$$

• **Heredity.** Let us bring to the forefront that this solution is continuous-in-time and satisfies $z(t_1) \in H^2(\Omega)\cap H_0^1(\Omega)$ so that system (3.18) considered on the next time interval $[t_1,t_2)$ has an initial condition in $H^2(\Omega)\cap H_0^1(\Omega)$ compatible with the homogeneous Dirichlet boundary data and a source term $i\alpha z(t_1) \in L^1(t_1,t_2;H^2(\Omega)\cap H_0^1(\Omega))$.

Hence, the same reasoning holds again and the heredity is proved similarly at any step $k \in \mathbb{N}$.

• **Conclusion.** By induction, the following regularity holds for any $k \in \mathbb{N}$,

$$z \in C^0([t_k,t_{k+1}];H^2(\Omega)\cap H_0^1(\Omega))\cap C^1([t_k,t_{k+1}];H_0^1(\Omega)).$$

Therefore, from the extension by continuity at the update instants t_k, one can conclude that (3.18) has a unique solution in the class (3.23). ♢

3.3.3 Avoidance of Zeno behavior

In event-triggering mechanism framework the Zeno free phenomenon is challenging particularly when the static and dynamical algorithms are used. For instance, in [33, Definition 2], [28, Definition 3], [5] the combination of the absolute and the relative threshold allowed the authors to ease the proof of the avoidance of the Zeno phenomenon. When the event-triggering law is built on the comparison between an error term (the difference of the state value at the last triggering instant and the current one) and a proportion of the energy, it was usually added a term exponentially decreasing and depending on the initial condition as in [5, 33, 55]. Some recent exceptions to these approaches is detailed in [66] for the wave equation presented in chapter 2. The current chapter deals with Schrödinger equation and follows the same route in order to prove the absence of Zeno phenomenon without any extra exponential term in the event-triggering law. Hence, using an observability inequality for the linear Schrödinger equation, the exponential stability of the closed-loop system under state-based event-triggered control is established. Furthermore, following the same reasoning as in [66] the avoidance of Zeno behavior is guaranteed thanks to the absence of accumulation points in the sequence of time updates.

In this section, the proof of the absence of Zeno behavior, based on the proof that the maximal time of existence of a solution to the closed-loop system can only be $T^* = +\infty$ is developed.

Let us show that the natural energy of the closed-loop system, defined in (3.9), has a useful property stated in the following lemma.

Lemma 3.7: Energy boundedness

Under the event-triggering law (3.20), for all $t \in (0, T^*)$, the solution to system (3.18) has an energy

$$E(t) = \frac{1}{2}\|z(t)\|^2$$

that satisfies:

$$E(0)e^{-2Ct} \leq E(t) \leq E(0)e^{2Ct} \tag{3.25}$$

with $C = \alpha_1(1 + \sqrt{\gamma}) > 0$.

Proof : Using (3.19), the closed-loop system (3.18) can be re-written as:

$$\begin{cases} i\partial_t z + \Delta z = -i\alpha z + i\alpha e_k, & \text{in } \Omega \times [t_k, t_{k+1}), \\ z = 0, & \text{on } \partial\Omega \times \mathbb{R}^+, \\ z(\cdot, 0) = z_0, & \text{in } \Omega. \end{cases} \tag{3.26}$$

The time-derivative of $E(t)$ along the trajectories of system (3.26) is given by

$$\begin{aligned} \dot{E}(t) &= Re\left(\int_\Omega \bar{z}(t)\partial_t z(t)\right) = Im\left(\int_\Omega i\bar{z}(t)\partial_t z(t)\right) \\ &= -Im\int_\Omega \left(\bar{z}(t)\Delta z(t) + i\alpha(x)|z(t)|^2 + i\alpha(x)e_k(t)\bar{z}(t)\right). \end{aligned}$$

By the Green's formula (Lemma A.3 in Appendix) with $z = 0$ on $\partial\Omega$, and since α

takes its values in $\mathbb{R}$,

$$\dot{E}(t) = -\int_\Omega \alpha(x)|z(t)|^2 + Re\left(\int_\Omega \alpha(x)\bar{e}_k(t)z(t)\right). \tag{3.27}$$

Recalling that $2E(t) = \|z(t)\|^2$ and using the inequality (3.21) which, is a consequence of the definition of the event-triggering law, one can calculate from (3.27) using the Cauchy Schwarz's inequality (see Lemma A.1 in Appendix):

$$\begin{aligned} |\dot{E}(t)| &\leq \alpha_1\|z(t)\|^2 + \alpha_1\|e_k(t)\|\|z(t)\| \\ &\leq 2\alpha_1 E(t) + 2\alpha_1\sqrt{\gamma}E(t) \\ |\dot{E}(t)| &\leq 2CE(t) \end{aligned} \tag{3.28}$$

with

$$C = \alpha_1(1+\sqrt{\gamma}). \tag{3.29}$$

This shows that $-2CE(t) \leq \dot{E}(t) \leq 2CE(t)$. By Gronwall's Lemma on $[t_k, t]$, the second inequality gives

$$E(t) \leq E(t_k)\exp\left(\int_{t_k}^t 2Cdu\right), \forall t \geq t_k,$$

that is $E(t) \leq E(t_k)e^{2C(t-t_k)}$. By applying also Gronwall's Lemma to the first inequality one gets: $E(t) \geq E(t_k)e^{-2C(t-t_k)}$. Hence, for any $t \in [t_k, t_{k+1})$

$$E(t_k)e^{-2C(t-t_k)} \leq E(t) \leq E(t_k)e^{2C(t-t_k)}. \tag{3.30}$$

Then taking $t = t_{k+1}$, inequality (3.30) becomes :

$$E(t_k)e^{-2C(t_{k+1}-t_k)} \leq E(t_{k+1}) \leq E(t_k)e^{2C(t_{k+1}-t_k)}.$$

Inferring (3.30) for $E(t_k)$ allows to deduce:

$$E(t_{k-1})e^{-2C(t_{k+1}-t_{k-1})} \leq E(t_{k+1}) \leq E(t_{k-1})e^{2C(t_{k+1}-t_{k-1})}.$$

Since $t_0 = 0$, by induction we get:

$$E(0)e^{-2Ct_{k+1}} \leq E(t_{k+1}) \leq E(0)e^{2Ct_{k+1}}.$$

Then inequality (3.30) yields:

$$E(0)e^{-2Ct_k}e^{-2C(t-t_k)} \leq E(t) \leq E(0)e^{2Ct_k}e^{2C(t-t_k)},$$

showing that (3.25) holds for all $t \in \mathbb{R}^+$. ♢

We will also require the following lemma, which provides an upper bound for the term $|\Delta z(t)|$. This lemma is analogous to Lemma 2.7 in Chapter 2.

Lemma 3.8: Intermediate result

For any $z_0 \in H^2(\Omega) \cap H_0^1(\Omega)$, the closed-loop system (3.18) with (3.20) has a unique solution satisfying $z \in C([0, T^*); H^2(\Omega) \cap H_0^1(\Omega))$, then there exists a constant $C_\Delta > 0$ such that $\forall t \in [0, T^*)$

$$\|\Delta z(t)\| \leq C_\Delta = C\|z_0\|^2_{H^2(\Omega)\cap H_0^1(\Omega)}. \tag{3.31}$$

Proof : Combining Lemma 3.7 and Theorem 3.6, for all solution to the system

$$\begin{cases} i\partial_t z + \Delta z = -i\alpha(x)z(t_k), \\ z(0) = z_0, \end{cases} \tag{3.32}$$

we obtain

$$\|z(t)\|^2 \leq C\|z_0\|^2 \quad \text{with } C = e^{2\alpha_1(1+\sqrt{\gamma})T^*}. \tag{3.33}$$

Let us set $w = \partial_t z$ so that w verifies

$$\begin{cases} i\partial_t w + \Delta w = 0 \\ w(0) = i\Delta z_0 - \alpha z_0. \end{cases} \tag{3.34}$$

Therefore, since $z_0 \in H^2 \cap H_0^1(\Omega)$ we have:

$$\begin{aligned} \|\partial_t z(t)\|^2 &= \|w(t)\|^2 \\ &\leq C\|z_0\|^2_{H^2(\Omega)\cap H_0^1(\Omega)} \end{aligned}$$

This leads to

$$\begin{aligned} \|\Delta z(t)\|^2 &= \| - i\partial_t z(t) - i\alpha z(t_k)\|^2 \\ &= C\|z_0\|^2_{H^2(\Omega)\cap H_0^1(\Omega)} = C_\Delta. \end{aligned}$$

♢

We can now present the main result of this section regarding the strong non-zenoness. This theorem is similar to Theorem 2.8.

Theorem 3.9: Zeno free

There is no Zeno phenomenon for the system (3.18) under the event-triggering mechanism (3.20).
Equivalently, the maximal time defined by (3.22) is $T^* = +\infty$.

Proof : Following the same reasoning as in [111, 41], the proof is based on the study of the function φ defined on $[t_k, t_{k+1})$ by the ratio

$$\varphi : t \mapsto \varphi(t) = \frac{\|e_k(t)\|^2}{2\gamma E(t)}.$$

φ is a non negative function that satisfies, $\forall k \in \mathbb{N}$, $\varphi(t_k^+) = 0$ and jumps from $\varphi(t_{k+1}^-) = 1$ to $\varphi(t_{k+1}^+) = 0$, where $\varphi(t_{k+1}^-)$ is the value of φ before the update in time t_k $\varphi(t_{k+1}^+)$ is the one after the update in time t_{k+1}. Of course, we need to assume that $E(t) \neq 0 \quad \forall t \in [t_k, t_{k+1})$, recalling that $E(t) = 0$ would mean stopping the updates since, then, E remains null. Let us study the time-derivative of φ :

$$\dot{\varphi}(t) = \frac{\mathcal{R}e\left(\int_\Omega \partial_t e_k(t)\bar{e}_k(t)\right)}{\gamma E(t)} - \frac{\dot{E}(t)\|e_k(t)\|^2}{2\gamma\left(E(t)\right)^2}. \tag{3.35}$$

We have from (3.19), $\partial_t e_k = \partial_t z$ a.e. in Ω, and using equation (3.26) and the Cauchy Schwarz's inequality we get, $\forall t \in [t_k, t_{k+1})$,

$$\begin{aligned}\mathcal{R}e\left(\int_\Omega \partial_t e_k(t)\bar{e}_k(t)\right) &= Im\left(\int_\Omega \Delta z(t)\bar{e}_k(t)\right) - \mathcal{R}e\left(\int_\Omega \alpha z(t)\bar{e}_k(t) + \alpha|e_k(t)|^2\right)\\ &\leq \|e_k(t)\|\|\Delta z(t)\| + \alpha_1\|e_k(t)\|\|z(t)\| + \alpha_1\|e_k(t)\|^2.\end{aligned}$$

Since for any $z_0 \in H^2 \cap H_0^1(\Omega)$, the closed-loop system (3.26) under the event-triggering mechanism (3.20) has a unique solution $z \in C^0([0,T^*); H^2(\Omega) \cap H_0^1(\Omega))$, then from Lemma 3.8, there exists a constant $C_\Delta > 0$ such that $\forall t \in [0, T^*)$,

$$\|\Delta z(t)\| \leq \|\Delta z\|_{L^\infty(0,T;L^2(\Omega))} \leq C_\Delta. \tag{3.36}$$

Then using $\|z(t)\|^2 = 2E(t)$ and (3.21) it follows :

$$\begin{aligned}\frac{\mathcal{R}e\left(\int_\Omega \partial_t e_k(t)\bar{e}_k(t)dx\right)}{\gamma E(t)} &\leq \frac{C_\Delta\sqrt{2\gamma E(t)}}{\gamma E(t)} + \frac{\alpha_1\sqrt{2\gamma E(t)}\sqrt{2E(t)}}{\gamma E(t)} + 2\alpha_1\varphi(t)\\ &\leq \frac{C_\Delta\sqrt{2}}{\sqrt{\gamma E(t)}} + \frac{2\alpha_1}{\sqrt{\gamma}} + 2\alpha_1\varphi(t).\end{aligned} \tag{3.37}$$

On the other hand, using (3.28) we get:

$$\frac{-\dot{E}(t)\|e_k(t)\|^2}{2\gamma\left(E(t)\right)^2} \leq 2\alpha_1(1+\sqrt{\gamma})\varphi(t). \tag{3.38}$$

Gathering the terms (3.37) and (3.38) we have:

$$\dot{\varphi}(t) \leq \frac{C_\Delta\sqrt{2}}{\sqrt{\gamma E(t)}} + \frac{2\alpha_1}{\sqrt{\gamma}} + 2\alpha_1(2+\sqrt{\gamma})\varphi(t).$$

Since $\varphi(t) \leq 1$ from the event-triggering law, it follows

$$\dot{\varphi}(t) \leq \frac{C_\Delta\sqrt{2}}{\sqrt{\gamma E(t)}} + \frac{2\alpha_1}{\sqrt{\gamma}} + 2\alpha_1(2+\sqrt{\gamma}),$$

or equivalently,

$$\dot{\varphi}(t) \leq A + \frac{B}{\sqrt{E(t)}}$$

with $A = \dfrac{2\alpha_1}{\sqrt{\gamma}} + 2\alpha_1(2+\sqrt{\gamma})$ and $B = C_\Delta\sqrt{\dfrac{2}{\gamma}}$.

Using Lemma 3.7 one has $\forall t \in [0,T^*], E(t) \geq E(0)e^{-2Ct} \geq E(0)e^{-2CT^*}$, and then $\dot{\varphi}(t) \leq A + \frac{Be^{CT^*}}{\sqrt{E(0)}}$. Therefore, $\forall k \in \mathbb{N}$, integrating on $[t_k, t_{k+1}]$ knowing that $\varphi(t_k) = 0$ and $\lim\limits_{k \to t_{k+1}} \varphi(t) = 1$ we obtain:

$$1 \leq \left[A + \frac{Be^{CT^*}}{\sqrt{E(0)}}\right](t_{k+1} - t_k). \tag{3.39}$$

Now let $t_k \to T^*$ as $k \to +\infty$ in (3.39), then we get a contradiction if $T^* \neq +\infty$. We therefore get $T^* = +\infty$ leading to the absence of any accumulation points. Hence, the avoidance of Zeno behavior is guaranteed. ♢

3.3.4 Exponential stability

Let us now propose sufficient conditions to ensure the exponential stability of system (3.18)-(3.20).

Inspired by the energy estimate (3.14) concerning the continuous setting, we start with the following Lemma for our event-triggered setting.

Lemma 3.10: Energy estimate

Consider the solution z to system (3.18) with α satisfying (3.3). For any $\tau > 0$ there exist some constants $K_1, K_2 > 0$ such that

$$E(\tau) \leq K_1 \int_0^\tau \int_\Omega \alpha(x)|z(t)|^2 dxdt + K_2 \int_0^\tau E(t)dt. \tag{3.40}$$

Proof : Let $\tau > 0$ and let us recall that the time-derivative of $E(t)$ is

$$\dot{E}(t) = -\int_\Omega \alpha(x)|z(t)|^2 + Re\left(\int_\Omega \alpha(x)\bar{e}_k(t)z(t)\right).$$

From the condition (3.3) on function α, we get

$$\dot{E}(t) \leq \alpha_1 \|e_k(t)\|\|z(t)\|.$$

Then, using (3.21) we get

$$\dot{E}(t) \leq \alpha_1\sqrt{\gamma}E(t).$$

Integrating this relation on we get:

$$E(\tau) \leq E(0) + 2\alpha_1\sqrt{\gamma}\int_0^\tau E(t)dt. \tag{3.41}$$

Inspired by the proof of Theorem 3.4, we introduce the variables y and ϕ such that $z = y + \phi$ where z is solution to (3.26) and $y = y(x,t)$ and $\phi = \phi(x,t)$ are solution to the following systems

$$\begin{cases} i\partial_t y + \Delta y = -i\alpha z + i\alpha e_k & \text{in } \Omega \times [t_k, t_{k+1}), \\ y = 0 & \text{on } \partial\Omega \times \mathbb{R}^+, \\ y(\cdot, 0) = 0 & \text{in } \Omega, \end{cases} \tag{3.42}$$

and

$$\begin{cases} i\partial_t \phi + \Delta\phi = 0 & \text{in } \Omega \times \mathbb{R}^+, \\ \phi = 0 & \text{on } \partial\Omega \times \mathbb{R}^+, \\ \phi(\cdot, 0) = z_0 & \text{in } \Omega. \end{cases} \tag{3.43}$$

Furthermore, for system (3.43) the observability inequality given Lemma 3.2 holds for $w \equiv \phi$ and $f \equiv 0$.

Hence, from (3.41), inequality (3.21), and the fact that $\phi = z - y$ and that for any $a, b \in \mathbb{R}$, $|a-b|^2 \leq 2(a^2+b^2)$, we have:

$$\begin{aligned}
E(\tau) &\leq \frac{1}{2}\|z_0\|^2 + 2\alpha_1\sqrt{\gamma}\int_0^\tau E(t)dt \\
&= \frac{1}{2}\|\phi(0)\|^2 + 2\alpha_1\sqrt{\gamma}\int_0^\tau E(t)dt \\
&\leq \frac{C_{obs}}{2\alpha_0}\int_0^\tau\int_\omega \alpha(x)|\phi(x,t)|^2dxdt + 2\alpha_1\sqrt{\gamma}\int_0^\tau E(t)dt \\
&\leq \frac{C_{obs}}{\alpha_0}\int_0^\tau\int_\Omega \alpha(x)|z(t)|^2dt + \frac{C_{obs}\alpha_1}{\alpha_0}\|y\|^2_{L^\infty(0,\tau;L^2(\omega))} + 2\alpha_1\sqrt{\gamma}\int_0^\tau E(t)dt.
\end{aligned}$$

Using classical energy estimate (3.13), on the Schrödinger equation (3.42), with source term $-i\alpha z + i\alpha e_k$, we have

$$\begin{aligned}
\|y\|^2_{L^\infty(0,\tau;L^2(\omega))} &\leq \|\alpha(e_k - z)\|^2_{L^1(0,\tau;L^2(\Omega))} \\
&\leq 2\tau\alpha_1^2\int_0^\tau \|e_k(t)\|^2dt + 2\tau\alpha_1\int_0^\tau\int_\Omega \alpha(x)|z(t)|^2dt.
\end{aligned}$$

From the event-triggering mechanism, one has (3.21), so that

$$\|y\|^2_{L^2(0,\tau;L^2(\omega))} \leq 4\tau\alpha_1^2\gamma\int_0^\tau E(t)dt + 2\tau\alpha_1\int_0^\tau\int_\Omega \alpha(x)|z(t)|^2dt.$$

Hence,

$$E(\tau) \leq \left(\frac{C_{obs}}{\alpha_0} + \frac{2\tau C_{obs}\alpha_1^2}{\alpha_0}\right)\int_0^\tau\int_\Omega \alpha(x)|z(t)|^2dt + \left(2\alpha_1\sqrt{\gamma} + \frac{4\tau C_{obs}\alpha_1^3\gamma}{\alpha_0}\right)\int_0^\tau E(t)dt.$$

Therefore we get inequality (3.40) with

$$K_1 = \frac{C_{obs}}{\alpha_0}\left(1 + 2\tau\alpha_1^2\right); K_2 = 2\alpha_1\sqrt{\gamma} + 4\tau C_{obs}\alpha_1^3\gamma\alpha_0^{-1}. \tag{3.44}$$

♢

Then we can state and prove the following main exponential stability result.

Theorem 3.11: Exponentially stability

There exists $\gamma_0 > 0$ such that for all $\gamma \in (0, \gamma_0)$, for any initial condition $z_0 \in H^2(\Omega) \cap H_0^1(\Omega)$, the closed-loop system (3.18) under the event-triggering mechanism (3.20) is exponentially stable with decay rate $\delta > 0$. In other words, there exists $K > 0$ such that

$$E(t) \leq KE(0)e^{-2\delta t}, \quad \forall t > 0. \tag{3.45}$$

Proof : In the sake of clarity, we will first prove this result in the simpler situation where $\omega = \Omega$ before detailing the general case where one only knows that α may vanish away from the Γ_0 boundary.

- *Globally non-vanishing damping.*

When the damping function does not vanish in Ω, one sets $\omega = \Omega$. In that case, one obtains from (3.27), inequality (3.21) on the damping, the Cauchy-Schwarz's inequality and the event-triggering law (3.20) that

$$\dot{E}(t) \leq (-2\alpha_0 + 2\alpha_1\sqrt{\gamma})\, E(t).$$

Thus, recalling that $\alpha_0 < \alpha_1$ and choosing γ small enough, there exists

$$\delta = \alpha_0 - \alpha_1\sqrt{\gamma} > 0$$

such that

$$\dot{E}(t) \leq -2\delta E(t)$$

and (3.45) holds.

• *Locally non-vanishing damping.*

In the general case, the damping may vanish outside ω and we will need to use Lemma 3.10. Integrating (3.27) on $[0, \tau]$, we obtain:

$$E(\tau) - E(0) \leq 2\alpha_1\sqrt{\gamma}\int_0^\tau E(t)dt - \int_0^\tau\int_\Omega \alpha(x)|z(t)|^2 dt. \tag{3.46}$$

We can rewrite (3.40) of Lemma 3.10 as follows

$$-\int_0^\tau\int_\Omega \alpha(x)|z(t)|^2 dxdt \leq -\frac{1}{K_1}E(\tau) + \frac{K_2}{K_1}\int_0^\tau E(t)dt,$$

Combining this last inequality with (3.46), we get

$$\left(1 + \frac{1}{K_1}\right) E(\tau) \leq E(0) + \left(2\alpha_1\sqrt{\gamma} + \frac{K_2}{K_1}\right)\int_0^\tau E(t)dt.$$

It brings by Gronwall's Lemma,

$$E(\tau) \leq \frac{K_1}{K_1+1}\exp\left[\frac{K_1}{K_1+1}\left(2\alpha_1\sqrt{\gamma} + \frac{K_2}{K_1}\right)\tau\right] E(0),$$

that can be written as $E(\tau) \leq pe^{K_3\tau}E(0)$ with

$$p = \frac{K_1}{K_1+1}, K_3 = \frac{K_1}{K_1+1}\left(2\alpha_1\sqrt{\gamma} + \frac{K_2}{K_1}\right).$$

Next, we use the fact that the linear Schrödinger equation is invariant by translation in time, and this argument applies on the interval $[(n-1)\tau, n\tau]$, for $n = 1, 2, \ldots$, yields (denoting $a = pe^{K_3\tau}$):

$$E(n\tau) \leq aE((n-1)\tau) \leq \cdots \leq a^n E(0) = e^{-n\tau\kappa}E(0),$$

where we set $a^n = \exp\left(-n\tau\frac{1}{\tau}\ln\left(\frac{1}{a}\right)\right)$ and $\kappa = \frac{1}{\tau}\ln\left(\frac{1}{a}\right)$. Note that $\kappa > 0$ if and only if $a < 1$, so that we must have $pe^{\tau K_3} < 1$ which is equivalent to

$$\tau < -\frac{\ln p}{K_3} = \frac{(K_1+1)\ln\left(\frac{K_1+1}{K_1}\right)}{\left(2K_1\alpha_1\sqrt{\gamma} + K_2\right)}.$$

Now, for every positive time t, there exists $n \in \mathbb{N}^*$ such that $(n-1)\tau < t \leq n\tau$. Using (3.46) and integration on $[(n-1)\tau, t]$ we have:

$$\begin{aligned} E(t) &\leq E((n-1)\tau) + 2\alpha_1\sqrt{\gamma}\int_{(n-1)\tau}^{t} E(s)ds \\ &\leq e^{-n\tau\kappa}e^{\tau\kappa}E(0) + 2\alpha_1\sqrt{\gamma}\int_0^t E(s)ds. \end{aligned}$$

Since $e^{-n\tau\kappa} \leq e^{-\kappa t}$ for $t \leq n\tau$, and $e^{\tau\kappa} = 1/a$, we get

$$E(t) \leq \frac{1}{a}e^{-\kappa t}E(0) + 2\alpha_1\sqrt{\gamma}\int_0^t E(s)ds.$$

Then by Gronwall's Lemma, it follows:

$$E(t) \leq \frac{1}{a}e^{-\kappa t}e^{2\alpha_1\sqrt{\gamma}t}E(0)$$

and if $\gamma \leq \dfrac{\kappa^2}{4\alpha_1^2}$ then

$$2\delta = \kappa - 2\alpha_1\sqrt{\gamma} \geq 0$$

and we obtain $E(t) \leq \frac{1}{a}e^{-2\delta t}E(0)$. The proof of Theorem 3.17 is complete. ♢

Remark 3.2 *The existence of a suitable design parameter γ depends on the domain ω.*

1. *If $\omega = \Omega$, then the design parameter has to satisfy $\gamma \in (0, \frac{\alpha_0^2}{\alpha_1^2})$ where α_0 and α_1 are given in* (3.3).
2. *If $\omega \subset \Omega$, then the design parameter γ is solution to the inequatility $\kappa - 2\alpha_1\sqrt{\gamma} \geq 0$, which can be also written:*

$$P(\beta) = \frac{4\tau C_{obs}\alpha_1^3}{\alpha_0(K_1+1)}\beta^2 + 4\alpha_1\beta + \frac{1}{\tau}\ln\left(\frac{K_1}{K_1+1}\right) \leq 0 \tag{3.47}$$

where $\beta = \sqrt{\gamma}$, K_1 is given by (3.44), *C_{obs} is the contant of observability* (3.12).

Since we have $\frac{\alpha_0(K_1+1)}{4\tau C_{obs}\alpha_1^3}\ln\left(\frac{K_1}{K_1+1}\right) < 0$, then it is guaranteed that (3.47) admits two opposit sign roots, allowing to find β (thus γ) such that $P(\beta) \leq 0$.

3.3.5 Numerical simulation

We consider the one-dimensional Schrödinger equation (3.18) on $\Omega = (0, \pi)$ with initial condition $z(x, 0) = z_0(x) = \sin(x)$, $x \in [0, \pi]$ compatible with the homogeneous Dirichlet boundary condition. For numerical simulations, we use the divided differences on a uniform grid for the space variable and the discretization with respect to time was done using the Crank Nicolson scheme.

We stabilize the system under the event-triggering mechanism (3.20).

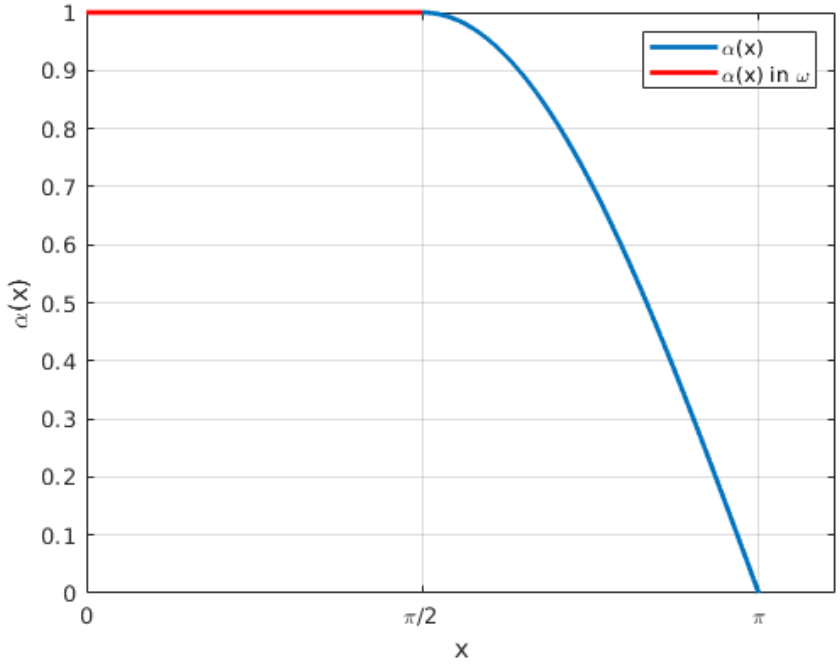

Figure 3.2: Damping coefficient α

Let $\omega = (0, \pi/2)$ be a neighborhood of $x_0 = 0$. We define

$$\alpha(x) = \begin{cases} 1, & \text{if } x \in [0, \pi/2], \\ \sin(x), & \text{if } x \in (\pi/2, \pi] \end{cases}$$

This α satisfies condition (3.3) with $\alpha_0 = \alpha_1 = 1$. Using [87, Theorem 2.2 and equation (5.5)] we select the constants $C_{obs} = 2.8$, $\tau = 5$ we get $K_1 = 30$ and from (3.47), $\gamma \in (0, 0.041)$.

A simulation is done with an appropriate $\gamma = 0.04$ and Figure 3.5 allows to compare the very much alike imaginary part $\mathcal{I}m(z)$ of the numerical solution z of the continuous-in-time closed-loop systems (3.2) (top) and the event-triggered one (3.18)-(3.20) (bottom). It also illustrates the guarantee of the exponential stability of the solution as proved by in Theorem 3.11. This is confirmed even more clearly with Figure 3.4 where we depicted the evolution of the energy of the solution to systems (3.18) and (3.2).

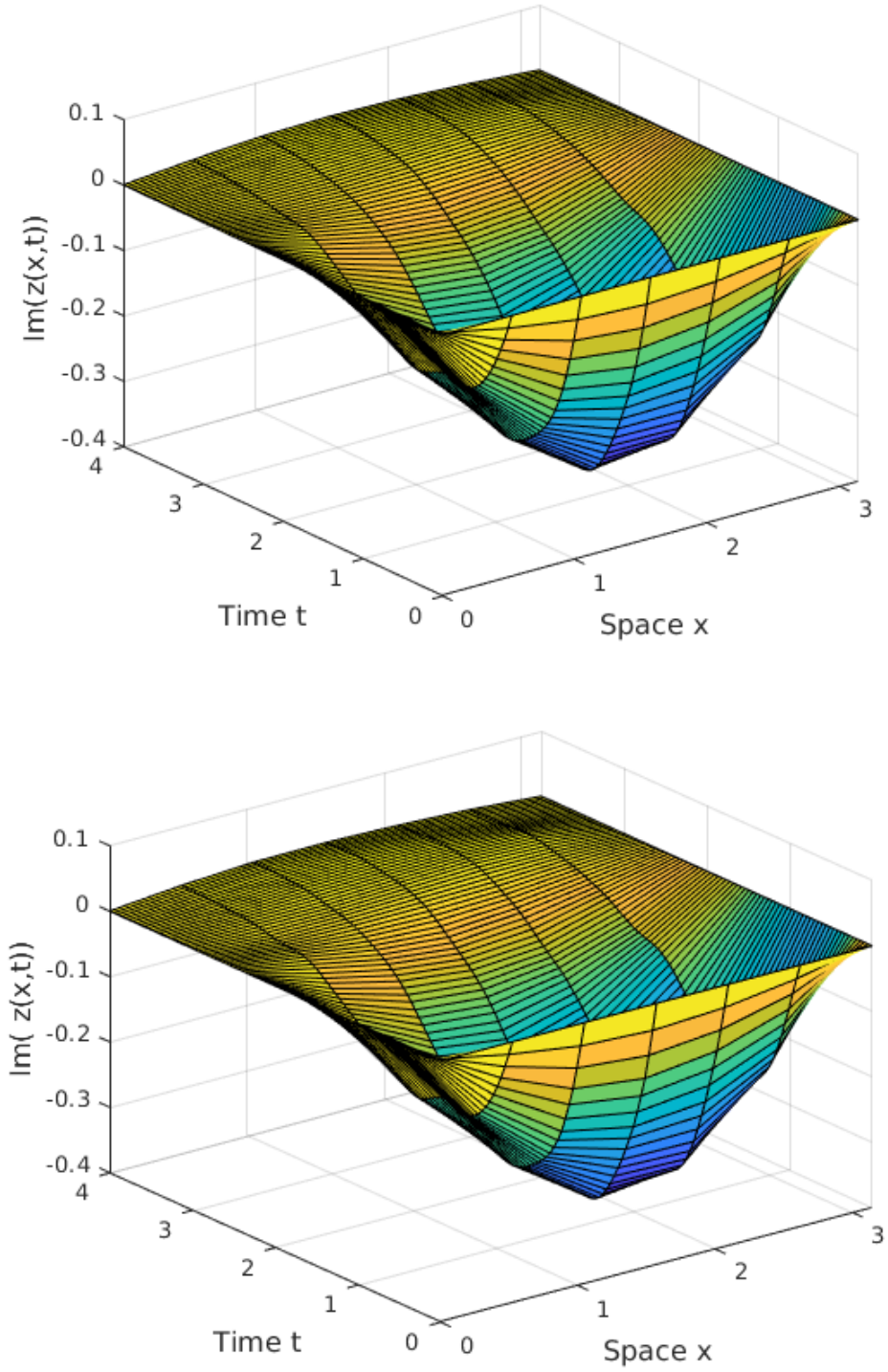

Figure 3.3: Imaginary part of the numerical solution to the closed-loop system (3.18) under the event-triggering mechanism (3.20) with $\gamma = 0.04$ (bottom), and the solution the continuous closed-loop system (3.2)(top).

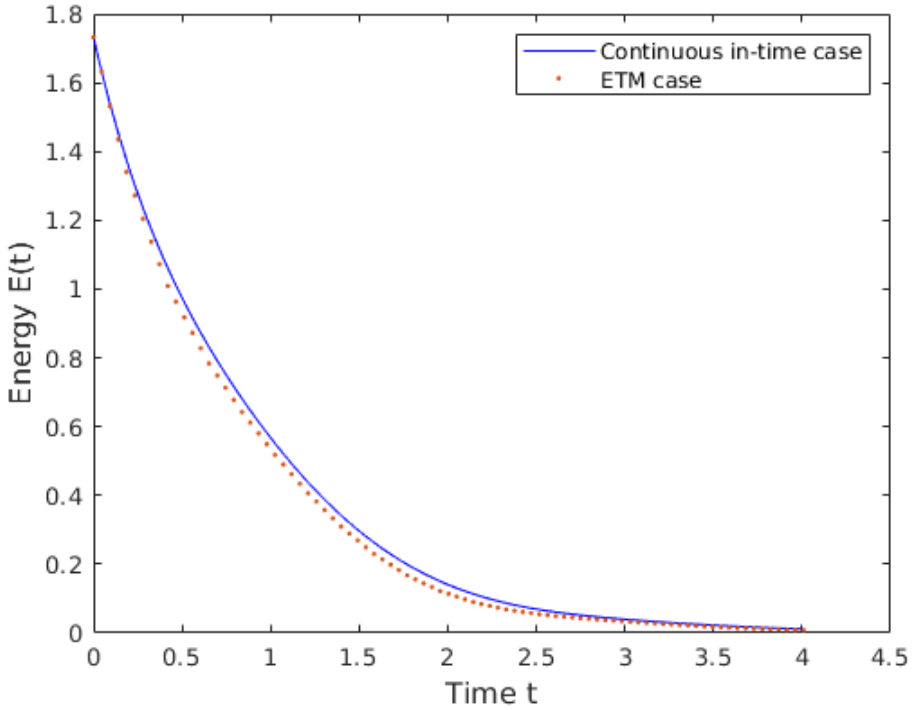

Figure 3.4: Time-evolution of the L^2-norm of the solution to the closed-loop system (3.18) under the event-triggering rule (3.20) (dotted black) and to the continuous-in-time system (3.2) (solid line).

3.4 Dynamic ETC for the Schrödinger equation subject to constant or localized damping

With the goal of enriching the static event-triggering mechanism (state-based triggering condition) designed in section 3.3, we propose here a dynamic event-triggering rule, similarly to the one introduced for general framework of nonlinear finite-dimentional control system in [41] and in [131, 85] using small-gain and high gain methods. This new dynamic rule consists in introducing an additional internal dynamic variable to the static law. It is worthwhile to mention that the dynamic event-triggering strategy has already been extended to PDE framework in [30] for a coupled 2×2 linear hyperbolic system, in [123] for sandwich hyperbolic PDE systems and in [93] for a class of reaction-diffusion PDEs with Robin actuation.

3.4.1 Definition of a dynamic event-triggering mechanism

Inspired by the emulation approach introduced in the context of ordinary differential equation in [111, 88, 41], a state-dependent criterion was proposed in the previous section taking the shape of the static event-triggering mechanism (3.20). In this section, we propose to enrich our event-triggering mechanism (3.20) by adding an internal scalar dynamic variable m satisfying the following differential equation

$$\dot{m}(t) = -\eta m(t) + 2\gamma E(t) - \|e_k(t)\|^2, \text{ for } t \geq t_k \tag{3.48}$$

with $m(t_0) = 0$ and $m(t_k^-) = m(t_k) = m(t_k^+)$ and $\eta > 0$ a design parameter.

Then, we can describe the event-triggering law under consideration, starting from $t_0 = 0$ by

$$t_{k+1} = \inf \left\{ t \geq t_k, \|e_k(t)\|^2 - 2\gamma E(t) > \frac{1}{\theta} m(t) \right\} \tag{3.49}$$

where $\gamma > 0$ and $\theta > 0$ are design parameters as well. This triggering law corresponds to the one defined by G given by (1.6).

Remark 3.3 *When the design parameter θ tends to $+\infty$ in the dynamic rule* (3.49)*, we obtain the static rule* (3.20)*. Note that the signal $m(t)$ can be considered as a filtered value of the difference $2\gamma E(t) - \|e_k(t)\|^2$.*

Similarly to [41], one gets the following result.

Lemma 3.12:

Using the definition of the event-triggering mechanism (3.49), it follows, for all $t \in [0, T^*), k \geq 0$ that

$$m(t) \geq 0 \quad \text{and} \quad \|e_k(t)\|^2 \leq \frac{1}{\theta} m(t) + 2\gamma E(t). \tag{3.50}$$

Proof : Indeed, between two triggering instants t_k, t_{k+1}, from (3.49), we have

$$\frac{1}{\theta} m(t) + 2\gamma E(t) - \|e_k(t)\|^2 \geq 0.$$

Combined to (3.48), this inequality brings

$$\frac{1}{\theta}m(t) + \dot{m}(t) + \eta m(t) \geq 0, \text{ i.e } \dot{m}(t) \geq -\left(\frac{1}{\theta} + \eta\right) m(t)$$

for all $t \in [t_k, t_{k+1})$. Therefore, for all $t \in [t_k, t_{k+1}]$

$$m(t) \geq m(t_k) e^{-\left(\frac{1}{\theta}+\eta\right)(t-t_k)}. \tag{3.51}$$

Since $m(t_0) = 0$, it follows from (3.51) that $m(t) \geq 0$ for all $t \in [0, t_1]$. The same reasonning will give $m(t) \geq 0$ for $t \in [t_1, t_2]$ and then we obtain successively in the future intervals $m(t) \geq 0$ for all $t \in [0, T^*)$. ♢

Remark 3.4 *The dynamic event-triggering mechanism is frequently constructed with $m(t) \leq 0$ as in [33, 30, 93, 123] but we will follow the same approach as in [41, 56] where $m(t)$ is positive.*

Remark 3.5 *For a given state $z(t_k)$ of the event-triggered control system* (3.18)*, since $m(t) \geq 0$, the next execution time t_{k+1} given by the dynamic rule* (3.49) *comes later than the one given by the static rule* (3.20)*. Thus, we can hope less frequent updates with this dynamic law.*

As in section 3.3, we want to ensure 1) the well-posedness, 2) the absence of Zeno and 3) the exponential stability of the closed loop subjected to the event-triggering mechanism (3.49)-(3.48).

3.4.2 Well-posedness and absence of Zeno behavior

Leveraging on some regularity of the classical solutions to the Schrödinger equation we get the following theorem.

Theorem 3.13: Well-posedness of the event-triggered control system

Let Ω be an open bounded domain of class C^2. For any initial conditions $z_0 \in H^2(\Omega) \cap H_0^1(\Omega)$, there exists a unique solution to (3.18) under the event-triggering mechanism (3.49), satisfying

$$z \in C^0([0, T^*); H^2(\Omega) \cap H_0^1(\Omega)) \cap C^1([0, T^*); L^2(\Omega)). \tag{3.52}$$

Proof : The proof is constructed by induction and is similar to the one that has been presented in for proving Theorem 3.3.2 for the static event-triggering law (3.20). ♢

Before proving that the Zeno phenomenon cannot occur, let us state the following intermediate result which is the equivalent version of Lemma 3.7 in the dynamic event-triggering mechanism framework.

Lemma 3.14: Intermediate result

Under the event-triggering law (3.49), for all $t \in [0, T^*)$ one has:

$$2\gamma\theta E(0) \leq e^{2Kt}\left(2\gamma\theta E(t) + m(t)\right), \tag{3.53}$$

with

$$K = \frac{1}{2}\max\left\{3\alpha_1 + \frac{2}{\theta} + \alpha_1\gamma \ ; \ \alpha_1\gamma + \frac{1}{\theta} + \eta\right\}. \tag{3.54}$$

Proof : Let us start with the fact that from (3.19), the event-triggered closed-loop system also reads:

$$\begin{cases} i\partial_t z + \Delta z = -i\alpha z + i\alpha e_k, & \text{in } \Omega \times [t_k, t_{k+1}), \\ z = 0, & \text{on } \partial\Omega \times \mathbb{R}^+, \\ z(\cdot, 0) = z_0, & \text{in } \Omega. \end{cases} \tag{3.55}$$

Performing the Green formula with $z = 0$ on $\partial\Omega$, the time-derivative of $E(t)$ along the trajectories of system (3.55) is given by:

$$\begin{aligned} \dot{E}(t) =& \mathcal{R}e\left(\int_\Omega \bar{z}(t)\partial_t z(t)\right) \\ =& \mathcal{I}m\left(-\int_\Omega \bar{z}(t)\Delta z(t)\right) - \mathcal{I}m\left(\int_\Omega i\alpha(x)|z(t)|^2\right) + \mathcal{I}m\left(i\int_\Omega \alpha(x)e_k(t)\bar{z}(t)\right) \\ \dot{E}(t) =& -\int_\Omega \alpha(x)|z(t)|^2 + \mathcal{R}e\left(\int_\Omega \alpha(x)\bar{e}_k(t)z(t)\right). \end{aligned}$$

Then, we use (3.3), along with Cauchy-Schwarz and Young's inequalities, to obtain

$$\begin{aligned} \left|\dot{E}(t)\right| &\leq \alpha_1\|z(t)\|^2 + \alpha_1\|e_k(t)\|\|z(t)\| \\ &\leq \frac{3\alpha_1}{2}\|z(t)\|^2 + \frac{\alpha_1}{2}\|e_k(t)\|^2. \end{aligned} \tag{3.56}$$

Hence,

$$|2\gamma\theta\dot{E}(t)+\dot{m}(t)| \leq \gamma\theta\left(3\alpha_1\|z(t)\|^2 + \alpha_1\|e_k(t)\|^2\right) + \eta m(t) + 2\gamma E(t) + \|e_k(t)\|^2. \tag{3.57}$$

From (3.50) and using $\|z(t)\|^2 = 2E(t)$, we get $\forall t \in [t_k, t_{k+1})$,

$$\begin{aligned} |2\gamma\theta\dot{E}(t) + \dot{m}(t)| &\leq 6\alpha_1\theta\gamma E(t) + \eta m(t) + 2\gamma E(t) + (\alpha_1\theta\gamma + 1)\left(\frac{1}{\theta}m(t) + 2\gamma E(t)\right), \\ &\leq \left(6\alpha_1\theta\gamma + 4\gamma + 2\alpha_1\theta\gamma^2\right)E(t) + \left(\alpha_1\gamma + \frac{1}{\theta} + \eta\right)m(t), \\ &\leq 2\theta\gamma\left(3\alpha_1 + \frac{2}{\theta} + \alpha_1\gamma\right)E(t) + \left(\alpha_1\gamma + \frac{1}{\theta} + \eta\right)m(t), \end{aligned}$$

so that with K defined by (3.54), we can write $\forall t \in [t_k, t_{k+1})$,

$$|2\gamma\theta\dot{E}(t) + \dot{m}(t)| \leq 2K\left(2\gamma\theta E(t) + m(t)\right).$$

From there,

$$-2K\left(2\gamma\theta E(t)+m(t)\right)\leq 2\gamma\theta\dot{E}(t)+\dot{m}(t)$$
$$\dot{F}(t)=2\gamma\theta\dot{E}(t)+\dot{m}(t)+2K\left(2\gamma\theta E(t)+m(t)\right)\geq 0$$

and one gets that $F(t)=e^{2Kt}(2\gamma\theta E(t)+m(t))$ satisfies $\dot{F}(t)\geq 0$ so that for any $t\geq t_k$, $F(t)\geq F(t_k)$. Inferring this inequality for $F(t_k)$ up to $t_0=0$, one obtains

$$F(t)\geq F(t_k)\geq F(t_{k-1})\geq\cdots F(t_1)\geq F(0).$$

Using the fact that $m(t_0)=0$, we get (3.53) for all $t\in[0,T^*)$ and the lemma is proved. ♢

We can now provide the main result on the fact that our event-triggering law does not generate some infinite sequence of updates in finite time.

Theorem 3.15: Avoidance of the Zeno phenomenon

There is no Zeno phenomenon for the system (3.18) under the event-triggering mechanism (3.49). In other words, following (3.49), there will not be infinitely many updates of the control of system (3.18) over any bounded time interval.

Proof : The proof is done by contradiction and mimics that one of Theorem 3.9. Let us assume that T^* defined by 3.22 is such that $T^*<+\infty$. Let us also define and study the evolution of the following function :

$$\varphi: t\in[t_k,t_{k+1})\mapsto\varphi(t)=\frac{\theta\|e_k(t)\|^2}{2\gamma\theta E(t)+m(t)}. \tag{3.58}$$

As in [41, 67], the proof is based on the study of φ in the interval $[0,T^*]$. This function φ is non negative and satisfies, $\forall k\in\mathbb{N}$, $\varphi(t_k)=0$ and jumps from $\varphi(t_{k+1}^-)=1$ to $\varphi(t_{k+1}^+)=0$, where $\varphi(t_{k+1}^-)$ is the value of φ before the update and $\varphi(t_{k+1}^+)$ is the one after. The time-derivative of φ reads:

$$\dot{\varphi}(t)=\frac{\theta\frac{d}{dt}\|e_k(t)\|^2}{2\gamma\theta E(t)+m(t)}-\varphi(t)\frac{2\gamma\theta\dot{E}+\dot{m}(t)}{2\gamma\theta E(t)+m(t)}. \tag{3.59}$$

On the one hand, (3.19) and (3.18) imply that

$$i\partial_t e_k(t)=i\partial_t z(t)=-\Delta z(t)-i\alpha(x)z(t)+i\alpha(x)e_k(t)$$

so that we have by Cauchy-Schwarz's inequality:

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|e_k(t)\|^2&=\mathcal{I}m\int_\Omega i\bar{e}_k(t)\partial_t e_k(t)\\&=\mathcal{I}m\int_\Omega\bar{e}_k(t)\left(-\Delta z(t)-i\alpha(x)z(t)+i\alpha(x)e_k\right)\\&\leq\|\Delta z(t)\|\|e_k(t)\|+\alpha_1\|e_k(t)\|\|z(t)\|+\alpha_1\|e_k(t)\|^2.\end{aligned}$$

From Theorem 3.13, it follows that for all $z_0\in H^2(\Omega)\cap H_0^1(\Omega)$, the closed-loop system (3.55) under the event-triggering mechanism (3.49) has a unique solution

$z \in C^0([0,T^*);H^2(\Omega)\cap H_0^1(\Omega))$. Then from Lemma 3.8 there exists a constant $C_\Delta = C_\Delta(T^*, \|z_0\|_{H^2(\Omega)\cap H^1(\Omega)}) > 0$ such that $\forall t \in [0,T^*)$,

$$\|\Delta z(t)\| \leq \|\Delta z\|_{L^\infty(0,T;L^2(\Omega))} \leq C_\Delta. \tag{3.60}$$

By Young's inequality and (3.3) it follows:

$$\frac{d}{dt}\|e_k(t)\|^2 \leq 2C_\Delta\|e_k(t)\| + 3\alpha_1\|e_k(t)\|^2 + \alpha_1\|z(t)\|^2.$$

On the other hand, dealing with the numerator of the second term of (3.59), we obtain (3.57). Re-organizing terms in (3.59), we get

$$\begin{aligned}(2\gamma\theta E(t) + m(t))\dot\varphi(t) \leq\ & 2C_0\theta\|e_k(t)\| + 3\alpha_1\theta\|e_k(t)\|^2 \\ & + 2\alpha_1\theta E(t) + \varphi(t)\left(6\theta\gamma\alpha_1 + 2\gamma\right)E(t) \\ & + \varphi(t)(\theta\alpha_1\gamma + 1)\|e_k(t)\|^2 + \varphi(t)\eta m(t).\end{aligned} \tag{3.61}$$

In (3.61), several terms have to be handled. First, from (3.50) we have

$$\theta\|e_k(t)\| \leq \sqrt{\theta}\sqrt{(2\theta\gamma E(t) + m(t))}$$

so that using Lemma 3.14 we can write, for all $t \in [0,T^*)$,

$$\frac{\theta\|e_k(t)\|}{2\gamma\theta E(t) + m(t)} \leq \frac{\sqrt{\theta}}{\sqrt{2\gamma\theta E(t) + m(t)}} \leq \frac{\sqrt{\theta}}{\sqrt{2\gamma\theta E(0)e^{-2Kt}}} \leq \frac{e^{KT}}{\sqrt{2\gamma E(0)}}. \tag{3.62}$$

Moreover, one should notice that

$$\frac{E(t)}{2\gamma\theta E(t) + m(t)} \leq \frac{1}{2\gamma\theta} \quad \text{and} \quad \frac{m(t)}{2\gamma\theta E(t) + m(t)} \leq 1. \tag{3.63}$$

Therefore, back to (3.61), deviding by $(2\gamma\theta E(t) + m(t))$, recalling

$$\varphi(t) = \frac{\theta\|e_k(t)\|^2}{2\gamma\theta E(t) + m(t)}$$

and using (3.62) and (3.63), we obtain

$$\dot\varphi(t) \leq \frac{2C_\Delta e^{KT^*}}{\sqrt{2\gamma E(0)}} + 3\alpha_1\varphi(t) + \frac{\alpha_1}{\gamma} + \left(3\alpha_1 + \frac{1}{\theta}\right)\varphi(t) + \frac{\theta\alpha_1\gamma + 1}{\theta}\varphi^2(t) + \eta\varphi(t).$$

Finally, denoting

$$a_0 = \frac{2C_\Delta e^{KT^*}}{\sqrt{2\gamma E(0)}} + \frac{\alpha_1}{\gamma},\ a_1 = 6\alpha_1 + \eta + \frac{1}{\theta},\ a_2 = \alpha_1\gamma + \frac{1}{\theta},$$

we can actually write

$$\dot\varphi(t) \leq a_0 + a_1\varphi(t) + a_2\varphi^2(t),$$

which gives, by integration over (t_k, t_{k+1})

$$t_{k+1} - t_k \geq \int_{t_k}^{t_{k+1}} \frac{\dot{\varphi}(t)}{a_0 + a_1\varphi(t) + a_2\varphi^2(t)} dt$$

and after a change of variable $s = \varphi(t)$

$$t_{k+1} - t_k \geq \int_0^1 \frac{ds}{a_0 + a_1 s + a_2 s^2}$$

using the fact that $\varphi(t_k) = 0$, $\lim_{t \to t_{k+1}^-} \varphi(t_{k+1}) = 1$.

One gets

$$1 \leq \frac{1}{A}(t_{k+1} - t_k) \tag{3.64}$$

where $A = \int_0^1 \frac{ds}{a_0 + a_1 s + a_2 s^2} > 0$ since $a_0, a_1, a_2 > 0$.

Since we assumed that $T^* < +\infty$, passing to the limit $t_k \to T^*$ as $k \to +\infty$ in (3.64) leads to a contradiction. We therefore obtained $T^* = +\infty$, ensuring the absence of any accumulation points and the avoidance of Zeno behavior. ♢

Remark 3.6 *Differently from the usual literature dealing with event-triggered control for finite-time dimension systems, the proof of Theorem 3.15 is not based on the existence of a dwell-time (see for example [41, 29]) which is a sufficient but not necessary condition. One could consider that* $A = \int_0^1 \frac{ds}{a_0 + a_1 s + a_2 s^2}$ *is a dwell time, but then one should notice that as* a_0*, it is dependent on* T^* *and the initial condition through* $E(0)$*. Then taking another route, proving that there exists no accumulation point for the sequence* $(t_k)_{k\geq 0}$ *is actually necessary and sufficient.*

3.4.3 Exponential stability analysis

This section addresses the problem of the exponential stability of system (3.18)-(3.49). In order to prove the stability of the closed loop, we consider the Lyapunov candidate function defined by :

$$W(t) = E(t) + m(t), \tag{3.65}$$

with the energy E defined in (3.9) and the internal state m defined in (3.48).

We can first take inspiration from [79, 80] in order to bound the functional W, defined in (3.65) as we did for the energy in Lemma 3.10. This is reported in the following intermediate Lemma.

Lemma 3.16: Energy-like estimate

Consider the solution z to system (3.18)-3.49. For any $\tau > 0$ there exist some constant K_1 and $K_2 > 0$ such that $W(t) = E(t) + m(t)$, satisfies:

$$W(\tau) \leq K_1 \int_0^\tau \int_\Omega \alpha(x)|z(t)|^2 dx dt + K_2 \int_0^\tau W(t) dt. \tag{3.66}$$

Proof : As for Lemma 3.5 and Lemma 3.10, the proof will rely strongly on an observability inequality proved for this Schrödinger equation with internally localized damping in [80] and recalled in Lemma 3.2. Beware that the damping's non-vanishing set ω has to be a neighborhood of $\Gamma_0 \subset \partial\Omega$.

Let $\tau > 0$ and let us recall that from the time-derivative of $E(t)$ in (3.56) we have, since $m(t) \geq 0$,

$$\begin{aligned}\dot{W}(t) &= -\int_\Omega \alpha(x)|z(t)|^2 + Re\left(\int_\Omega \alpha(x)\bar{e}_k(t)z(t)\right) \\ &\quad - \eta m(t) + 2\gamma E(t) - \|e_k(t)\|^2 \\ &\leq Re\left(\int_\Omega \alpha(x)\bar{e}_k(t)z(t)\right) + 2\gamma E(t).\end{aligned} \tag{3.67}$$

From (3.3), (3.50), Cauchy-Schwarz and Young's inequalities we get

$$\begin{aligned}\dot{W}(t) &\leq \alpha_1\left(\frac{\|e_k(t)\|^2}{2\sqrt{\gamma}} + \frac{\sqrt{\gamma}\|z(t)\|^2}{2}\right) + 2\gamma E(t) \\ &\leq \frac{\alpha_1}{2\sqrt{\gamma}}\left(2\gamma E(t) + \frac{1}{\theta}m(t)\right) + (\alpha_1\sqrt{\gamma} + 2\gamma)E(t),\end{aligned}$$

yielding,

$$\dot{W}(t) \leq C_1 W(t), \text{ with } C_1 = \max\{2\alpha_1\sqrt{\gamma} + 2\gamma \ ; \ \alpha_1/(2\theta\sqrt{\gamma})\}.$$

Integrating on $[0, \tau]$, and knowing that $W(0) = E(0)$ since $m(0) = 0$ we get

$$W(\tau) \leq E(0) + C_1\int_0^\tau W(t)dt. \tag{3.68}$$

Consider the solution z to (3.18) as $z = y + \phi$ the sum of two variables $y = y(x,t)$ and $\phi = \phi(x,t)$ satisfying (3.10) and (3.11).

Taking advantage of this important result, from (3.68), under assumption (3.3) and the fact that $\phi = z - y$, recalling that for any $a, b \in \mathbb{R}$, $|a-b|^2 \leq 2(a^2+b^2)$, we can write using the observability inequality (3.12):

$$\begin{aligned}W(\tau) &\leq \frac{1}{2}\|z_0\|^2 + C_1\int_0^\tau W(t)dt \\ &\leq \frac{1}{2}\|\phi(0)\|^2 + C_1\int_0^\tau W(t)dt \\ &\leq \frac{C_{obs}}{2\alpha_0}\int_0^\tau\int_\omega \alpha(x)|\phi(x,t)|^2dxdt + C_1\int_0^\tau W(t)dt \\ &\leq \frac{C_{obs}}{\alpha_0}\int_0^\tau\int_\Omega \alpha(x)|z(t)|^2dt + \frac{C_{obs}\alpha_1}{\alpha_0}\|y\|^2_{L^\infty(0,\tau;L^2(\omega))} + C_1\int_0^\tau W(t)dt.\end{aligned}$$

Using classical energy estimate (3.13), on the Schrödinger equation (3.10), for a source term $-i\alpha z + i\alpha e_k$, one obtains

$$\begin{aligned}\|y\|^2_{L^\infty(0,\tau;L^2(\omega)} &\leq \|\alpha(e_k - z)\|^2_{L^1(0,\tau;L^2(\Omega))} \\ &\leq 2\tau\alpha_1^2\int_0^\tau \|e_k(t)\|^2dt + 2\tau\alpha_1\int_0^\tau\int_\Omega \alpha(x)|z(t)|^2dt.\end{aligned}$$

From (3.50), we have $\forall t \in [t_k, t_{k+1}), \|e_k(t)\|^2 \leq C_2 W(t)$ with $C_2 = \max\{2\gamma; \frac{1}{\theta}\}$, thus

$$\|y\|^2_{L^\infty(0,\tau;L^2(\omega))} \leq 2\tau\alpha_1^2 C_2 \int_0^\tau W(t)dt + 2\tau\alpha_1 \int_0^\tau \int_\Omega \alpha(x)|z(t)|^2 dt.$$

Hence,

$$W(\tau) \leq \left(\frac{C_{obs}}{\alpha_0} + \frac{2\tau C_{obs}\alpha_1^2}{\alpha_0}\right) \int_0^\tau \int_\Omega \alpha(x)|z(t)|^2 dt + \left(C_1 + \frac{2\tau C_{obs}\alpha_1^3 C_2}{\alpha_0}\right) \int_0^\tau W(t)dt.$$

Therefore we get inequality (3.66) with

$$K_1 = \frac{C_{obs}}{\alpha_0}\left(1 + 2\tau\alpha_1^2\right); K_2 = C_1 + \frac{2\tau C_{obs}\alpha_1^3 C_2}{\alpha_0}. \tag{3.69}$$

♢

Finally, the main exponential stability result is proven by using the Lyapunov functional candidate W defined in (3.65) and by studying its time-derivative along the trajectories of the closed-loop system. The result is reported in the theorem below.

Theorem 3.17: Exponential stability

There exists $\gamma > 0$ such that for any initial condition $z_0 \in H^2(\Omega) \cap H_0^1(\Omega)$, the closed-loop system (3.18) with damping coefficient satisfying (3.3), and under the event-triggering mechanism (3.49), is exponentially stable. In other words, there exist an overshoot constant $K > 0$ and a decay rate $\delta > 0$ such that

$$E(t) \leq KE(0)e^{-2\delta t}, \quad \forall t > 0. \tag{3.70}$$

Proof : We use the following Lyapunov functional candidate $W(t) = E(t) + m(t)$, (also defined in (3.65)). In the proof we consider two cases depending on the damping.

• *Globally non-vanishing damping.* Let us discuss the case where the damping does not vanish in Ω (corresponding to $\omega = \Omega$). Performing the Cauchy-Schwarz and Young's inequalities and using (3.3) we get from (3.50) and (3.67)

$$\begin{aligned}
\dot{W}(t) &\leq -\alpha_0\|z(t)\|^2 + \alpha_1\|z(t)\|\|e_k(t)\| - \eta m(t) + 2\gamma E(t) - \|e_k(t)\|^2 \\
&\leq (2\gamma - 2\alpha_0)E(t) + \frac{\alpha_1}{2\varepsilon}\|z(t)\|^2 + \left(\frac{\alpha_1\varepsilon}{2} - 1\right)\|e_k(t)\|^2 - \eta m(t) \\
&\leq \left(-2\alpha_0 + \alpha_1\varepsilon\gamma + \frac{\alpha_1}{\varepsilon}\right)E(t) + \left(-\eta + \frac{\alpha_1\varepsilon}{2\theta} - \frac{1}{\theta}\right)m(t),
\end{aligned}$$

with $\alpha_1\varepsilon > 2$. Setting $\delta_1 = \frac{1}{2}\min\left\{2\alpha_0 - \alpha_1\varepsilon\gamma - \frac{\alpha_1}{\varepsilon}\ ;\ \eta - \frac{\alpha_1\varepsilon}{2\theta} + \frac{1}{\theta}\right\}$ we obtain

$$\dot{W}(t) \leq -2\delta_1 W(t). \tag{3.71}$$

Choosing $\varepsilon = 1/\sqrt{\gamma}$, and in order to have $\delta_1 > 0$, easy calculations prove that we can pick the tuning parameters γ, η and θ such that

$$0 < \gamma < \frac{\alpha_0^2}{\alpha_1^2}, \quad \gamma < \frac{\alpha_1^2}{4} \quad \text{and} \quad \eta\theta > \frac{\alpha_1^2}{2\alpha_0} - 1. \tag{3.72}$$

Noticing that from the definition of $W(t)$ one gets $E(t) \leq W(t)$ and performing the usual integration calculations, we obtain that for all $t \geq 0$,

$$E(t) \leq e^{-2\delta_1 t} W(0).$$

Finally, since $m(0) = 0$, we get $E(t) \leq e^{-2\delta_1 t} E(0)$ proving that (3.70) holds with $K = 1$ and $\delta = \delta_1$ in the case of non-vanishing damping in Ω.

• *Locally non-vanishing damping.* In the general case, one has $\omega \subsetneq \Omega$, with ω being only a neighborhood of Γ_0, and the damping $\alpha = \alpha(x)$ may vanish outside ω. We will thus need to use Lemma 3.16. Let $\tau > 0$. Integrating (3.67) on $[0, \tau]$, we can write:

$$W(\tau) - W(0) = -\int_0^\tau \int_\Omega \alpha(x)|z(t)|^2 dx + \mathcal{R}e\left(\int_0^\tau \int_\Omega \alpha(x)\bar{e}_k(t)z(t)\right) \tag{3.73}$$

$$- \eta \int_0^\tau m(t) + 2\gamma \int_0^\tau E(t) - \int_0^\tau \|e_k(t)\|^2. \tag{3.74}$$

We can rewrite (3.66) of Lemma 3.16 as follows

$$-\int_0^\tau \int_\Omega \alpha(x)|z(t)|^2 dxdt \leq -\frac{1}{K_1} W(\tau) + \frac{K_2}{K_1} \int_0^\tau W(t)dt.$$

Combining this inequality with (3.73), and using the usual tricks, we get

$$\left(1 + \frac{1}{K_1}\right) W(\tau) \leq W(0) + \frac{K_2}{K_1} \int_0^\tau W(t)dt - \eta \int_0^\tau m(t) + \left(\frac{\alpha_1 \varepsilon}{2} - 1\right) \int_0^\tau \|e_k(t)\|^2 + \left(2\gamma + \frac{\alpha_1}{\varepsilon}\right) \int_0^\tau E(t)$$

so that using (3.50),

$$\left(1 + \frac{1}{K_1}\right) W(\tau) \leq W(0) + \frac{K_2}{K_1} \int_0^\tau W(t) + \left(\alpha_1 \gamma \varepsilon + \frac{\alpha_1}{\varepsilon}\right) \int_0^\tau E(t)dt + \left(-\eta + \frac{\alpha_1 \varepsilon}{2\theta} - \frac{1}{\theta}\right) \int_0^\tau m(t).$$

Since $W(0) = E(0)$, using (3.69) and by selecting $\varepsilon = \frac{1}{\sqrt{\gamma}}$ and $K_3 = 2\alpha_1\sqrt{\gamma}$ we assume that

$$\eta\theta > \frac{\alpha_1}{2\sqrt{\gamma}} - 1$$

so that we can write:

$$\left(1 + \frac{1}{K_1}\right) W(\tau) \leq E(0) + \left(\frac{K_2}{K_1} + K_3\right) \int_0^\tau W(t). \tag{3.75}$$

It brings by Gronwall's Lemma,

$$W(\tau) \leq \frac{K_1}{K_1+1} \exp\left[\frac{K_1}{K_1+1}\left(K_3 + \frac{K_2}{K_1}\right)\tau\right] E(0),$$

that can be written as

$$W(\tau) \leq pe^{\mathbf{c}\tau} E(0)$$

with

$$p = \frac{K_1}{K_1+1}, \mathbf{c} = \frac{K_1}{K_1+1}\left(K_3 + \frac{K_2}{K_1}\right) = \frac{K_1K_3+K_2}{K_1+1}.$$

Next, we apply the invariance by translation in time of the linear Schrödinger equation on the interval $[(n-1)\tau, n\tau]$, for $n = 1, 2, \ldots$, to get (denoting $a = pe^{\mathbf{c}\tau}$):

$$W(n\tau) \leq aW((n-1)\tau) \leq \cdots \leq a^n E(0) = e^{-n\tau\kappa} E(0),$$

where we set $a^n = \exp\left(-n\tau\frac{1}{\tau}\ln\left(\frac{1}{a}\right)\right)$ and $\kappa = \frac{1}{\tau}\ln\left(\frac{1}{a}\right)$. Note that $\kappa > 0$ if and only if $a < 1$, so that we must have $pe^{\tau\mathbf{c}} < 1$ which is equivalent to

$$\tau < -\frac{\ln p}{\mathbf{c}} = \frac{(K_1+1)\ln\left(\frac{K_1+1}{K_1}\right)}{(K_1K_3+K_2)}. \tag{3.76}$$

Now, for every positive time t, there exists $n \in \mathbb{N}^*$ such that $(n-1)\tau < t \leq n\tau$. Using integration on $[(n-1)\tau, t]$ we have:

$$W(t) \leq W((n-1)\tau) + C_1 \int_{(n-1)\tau}^{t} W(s)ds \leq e^{-n\tau\kappa} e^{\tau\kappa} E(0) + C_1 \int_0^t W(s)ds. \tag{3.77}$$

Since $e^{-n\tau\kappa} \leq e^{-\kappa t}$ for $t \leq n\tau$, and $e^{\tau\kappa} = 1/a$, we get

$$W(t) \leq \frac{1}{a} e^{-\kappa t} E(0) + C_1 \int_0^t W(s)ds.$$

Then by Gronwall's Lemma, it follows, for $2\delta = \kappa - C_1$,

$$E(t) \leq W(t) \leq \frac{1}{a} e^{-2\delta t} E(0)$$

and some calculations prove that we can insure $\delta > 0$ if

$$\frac{1}{\tau}\ln\left(\frac{K_1+1}{K_1}\right) - \frac{K_1K_3+K_2}{K_1+1} > C_1 \tag{3.78}$$

where K_1 and K_2 are defined by (3.69) and $K_3 = 2\alpha_1\sqrt{\gamma}$ appears in (3.75).

The proof of Theorem 3.17 is complete as soon as we can ensure that (3.78) can be obtained for a good choice of the tuning parameters γ, η and θ of the event-triggering law.
Notice first that (3.76) gives

$$\frac{1}{\tau}\ln\left(\frac{K_1+1}{K_1}\right) > \frac{(K_1K_3+K_2)}{K_1+1} \tag{3.79}$$

so that (3.78) becomes true if $C_1 = \max\{2\alpha_1\sqrt{\gamma} + 2\gamma \ ; \ \alpha_1/(2\theta\sqrt{\gamma})\}$ can be chosen small enough. Then let us take $\theta > 0$ large enough to have $C_1 = 2\alpha_1\sqrt{\gamma} + 2\gamma$, positive constant that can be as small as needed when choosing $\gamma > 0$ small enough to satisfies:

$$2\tau\alpha_1(K_1+1)\sqrt{\gamma} + 2\tau(1 + 2\tau C_{obs}\alpha_0^{-1}\alpha_1^3)\gamma - (K_1+1)\ln\left(\frac{K_1+1}{K_1}\right) < 0. \quad (3.80)$$

♢

3.4.4 Numerical example

Consider the one-dimensional Schrödinger equation (3.18) under the event-triggering mechanism (3.49) on $\Omega = (0, \pi)$ with initial condition

$$z_0(x) = \sin(x), \ x \in [0, \pi].$$

We use the divided differences on a uniform grid for the space variable and the discretization with respect to time through Crank Nicolson scheme is performed. Let $\omega = (0, \pi/10)$ be a neighborhood of $x_0 = 0$. We define $\alpha_0 = \alpha_1 = 1$. With respect to (3.3), we select the same damping coefficient as section 3.3.5, as follows:

$$\alpha(x) = \begin{cases} 1, & \text{if } x \in [0, \pi/2], \\ \sin(x), & \text{if } x \in (\pi/2, \pi]. \end{cases}$$

For $\gamma = 0.13$, $\eta = 0.7$ and $\theta = 15$, in the case of a globally non-vanishing damping (corresponding to $\omega = \Omega$ and the damping does not vanish in Ω), the inequality (3.72) is verified. In Figure 3.5 we compare the imaginary part Imz of the numerical solution z of the continuous-in-time closed-loop systems (3.2) (top) and the dynamic event-triggered one (3.18)-(3.49) (bottom). It also illustrates the guarantee of the exponential stability of the solution as studied in Theorem 3.17. This is confirmed even more clearly with Figure 3.6 where we depicted the time-evolution of the energy of the solution to systems (3.18) under the static (3.20) and dynamic (3.49) event-triggering mechanism (ETM).

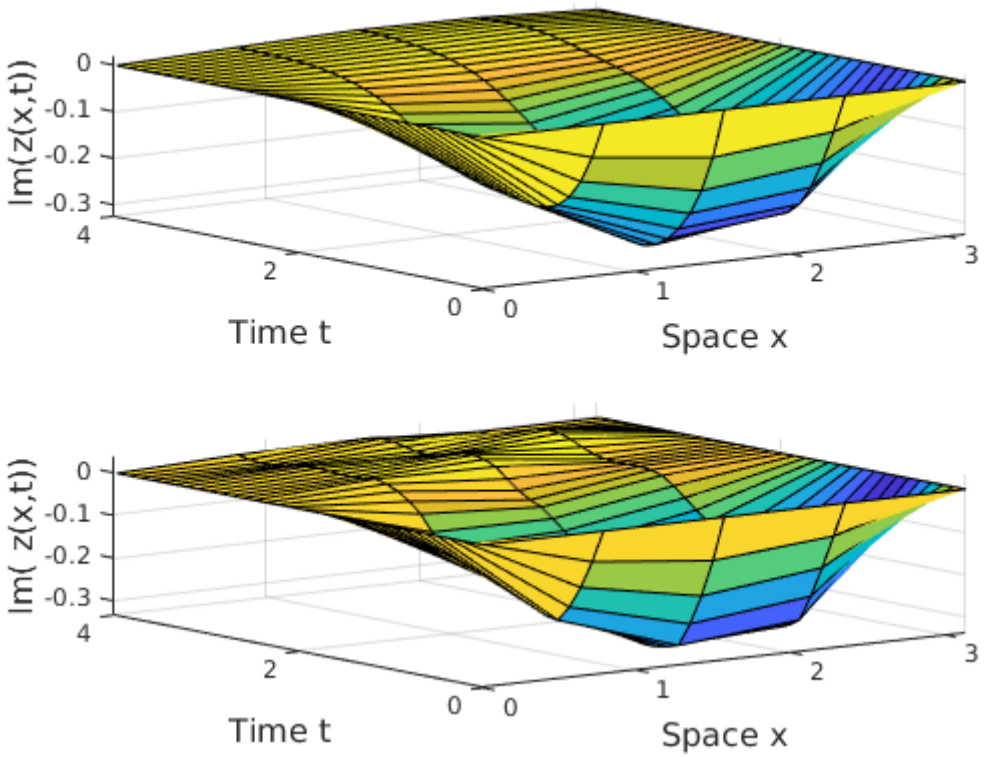

Figure 3.5: Imaginary part of the solution: of the closed-loop system (3.18) under the event-triggering mechanism (3.49) (bottom), and of the solution of the continuous-in-time closed-loop system (3.2) (top).

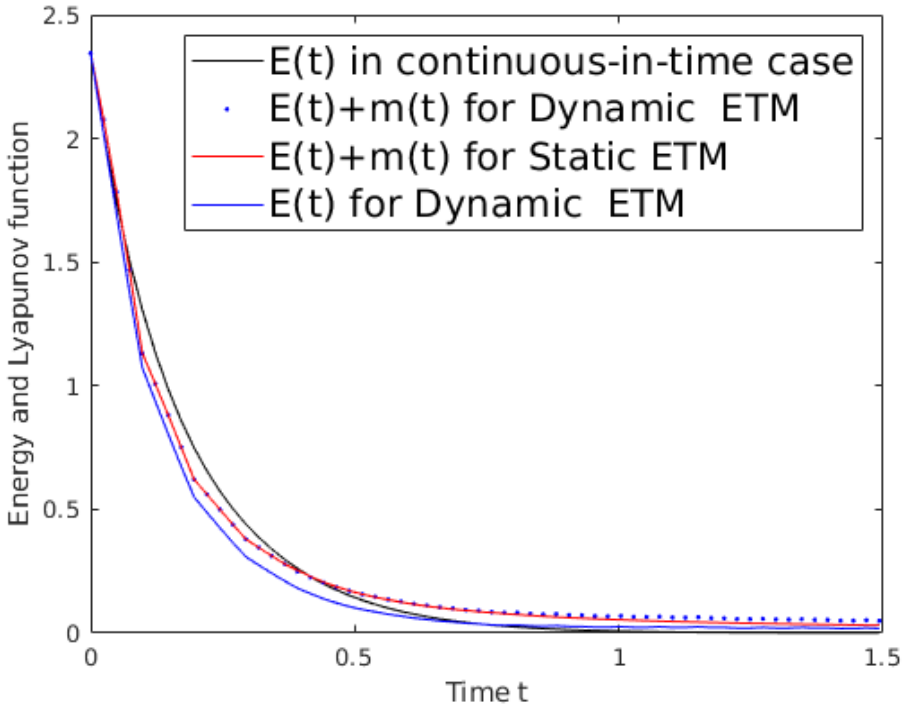

Figure 3.6: Time-evolution of the energy $E(t)$ and the Lyapunov function $W(t)$.

3.5 Conclusion and perspectives

We considered the problem of exponential stabilization for a locally damped linear Schrödinger equation under static and dynamic event-triggering mechanisms. Thanks to some regularity of the classical solution to the Schrödinger equation we prove the well-posedness of the closed-loop system. We also proved absence of accumulation points in the updates sequence leading to the avoidance of the Zeno behavior. Furthermore, in order to ensure the exponential stability of the closed loop we exploited classical observability inequality results. An illustrative example based on the one-dimensional Schrödinger equation demonstrates the efficiency of the results that were the source of geometrical conditions for the location of the damping's action. This work paves the way for future works on event-triggering control as described bellow.

- *The boundary damping control for the Schrödinger PDE*

The context we studied until now is corresponding to a bounded control operator that was sampled according to a event-triggering law. What about studying a case where the control operator is unbounded. For example starting from boundary control action.

Event-triggering mechanism can be designed for the boundary damping control for the Schrödinger PDE

$$\begin{cases} i\partial_t z + \Delta z = 0 \qquad in \qquad \Omega \times [0,\infty) \\ \partial_\nu z = -(x-x_0)\cdot\nu(x)\partial_t z \quad on \quad \Gamma_0 \times [0,\infty) \\ z = 0 \qquad\qquad on \qquad \Gamma_1 \times [0,\infty) \\ z(x,0) = z_0(x) \quad in \qquad \Omega \end{cases}$$

where $\nu(x)$ denotes the unit outward normal vector to Ω at $x \in \Gamma$,

$$\Gamma_0 = \{x \in \Gamma, (x-x_0)\cdot\nu(x) > 0\}$$
$$\Gamma_1 = \Gamma \setminus \Gamma_0 = \{x \in \Gamma, \quad (x-x_0)\cdot\nu(x) \leq 0\}$$

and the initial data's space is

$$H^1_{\Gamma_0}(\Omega) = \{z \in H^1(\Omega), \quad z = 0 \quad on \quad \Gamma_1\}.$$

Let us consider the event-triggering law defined by $t_0 = 0$ and

$$t_{k+1} = \inf\left\{t \geq t_k, \|z(x,t) - z(x,t_k)\|^2_{L^2(\Gamma_1)} - \gamma E^b(t) - \eta_0 \geq 0\right\},$$

with

$$E^b(t) = \frac{1}{2}\int_\Omega |\nabla z(x,t)|^2 dx.$$

This problem is very challenging since

a- For the **avoidance of the Zeno behavior**, Lemma 3.7 or Lemma 3.14 will no longer be obtained as in this boundary case the corresponding control

operator is unbounded. One way to deal with this issue could be to consider, as proposed, the combination of absolute and relative threshold event-triggering law.

b- We could only expect pratical stability, that is, a convergence to an attractor of size depending on η_0 and not an exponential stability forwards 0.

• $1-D$ *boundary control of linear Schrödinger equation via Backstepping approach*

Consider in $(0,1)\times\mathbb{R}^+$ the system

$$\begin{cases} \partial_t z(x,t) + i\partial^2_{xx} z(x,t) = 0 \\ \partial_x z(0,t) = 0 \\ z(1,t) = U(t). \end{cases}$$

A backstepping transformation is used in [70] to exponentially stabilize this system by considering the target plant:

$$\begin{cases} \partial_t u(x,t) + i\partial^2_{xx} u(x,t) + \lambda u(x,t) = 0, \\ \partial_x u(0,t) = 0 \\ u(1,t) = 0. \end{cases}$$

The approach is then based spectral theory. The obtained controller is given by

$$U(t) = z(1,t) = \int_0^1 k(1,x) z(x,t) dx \tag{3.81}$$

where k is the kernel of the backstepping transformation.

We could propose to construct an event-triggering mechanism (t_k) allowing to update the feedback law (3.81) and to ensure the exponential decay result and avoidance Zeno behavior of the closed-loop system.

One can consider the Schrödinger equation formally as a heat equation with an imaginary diffusion coefficient and solve the stabilization problem using the method presented in [31] for reaction-diffusion equation.

• *ODE-Schrödinger cascade system*

Consider the cascade ODE-Schrödinger equation defined by

$$\begin{cases} \dot{X}(t) = AX(t) + Bu(0,t), t > 0 \\ \partial_t u(x,t) = -i\partial^2_{xx} u(x,t), x \in (0,1), t > 0 \\ \partial_x u(0,t) = CX(t) \\ u(1,t) = U(t). \end{cases}$$

In [95] a two step backstepping transformation,

$$U(t) = \int_0^1 [k(1,y) + q(1,y) - \int_0^1 k(1,l) q(l,y) dl] u(y,t) dy + \gamma(t) - \int_0^1 k(1,y)\gamma(t)) dy X(t)$$

exponentially stabilizes the system. Then the question to construct an event-triggering mechanism $U(t) := U(t_k) \quad \forall t \in (t_k, t_{k+1})$, where (t_k) is appropriately chosen according to some event/threshold.

- *The Schrödinger equation with saturating distributed input*

Consider for $\alpha > 0$, the one dimentional linear Schrödinger equation subject to a cone-bounded nonlinearity (for example a saturation input)

$$\begin{cases} \partial_t z(x,t) = i\partial_{xx} z(x,t) - \alpha\sigma(z(x,t)) & (x,t) \in (0,1) \times \mathbb{R}^+, \\ z(0,t) = z(1,t) = 0 & \forall t \in \mathbb{R}^+ \\ z(x,0) = z_0(x) & \forall x \in (0,1). \end{cases} \tag{3.82}$$

Thanks to the nonlinear semigroup theory (see [83] for a good introduction), for any initial conditions $z_0 \in H^2(\Omega) \cap H_0^1(0,1)$, there exists a unique strong solution to (3.82) satisfying

$$z \in C^0([0,T]; H^2(\Omega) \cap H_0^1(0,1)) \cap C^1([0,T]; L^2(0,1)). \tag{3.83}$$

Moreover one can prove that the system is semi-globally exponentially stable. An event-triggering mechanism (t_k) can be designed to update the feedback law $-\alpha\sigma(z)$ and to ensure the exponential decay result and avoidance Zeno behavior of the closed-loop system.

4

ETC of reaction-diffusion PDE with input delay

4.1 Introduction

A delay system refers to a dynamic setup where the output behavior depends not only on the current input but also on past inputs or states, introducing a time delay into the system response. This time-delay effect is common in various natural and engineered processes such as in network-controlled systems, teleoperation,biology machining processes, rolling mills, cooling systems, chemical processes, traffic dynamics, supply networks, automotive propulsion, 3D printing and additive manufacturing, irrigation channels, and population dynamics [107, 104, 8] and can significantly affect the system stability [103]. In this chapter, we will focus on a particular class of systems described reaction-diffusion partial differential equations (PDEs):

$$u_t(x,t) = u_{xx}(x,t) + \lambda u(x,t)$$

which model several physical phenomena arising in biology [118, 86]; in chemistry [44] etc, the consideration of long delay inputs is particularly important. The presence of delay in the inputs may imply situations where there is a substantial time lag between an event occurring and its impact being felt in the system. This phenomenon is often observed in processes involving propagation, transport, communication delays, distribution of chemicals in biological tissues as highlighted in [81, 103, 35]. For real-world examples of input delays, during the COVID-19 pandemic, before the development of commercial rapid tests, it would take a couple of days for both an individual and public health authorities to receive test results. That may be viewed as a typical sensor delay. And then it would take even longer for the authorities to agree on a public health action. Some drugs also take days to produce their effects. These are all input delays in the context of reaction-diffusion PDEs.

The design of an event-triggering mechanism becomes crucial in this context due to the inherent complexities of delay systems. Event-triggered control ensures that responses are initiated only when specific conditions are met or when an event occurs, which can be particularly effective in situations where the time delay is significant. By judiciously activating responses based on these triggers, system resources can be conserved, and the system's performance and stability can be improved.

The stabilization of the unstable reaction-diffusion PDEs under arbitrarily long input delay is a challenging problem, first formulated and solved in [68] using the *backstepping method for PDEs* as the stabilization of a hyperbolic (transport) PDE, modeling the delay, which cascades into the reaction-diffusion PDE. Since then, control design for delay compensation (including known or unknown constant/time-varying delays) has evolved considerably and several results have been proposed for reaction-diffusion PDEs, see, e.g., [36, 45, 90, 62, 98]and [4, 22, 126] and the references therein.

4.2 Continuous-in-time framework overview

We consider the following scalar reaction-diffusion PDE with known constant input delay $D > 0$; and a state $u = u(x,t)$ evolving over the space domain $(0,1)$

$$\begin{cases} u_t(x,t) = u_{xx}(x,t) + \lambda u(x,t) & \forall\ (x,t) \in (0,1) \times \mathbb{R}^+, \\ u(0,t) = 0 & \forall\ t \in \mathbb{R}^+, \\ u(1,t) = U(t-D) & \forall\ t \geq D \\ u(x,0) = u_0(x) & \forall\ x \in (0,1), \end{cases} \tag{4.1}$$

where $\lambda \in \mathbb{R}$, and $U(t) \in \mathbb{R}$ is the control input.

We pose this delay problem as an actuated transport PDE (modeling the delay phenomenon) which cascades into the boundary of the reaction-diffusion PDE,

$$\begin{cases} u_t(x,t) = u_{xx}(x,t) + \lambda u(x,t) & \forall\ (x,t) \in (0,1) \times \mathbb{R}^+, \\ u(0,t) = 0 & \forall\ t \in \mathbb{R}^+, \\ u(1,t) = v(0,t) & \forall\ t \in \mathbb{R}^+, \\ v_t(x,t) = \frac{1}{D} v_x(x,t) & \forall\ (x,t) \in (0,1) \times \mathbb{R}^+, \\ v(1,t) = U(t) & \forall\ t \in \mathbb{R}^+, \\ u(x,0) = u_0(x) & \forall\ x \in (0,1), \\ v(x,0) = v_0(x) & \forall\ x \in (0,1), \end{cases} \tag{4.2}$$

where $u(\cdot,t)$ and $v(\cdot,t)$ are respectively, the reaction-diffusion PDE and the transport PDE states at time t, u_0 , v_0 are given functions, belonging to appropriate spaces to be specified later from Subsection 4.3.2.

For $x \in [0,1]$ and $t \in \mathbb{R}_+$, the solution of the input delay dynamics is given as

$$v(x,t) = v_0\left(\tfrac{1}{D}t + x\right) \text{ for } t \leq D(1-x)$$

and

$$v(x,t) = U(t + D(x-1)) \text{ for } t \geq D(1-x)$$

so that the output

$$v(0,t) = U(t-D)$$

gives the delayed input.

4.2.1 Backstepping stabilization

PDE backstepping method [69] makes use of a Volterra (or a Fredholm) transformation to map the PDE system into a suitable target PDE system on which one can perform Lyapunov stability analysis. An alternative method for stabilization of a parabolic PDE with input delay is *modal decomposition* [90, 62, 22] which relies on separating a finite-dimensional unstable part from a stable infinite-dimensional part of the PDE. Then, one applies the classical predictor-based techniques to the finite-dimensional system and uses for example, spectral analysis, the pole-shifting theorem, and Lyapunov-based techniques. Let us also mention the Both of the aforementioned methods have been the object of further advances, which include, on the one hand, the Fredholm backstepping control for coupled parabolic PDEs with input/output delays [19], and on the other hand, finite-dimensional observer-based control design for parabolic PDEs with delays and sampled-data (using spectral reduction and LMIs-based stability conditions) [61], among others.

In this chapter, we consider the backstepping approach. Then, consider the backstepping transformation

$$w(x,t) = u(x,t) - \int_0^x k(x,y)u(y,t)dy, \tag{4.3}$$

$$z(x,t) = v(x,t) - D\int_0^x q(x,y)v(y,t)dy - \int_0^1 \gamma(x,y)u(y,t)dy, \tag{4.4}$$

for $x \in [0,1]$, where $\gamma(x,y)$, $k(x,y)$ and $q(x,y)$ are the kernels and will be given later. With this transformation we want to map the system (4.2) into the target system:

$$\begin{cases} w_t(x,t) = w_{xx}(x,t) \\ w(0,t) = 0 \\ w(1,t) = z(0,t) \\ Dz_t(x,t) = z_x(x,t) \\ z(1,t) = 0 \end{cases} \tag{4.5}$$

with initial conditions

$$w_0(x) = u_0(x) - \int_0^x k(x,y)u_0(y)dy, \tag{4.6}$$

$$z_0(x) = v_0(x) - \int_0^1 \gamma(x,y)u_0(y)dy - D\int_0^x q(x,y)v_0(y)dy. \tag{4.7}$$

This backstepping transformation is a Voltera invertible transformation whose inverse reads:

$$u(x,t) = w(x,t) + \int_0^x l(x,y)w(y,t)dy, \tag{4.8}$$

$$v(x,t) = z(x,t) + \int_0^1 \delta(x,y)w(y,t)dy + D\int_0^x p(x,y)z(y,t)dy. \tag{4.9}$$

The controller $U(t)$ can be determined thanks to the direct and inverse backstepping transformations. To do so, we will need the explicit expressions of the kernels k, γ and q for the direct transformation (4.3) and l, δ and p for the inverse one.

Let us start with the kernels l, δ and p by calculating the time and spatial derivatives of the transformation (4.8)

$$\begin{aligned}
u_t(x,t) =& w_t(x,t) + \int_0^x l(x,y) w_t(y,t) dy \\
=& w_{xx}(x,t) + \int_0^x l(x,y) w_{yy}(y,t) dy \\
=& w_{xx}(x,t) + [l(x,y) w_x(y,t)]_0^x - \int_0^x l_y(x,y) w_y(y,t) dy \\
=& w_{xx}(x,t) + [l(x,y) w_x(x,t) - l_y(x,y) w(y,t)]_0^x + \int_0^x l_{yy}(x,y) w(y,t) dy \\
u_t(x,t) =& w_{xx}(x,t) + l(x,x) w_x(x,t) - l_x(x,x) w(x) - l(x,0) w_x(0,t) \\
&+ l_y(x,0) w(0,t) + \int_0^x l_{yy}(x,y) w(y,t) dy,
\end{aligned}$$

and using the classical formula:

$$\frac{d}{dx} \int_0^x f(x,y) dy = f(x,x) + \int_0^x f_x(x,y) dy,$$

the spatial derivatives of u are given by:

$$\begin{aligned}
u_x(x,t) =& w_x(x,t) + l(x,x) w(x,t) + \int_0^x l_x(x,y) w(y,t) dy \\
u_{xx}(x,t) =& w_{xx}(x,t) + w(x,t) \frac{d}{dx} l(x,x) + l(x,x) w_x(x,t) + l_x(x,x) w(x,t) \\
&+ \int_0^x l_{xx}(x,y) w(y,t) dy
\end{aligned}$$

Therefore, from the reaction diffusion subsystem we obtain

$$\begin{aligned}
u_t(x,t) - u_{xx}(x,t) - \lambda u(x,t) =& w_{xx}(x,t) + l(x,x) w_x(x,t) - l_y(x,x) w(x,t) \\
&- l(x,0) w_x(0,t) + l_y(x,0) w(0,t) + \int_0^x l_{yy}(x,y) w(y,t) dy - w_{xx}(x,t) \\
&- w(x,t) \frac{d}{dx} l(x,x) - l(x,x) w_x(x,t) - l_x(x,x) w(x,t) - \int_0^x l_{xx}(x,y) w(y,t) dy \\
&- \lambda w(x,t) - \lambda \int_0^x l(x,y) w(y,t) dy,
\end{aligned}$$

which gives, using

$$\frac{d}{dx} l(x,x) = l_x(x,x) + l_y(x,x),$$

and the boundary condition $w(0,t) = 0$:

$$\begin{aligned}
u_t - u_{xx} - \lambda u =& - \left(\lambda + 2 \frac{d}{dx} l(x,x) \right) w(x,t) \\
&+ \int_0^x [l_{yy}(x,y) - l_{xx}(x,y) - \lambda l(x,y)] w(y,t) dy - l(x,0) w_x(0,t) = 0.
\end{aligned}$$

This equation should be valid for all u, so we must have:

$$\begin{cases} l_{yy}(x,y) = l_{xx}(x,y) + \lambda l(x,y) \\ l(x,0) = 0 \\ \dfrac{d}{dx} l(x,x) = -\dfrac{\lambda}{2}. \end{cases} \tag{4.10}$$

The two last equations imply $l(x,x) = -\frac{\lambda}{2}x$. The solution to system (4.10) is given by:

$$l(x,y) = -\lambda y \frac{J_1\left(\sqrt{\lambda(x^2-y^2)}\right)}{\sqrt{\lambda(x^2-y^2)}}, \tag{4.11}$$

on $\mathcal{T} := \{(x,y) : 0 \leq y \leq x \leq 1\}$ where $\mathrm{J}_1(\cdot)$ denotes the Bessel function of first kind (we refer to [71, Appendix A] for an introduction to Bessel function).

For the transport subsystem, from

$$v(x,t) = z(x,t) + \int_0^1 \delta(x,y) w(y,t) dy + D \int_0^x p(x,y) z(y,t) dy \quad \text{and} \quad D z_t = z_x$$

one has:

$$\begin{aligned} v_t(x,t) =& \frac{1}{D} z_x(x,t) + \int_0^1 \delta(x,y) w_{xx}(y,t) dy + D \int_0^x p(x,y) \frac{1}{D} z_y(y,t) dy \\ =& \frac{1}{D} z_x(x,t) + \left[\delta(x,y) w_x(y,t)\right]_0^1 - \int_0^1 \delta_y(x,y) w_y(y,t) dy + \int_0^x p(x,y) z_y(y,t) dy \\ =& \frac{1}{D} z_x(x,t) + \left[\delta(x,y) w_x(y,t) - \delta_y(x,y) w(y,t)\right]_0^1 + \int_0^1 \delta_{yy}(x,y) w(y,t) dy \\ & + \int_0^x p(x,y) z_y(x,y) dy \\ =& \frac{1}{D} z_x(x,t) + \int_0^1 \delta_{yy}(x,y) w(y,t) dy + \int_0^x p(x,y) z_y(y,t) dy \\ & + \delta(x,1) w_x(1,t) - \delta_y(x,1) w(1,t) - \delta(x,0) w_x(0,t) + \delta_y(x,0) w(0). \end{aligned}$$

Therefore, multiplying by D we obtain:

$$\begin{aligned} D v_t =& z_x(x,t) + D \int_0^1 \delta_{yy}(x,y) w(y,t) dy \\ & + D p(x,x) z(x,t) - D p(x,0) z(t,0) \\ & - D \int_0^x p_y(x,y) z(y,t) dy + D \delta(x,1) w_x(1,t) \\ & - D \delta_y(x,1) w(1,t) - D \delta(x,0) w_x(0,t). \end{aligned}$$

Moreover the space derivative of v is given by:

$$v_x(x,t) = z_x(x,t) + \int_0^1 \delta_x(x,y) w(y,t) dy + D p(x,x) z(x,t) + D \int_0^x p_x(x,y) z(y,t) dy$$

so that for all v one gets

$$\begin{aligned} Dv_t(x,t) - v_x(x,t) = &\int_0^1 \left[D\delta_{yy}(x,y) - \delta_x(x,y)\right] w(x,t)dy \\ &- D\int_0^x \left[p_y(x,y) + p_x(x,y)\right] z(x,t)dy \\ &- Dp(x,0)z(0,t) + D\delta(x,1)w_x(1,t) \\ &- D\delta_y(x,1)z(0,t) - D\delta(x,0)w_x(0,t), \end{aligned}$$

where we use the boundary condition $w(t,0) = z(t,0)$. Then we obtain

$$\begin{cases} D\delta_{yy}(x,y) = \delta_x(x,y) \\ p_y(x,y) = -p_x(x,y) \\ p(x,0) = -\delta_y(x,1) \\ \delta(x,1) = 0, \quad \delta(x,0) = 0 \end{cases}$$

whose solution is given (using Separation of Variables [71, Chapter 3]) by

$$\delta(x,y) = 2\sum_{n=1}^{\infty} e^{-Dn^2\pi^2 x} \sin(n\pi y) \int_0^1 \sin(n\pi\zeta) l(1,\zeta)d\zeta, \tag{4.12}$$

and

$$p(x,y) = -\delta_y(x-y,1). \tag{4.13}$$

Using the same strategy on the direct transformation (4.3)-(4.4), we will obtain

$$\gamma(x,y) = 2\sum_{n=1}^{\infty} e^{D(\lambda - n^2\pi^2)x} \sin(n\pi y) \int_0^1 \sin(n\pi\zeta) k(1,\zeta)d\zeta, \tag{4.14}$$

with

$$k(x,y) = -\lambda y \frac{I_1\left(\sqrt{\lambda(x^2-y^2)}\right)}{\sqrt{\lambda(x^2-y^2)}}, \tag{4.15}$$

on $\mathcal{T} := \{(x,y) : 0 \le y \le x \le 1\}$, where $\mathrm{I}_1(\cdot)$ denotes the modified Bessel function of first kind. In addition

$$q(x,y) = -\gamma_y(x-y,1). \tag{4.16}$$

From the boundary condition $z(t,1) = 0$ (in (4.4) and (4.2)) one has

$$v(1,t) - D\int_0^1 q(1,y)v(y,t)dy - \int_0^1 \gamma(1,y)u(y,t)dy = 0$$

which leads to the controller

$$U(t) = \int_0^1 \gamma(1,y)u(y,t)dy + D\int_0^1 q(1,y)v(y,t)dy. \tag{4.17}$$

This continuous-time controller obtained by the Backstepping approach (see also [68]) is used to guarantee the global exponential stability of the closed-loop system (4.2) in H^1-norm as stated in the following theorem.

Theorem 4.1: Existence of solution and global exponential stability ([68])

Consider the closed-loop system (4.2) and the control law (4.17). If the initial conditions are such that $(u_0, v_0) \in L^2(0,1) \times H^1(1, 1+D)$, then the system has a unique solution

$$(u, v) \in C((0, \infty); L^2(0,1) \times H^1(1, 1+D))$$

and there exists a positive continuous function $M : \mathbb{R}^2 \to \mathbb{R}_+$ such that

$$E(t) \leq M(\lambda, D) e^{cD} E(0) e^{-\min(2,c)t}, \quad \forall t \geq 0$$

for any $c > 0$, where

$$E(t) = \int_0^1 u^2(x,t)dx + \int_1^{1+D} \left(v^2(x,t) + v_x^2(x,t)\right) dx.$$

A H^1-norm based Lyapunov approach is used in [68] to prove this theorem, but for the design of the triggering policy and for the stability analysis we will base our approach on Input-to-State stability and small-gain arguments [58].

4.3 Event-triggering boundary control

4.3.1 Problem formulation

Under the emulation approach, the boundary controller is perfectly known (i.e., the nominal control given in (4.17)). We aim at stabilizing closed-loop system (4.2) on events while updating the controller $U(t)$ (4.17) at certain time $\{t_j\}_j$ defined by an event-triggered mechanism. To that end, we consider the following event-triggered boundary control:

$$U_d(t) = \int_0^1 \gamma(1,y)u(y,t_j)dy + D\int_0^1 q(1,y)v(y,t_j)dy, \quad \forall t \in [t_j, t_{j+1}). \tag{4.18}$$

The updates times $\{t_j\}_j$ form an increasing sequence and are such that the value of the control is held constant between two successive events and is updated when some triggering condition is verified. The chosen event-triggering law will be given later and depends on the evolution of the system's state. Thus, the boundary value of the state is modified as $v(t,1) = U_d(t)$, for all $t \in [t_j, t_{j+1})$, $j \geq 0$.

Note that $U_d(t) = U(t) + d(t)$ with $U(t)$ given by (4.17) and d given by:

$$d(t) = \int_0^1 \gamma(1,y)\left(u(y,t_j) - u(y,t)\right)dy + D\int_0^1 q(1,y)\left(v(y,t_j) - v(y,t)\right)dy \tag{4.19}$$

where d can be viewed as an actuation deviation (or input holding error).

Therefore, the control problem we aim at handling can be reformulated as follows:

$$\begin{cases} u_t(x,t) = u_{xx}(x,t) + \lambda u(x,t), \\ u(0,t) = 0, \\ u(1,t) = v(0,t), \\ v_t(x,t) = \frac{1}{D} v_x(x,t), \\ v(1,t) = U_d(t), \end{cases} \tag{4.20}$$

with $U_d(t)$ being defined in (4.18) for all $t \in [t_j, t_{j+1})$, $j \geq 0$.

Since we need to assess the impact of the deviation $d(t)$ to the closed-loop system under the event-triggered implementation, we use the previous backstepping transformations so that we can work on a target system with desired stability properties and that exhibits the deviation $d(t)$ at the boundary. The backstepping transformation is defined in (4.3)-(4.4). Hence, the system (4.20) is transformed into the following target system:

$$\begin{cases} w_t(x,t) = w_{xx}(x,t), \\ w(0,t) = 0, \\ w(1,t) = z(0,t), \\ z_t(x,t) = \frac{1}{D} z_x(x,t), \\ z(1,t) = d(t), \end{cases} \tag{4.21}$$

with initial conditions w_0 (4.6) and z_0 (4.7). Notice that when $d(t) = 0$, the target system (4.21) is evidently globally exponential stable.

It is worth recalling that the backstepping transformation (4.3)-(4.4) is invertible with inverse transformation given by (4.8)-(4.9). Using the inverse transformation, we can rewrite (4.18) and (4.19) as a function of the states w and z, i.e.,

$$U_d(t) = \int_0^1 \delta(1,y) w(y,t_j) dy + D \int_0^1 p(1,y) z(y,t_j) dy, \tag{4.22}$$

and for all $t \in [t_j, t_{j+1})$, $j \geq 0$

$$d(t) = \int_0^1 \delta(1,y) \left(w(y,t_j) - w(y,t)\right) dy + D \int_0^1 p(1,y) \left(z(y,t_j) - z(y,t)\right) dy. \tag{4.23}$$

4.3.2 Well-posedness

System (4.21) can be subdivided into two subsystems for all $t \in [t_j, t_{j+1})$, $j \geq 0$ and $x \in (0,1)$: the heat subsystem

$$\begin{cases} u_t(x,t) = u_{xx}(x,t) + \lambda u(x,t) \\ u(0,t) = 0 \\ u(1,t) = v(0,t) \\ u(0,x) = u_0(x) \end{cases} \tag{4.24}$$

and the transport subsytem

$$\begin{cases} D v_t(x,t) = v_x(x,t) \\ v(1,t) = U_d(t) \\ v(0,x) = v_0(x). \end{cases} \tag{4.25}$$

Similar to [28] and [57], in this chapter we deal with a linear hyperbolic equation subject to a discontinuous boundary input. The discontinuous signal gets into the reaction-diffusion PDE through the boundary. Consequently, the well-posedness study requires to extend the case in [32] along with [28, 57] in order to be able to construct the solution for the closed-loop PDE-PDE system. This is done by means of the following proposition.

Proposition 4.2: Well-posedness

For every initial data $v_0 \in \mathcal{C}^1_{rpw}([0,1],\mathbb{R})$ and $u_0 \in L^2(0,1)$, there exist unique solutions u, v to (4.20) with the following properties:

- v is the unique solution to (4.25) in the sense of characteristics on $[0,1]\times[0,T^*)$. Moreover,

$$\forall t \in [0,T^*), \quad v(\cdot,t) \in \mathcal{C}^1_{rpw}([0,1],\mathbb{R})$$

  and for all $x \in [0,1]$, $v(\cdot,x) \in \mathcal{C}^1_{rpw}([0,T^*),\mathbb{R})$.

- $u \in C^0\left([0,T^*);L^2(0,1)\right)$ with $u(\cdot,t) \in C^2([0,1])$ for $t \in (0,T^*)$ and $u \in C^1(\tilde{I}\times[0,1])$ where $\tilde{I} = [0,T^*)\setminus\{t_j : j = 0,1,2,...\}$, which also satisfies (4.20) for $t \in \tilde{I}$.

Proof : Let us focus first on the v-system of (4.25). Following similar arguments as in [28], let us define for $k \in \mathbb{N}$, the interval

$$\Delta_k := [kD,(k+1)D] \subset [0,T^*),$$

where D is the time for the transport equation with velocity $1/D$ to cross the spatial domain $[0,1]$. By the method of characteristics, the explicit solution of (4.25), for a given initial data $v(\cdot,kD) \in \mathcal{C}^1_{rpw}([0,1],\mathbb{R})$ is as follows:

$$v(x,t) = \begin{cases} v\Big(\frac{1}{D}(t-kD)+x, kD\Big), & kD \le t < kD + D(1-x) \\ U_d(t+D(x-1)), & kD + D(1-x) \le t \le (k+1)D, \end{cases} \tag{4.26}$$

for all $t \in \Delta_k$, $k \in \mathbb{N}$. It follows then, from (4.26), that v is well-defined on $\Delta_k \times [0,1]$. Moreover, by definition of U_d (being a piecewise constant function and assuming $U_d(t) \in \mathcal{C}^1_{rpw}(\Delta_k,\mathbb{R})$) we have that $U_d(t+D(x-1))$ belongs to $\mathcal{C}^1_{rpw}$ with respect to both t and x.

In addition, $v\Big(\frac{1}{D}(t-kD)+x, kD\Big)$ belongs to $\mathcal{C}^1_{rpw}$ with respect to both t and x. Therefore, from (4.26) it holds that $v(\cdot,t) \in \mathcal{C}^1_{rpw}([0,1],\mathbb{R})$ for all $t \in \Delta_k$ and $v(\cdot,x) \in \mathcal{C}^1_{rpw}(\Delta_k,\mathbb{R})$ for all $x \in [0,1]$.

This yields $v(0,t) \in \mathcal{C}^1_{rpw}(\Delta_k,\mathbb{R})$ which constitutes an allowable boundary input for the u-system (4.24). Indeed, since it is piecewise continuous with the required regularity properties, we can apply [59, Theorem 4.10] for system (4.20) on the interval Δ_k. We obtain then, that for any initial data $u(kD,\cdot) \in L^2(0,1)$, there exists a unique function $u \in C^0\left(\Delta_k;L^2(0,1)\right)$ with $u(\cdot,t) \in C^2([0,1])$ for $t \in$

$\Delta_k \backslash \{kD, (k+1)D\}$ and $u \in C^1(\tilde{I}_k \times [0,1])$ where $\tilde{I}_k = \Delta_k \backslash \{t_j : j = 0, 1, 2, ...\}$, $k \in \mathbb{N}$ which also satisfies (4.20) for $t \in \tilde{I}_k$.

Therefore, by the step-by-step method, we can construct the solution for all $[0, T^*)$, i,e.,

• solutions v on $[0, T^*) \times [0, 1]$ such that for all $t \in [0, T^*)$, $\quad v(\cdot, t) \in \mathcal{C}^1_{rpw}([0,1], \mathbb{R})$ and for all $x \in [0,1]$, $v(\cdot, x) \in \mathcal{C}^1_{rpw}([0, T^*), \mathbb{R})$;

• $u \in C^0([0, T^*); L^2(0,1))$ with $u(\cdot, t) \in C^2([0,1])$ for $t \in (0, T^*)$ and $u \in C^1(\tilde{I} \times [0,1])$ where $\tilde{I} = [0, T^*) \setminus \{t_j : j = 0, 1, 2, ...\}$ which also satisfies (4.24) for $t \in \tilde{I}$.

This concludes the proof. ♢

4.3.3 Event-triggered control strategy

In this section we introduce the event-triggered boundary control law and the main results: avoidance of the Zeno phenomenon and exponential stability of the event-triggered controlled system. The event-triggered boundary control considered in this chapter involves a triggering condition and the backstepping boundary feedback which is applied as Zero-Order Hold (the value of the feedback is held constant during each discrete time interval). The proposed event-triggering condition is based on the evolution of the magnitude of the actuation deviation and the energy of the coupled reaction-diffusion and transport system's state.

Definition 4.3: Definition of the event-triggered boundary control

Let $\beta > 0$ be a design parameter and define the following set:

$$\mathcal{H}(t_j) := \{t > t_j : |d(t)| > \beta \max_{t_j \leq s \leq t}(\|w(\cdot, s)\|) + \beta \max_{t_j \leq s \leq t}(\|z(\cdot, s)\|_\infty)\} \quad (4.27)$$

where w and z are the solution of (4.21) for all $t \geq t_j$ and $d(t)$ is defined by (4.23).
The event-triggered boundary control is defined by considering the following components:
I) (The event-triggering condition) The times of the events $t_j \geq 0$ with $t_0 = 0$ form a finite or countable set of times which is determined by the following rules for any $j \geq 0$:

a) if $\mathcal{H}(t_j) = \emptyset$ then the set of the times of the events is $\{t_0, ..., t_j\}$.

b) if $\mathcal{H}(t_j) \neq \emptyset$, then the next event time is given by:

$$t_{j+1} := \inf \mathcal{H}(t_j). \quad (4.28)$$

II) (The control action) The boundary feedback law is defined by,

$$U_d(t) = \int_0^1 \delta(1, y) w(y, t_j) dy + D \int_0^1 p(1, y) z(y, t_j) dy, \quad (4.29)$$

for all $t \in [t_j, t_{j+1})$.

Remark 4.1 *In Definition 4.3, we use the L^∞- norm for the transport PDE subsystem. It is worth recalling that in [68], a Lyapunov-based stability analysis is performed using the H^1-norm for the transport PDE subsystem (instead of the L^2- norm) mainly due to the unboundedness of the operator in the interconnection parabolic-transport PDEs. One may indeed have trace terms that cannot be estimated using the L^2- norm. Therefore, H^1-norm turned out to be suitable in that work. Nevertheless, in this event-triggered framework, working with Lyapunov-based techniques using the H^1-norm would be problematic as this needs to have regularity*

on the initial data and solutions to be at least absolutely continuous. This is not possible in the present setting as we deal with piecewise-constant input signals in the transport PDE and discontinuities propagating through the spatial domain. Therefore, a L^∞- norm is suitable for the transport PDE subsystem, thanks to which it is possible: i) to rely on the required regularity for the analysis for the well-posedness (as established in Subsection 4.3.2) of the overall closed-loop system, ii) to obtain a suitable ISS estimate allowing a subsequent small-gain-based stability analysis (see Section 4.3.5).

4.3.4 Avoidance of the Zeno Phenomenon

In contrast to the method presented in Chapters 2 and 3, where we establish the avoidance of Zeno behavior through proof by contradiction, the focus of this chapter shifts. Here, we will give a demonstration showcasing the existence of a minimum dwell-time between two distinct triggering times. Importantly, this dwell-time is entirely independent of both the initial condition and the time $T^* = \lim_{j\to+\infty}(t_j)$.

Theorem 4.4: Minimal dwell-time

Consider the closed-loop system (4.20) with the event-triggered boundary control (4.28)-(4.29) in Definition 4.3 with $\beta > 0$ be given. Then, there exists a minimal dwell-time between two triggering times, i.e. there exists a constant $\tau > 0$ (independent of the initial conditions u_0, v_0 and of T^*) such that

$$t_{j+1} - t_j \geq \tau, \text{ for all } j \geq 0.$$

Proof : Let us focus on the deviation of actuation given in (4.23), expressed in terms of the dynamics of the target system (4.21) and the kernels of the inverse transformation. We recall the formula here:

$$d(t) = \int_0^1 \delta(1,y)\left(w(y,t_j) - w(y,t)\right)dy + D\int_0^1 p(1,y)\left(z(y,t_j) - z(y,t)\right)dy.$$

Proposition 4.2 in conjunction with the backstepping transformations (4.3) allow to assert that target system is well-posed as $d(t)$ can be proved to belong to $\mathcal{C}^1_{rpw}([0,T^*),\mathbb{R})$. Following similar arguments as in [57, Section 3], it can be further proved that the following differential equation holds, for $t \in (t_j, t_{j+1})$, $j \geq 0$:

$$\begin{aligned}
\dot{d}(t) =& -\int_0^1 \delta(1,y)w_t(y,t)dy - D\int_0^1 p(1,y)z_t(y,t)dy \\
=& -\int_0^1 \delta(1,y)w_{yy}(y,t)dy - \int_0^1 p(1,y)z_y(y,t)dy \\
=& -\delta(1,1)w_y(1,t) + \delta(1,0)w_y(0,t) + \delta_y(1,1)w(1,t) \\
& -\delta_y(1,0)w(0,t) - p(1,1)z(1,t) + p(1,0)z(0,t) \\
& -\int_0^1 \delta_{yy}(1,y)w(y,t)dy + \int_0^1 p_y(1,y)z(y,t)dy. \qquad (4.30)
\end{aligned}$$

Knowing that $w(0,t)=0$, and from (4.12)-(4.13), one has $\delta(1,1)=\delta(1,0)=0$ and $-\delta_y(1,1)=p(1,0)$. Moreover, $w(1,t)=z(0,t)$ and $z(1,t)=d(t)$, we get

$$\dot{d}(t)=-p(1,1)d(t)-\int_0^1\delta_{yy}(1,y)w(y,t)dy+\int_0^1 p_y(1,y)z(y,t)dy. \tag{4.31}$$

Thus, the following inequality holds for $t\in(t_j,t_{j+1})$:

$$|\dot{d}(t)|\leq a_0|d(t)|+\int_0^1|\delta_{yy}(1,y)w(y,t)|dy+\int_0^1|p_y(1,y)z(y,t)|dy. \tag{4.32}$$

with $a_0=p(1,1)>0$ as we will show in the following.

Since from (4.13), $p(1,1)=-\delta_y(0,1)$ and from the boundary condition and the inverse transformation (4.11), we obtain:

$$\begin{aligned}u(1,t)=v(0,t)&\Rightarrow w(1,t)+\int_0^1 l(1,y)w(y,t)dy=z(0,t)+\int_0^1\delta(0,y)w(y,t)dy\\&\Rightarrow\int_0^1\left[l(1,y)-\delta(0,y)\right]w(y,t)dy=0 \text{ since } w(1,t)=z(0,t) \text{ from (4.21)}\\&\Rightarrow l(1,y)=\delta(0,y)\qquad\forall y\in[0,1]\\&\Rightarrow l_y(1,y)=\delta_y(0,y),\qquad\forall y,\end{aligned}$$

then $a_0:=p(1,1)=-\delta_y(0,y)=-l_y(1,1)$. Using (4.11) and (4.13) together with the fact that $\frac{d}{d\rho}(\rho^{-1}J_1(\rho))=-\rho^{-1}J_2(\rho)$ and $\lim\limits_{\rho\to0}\frac{J_n(\rho)}{\rho^n}=\frac{1}{2^n n!}$ (see for example [71]) we get:

$$l(1,y)=-\lambda y\rho^{-1}(y)J_1(\rho(y)) \text{ with } \rho(y)=\sqrt{\lambda(1-y^2)}$$

and

$$\begin{aligned}l_y(1,y)&=-\lambda\frac{J_1\left(\sqrt{\lambda(1-y^2)}\right)}{\sqrt{\lambda(1-y^2)}}-\lambda^2y^2\frac{J_2\left(\sqrt{\lambda(1-y^2)}\right)}{\lambda(1-y^2)}\\l_y(1,1)&=-\lambda\lim_{y\to1}\frac{J_1\left(\sqrt{\lambda(1-y^2)}\right)}{\sqrt{\lambda(1-y^2)}}-\lambda^2\lim_{y\to1}\frac{J_2\left(\sqrt{\lambda(1-y^2)}\right)}{\lambda(1-y^2)}\\&=-\lambda\lim_{\rho\to0}\frac{J_1(\rho)}{\rho}-\lambda^2\lim_{\rho\to1}\frac{J_2(\rho)}{\rho^2}=-\frac{\lambda}{2}-\frac{\lambda^2}{8},\end{aligned}$$

so that we obtain the explicit value of a_0:

$$a_0=\frac{\lambda}{2}+\frac{\lambda^2}{8}. \tag{4.33}$$

Using the absolute continuity of $d(t)$ on (t_j,t_{j+1}), we get from (4.32), for all $t\in[t_j,t_{j+1})$

$$\begin{aligned}|d(t)|&\leq e^{a_0(t-t_j)}|d(t_j)|\\&\quad+\int_{t_j}^t e^{a_0(t-t_j)}\times\Big(\int_0^1|\delta_{yy}(1,y)w(y,s)|dy+\int_0^1|p_y(1,y)z(y,s)|dy\Big)ds.\end{aligned} \tag{4.34}$$

Therefore, from the Cauchy–Schwarz inequality and the fact $d(t_j) = 0$, we get the following estimate:

$$\begin{aligned} |d(t)| \leq \Big(\int_0^1 |p_y(y,t)|dy\Big) \int_{t_j}^t \|z(\cdot,s)\|_\infty e^{a_0(t-s)}ds \\ + \|\delta_{yy}(1,\cdot)\|_{L^2} \int_{t_j}^t \|w(\cdot,s)\| e^{a_0(t-s)}ds. \end{aligned} \tag{4.35}$$

Moreover, it holds for all $t \in [t_j, t_{j+1})$

$$|d(t)| \leq \Gamma_{a_0}(t-t_j)\Big(a_1 \max_{t_j\leq s\leq t}(\|z(s,\cdot)\|_\infty) + a_2 \max_{t_j\leq s\leq t}(\|w(s,\cdot)\|_{L^2})\Big), \tag{4.36}$$

where

$$\Gamma_{a_0}(s) := \tfrac{1}{a_0}\Big(e^{a_0 s} - 1\Big) > 0, \tag{4.37}$$

$$a_1 := \int_0^1 |p_y(1,y)|dy, \tag{4.38}$$

$$a_2 := \|\delta_{yy}(1,\cdot)\|_{L^2}. \tag{4.39}$$

Using (4.36) and assuming that an event is triggered at $t = t_{j+1}$, we have

$$|d(t_{j+1})| \leq \Gamma_{a_0}(t_{j+1}-t_j)\Big(a_1 \max_{t_j\leq s\leq t_{j+1}}(\|z(\cdot,s)\|_\infty) + a_2 \max_{t_j\leq s\leq t_{j+1}}(\|w(\cdot,s)\|_{L^2})\Big), \tag{4.40}$$

which, together with Definition 4.3, yields the following inequality:

$$\begin{aligned} &\beta\Big(\max_{t_j\leq s\leq t_{j+1}}(\|z(s,\cdot)\|_\infty) + \max_{t_j\leq s\leq t_{j+1}}(\|w(s,\cdot)\|_{L^2})\Big) \\ &\leq \Gamma_{a_0}(t_{j+1}-t_j)\Big(a_1 \max_{t_j\leq s\leq t_{j+1}}(\|z(s,\cdot)\|_\infty) + a_2 \max_{t_j\leq s\leq t_{j+1}}(\|w(s,\cdot)\|_{L^2})\Big), \end{aligned}$$

Therefore, we get easily

$$0 < \frac{\beta}{\max\{a_1,a_2\}} \leq \Gamma_{a_0}(t_{j+1}-t_j). \tag{4.41}$$

Using the definition (4.37) and from (4.41), we can conclude, for all $j \geq 0$

$$t_{j+1} - t_j \geq \frac{1}{a_0}\ln\Big(1 + \frac{a_0\beta}{\max\{a_1,a_2\}}\Big) =: \tau > 0, \tag{4.42}$$

which is a minimal dwell-time (independent on the initial conditions and of T^*). This concludes the proof. ♢

Theorem 4.4 allows to conclude that $T^* = \lim_{j\to+\infty}(t_j) = +\infty$ and therefore we can apply Proposition 4.2 to finally get the following well-posedness result of the closed-loop system (4.20).

Corollary 4.5:

For every initial data $v_0 \in \mathcal{C}^1_{rpw}([0,1],\mathbb{R})$ and $u_0 \in L^2(0,1)$, there exist a unique solution (u,v) to (4.20) with the following properties:

- v is the unique solution to (4.25) on $\mathbb{R}_+ \times [0,1]$. Moreover, for all $t \in \mathbb{R}_+$, $v(\cdot,t) \in \mathcal{C}^1_{rpw}([0,1],\mathbb{R})$ and for all $x \in [0,1]$, $v(x,\cdot) \in \mathcal{C}^1_{rpw}(\mathbb{R}_+,\mathbb{R})$.
- $u \in C^0\left(\mathbb{R}_+; L^2(0,1)\right)$ with $u(t,\cdot) \in C^2([0,1])$ for $t \in \mathbb{R}_+$ and $u \in C^1(I \times [0,1])$ where $I = \mathbb{R}_+ \backslash \{t_j : j = 0,1,2,...\}$ which also satisfies (4.20) for $t \in I$.

Proof : It is an immediate consequence of Proposition 4.2 and Theorem 4.4 (which guarantees that no Zeno solution can appear). ♢

4.3.5 Exponential stability

In this section, we derive the exponential stability result for the closed-loop system (4.20). To that end, we seek an Input-to-State Stability (ISS) property of the target system (4.21) with respect to the deviation $d(t)$, and we follow small-gain arguments.

We will begin by establishing intermediary outcomes that we will subsequently leverage to demonstrate the exponential stability of the system. These outcomes will pertain to the following heat subsystem defined in $(0,1) \times \mathbb{R}$

$$\begin{cases} w_t(x,t) = w_{xx}(x,t) \\ w(0,t) = 0 \\ w(1,t) = z(0,t) \\ w(0,x) = w_0(x) \end{cases} \tag{4.43}$$

and the transport subsytem

$$\begin{cases} Dz_t(x,t) = z_x(x,t) \\ z(1,t) = 0 \\ z(0,x) = z_0(x). \end{cases} \tag{4.44}$$

We define the following quantities for all $t \geq 0$:

$$\|w\|_{[0,t]} := \max_{0 \leq s \leq t} \left(\|w(\cdot,s)\|_{L^2} e^{\delta s} \right), \tag{4.45}$$

$$\|z\|_{[0,t]} := \max_{0 \leq s \leq t} \left(\|z(\cdot,s)\|_{\infty} e^{\delta s} \right). \tag{4.46}$$

Lemma 4.6: ISS-like estimate for the heat and transport PDEs

For every $\varepsilon > 0$, there exists $\delta > 0$ such that the solution to systems (4.44) and (4.43) the following inequalities hold:

$$\|w\|_{[0,t]} \leq \|w_0\|_{L^2} + \tfrac{1}{\sqrt{3}}(1+\varepsilon)\|z\|_{[0,t]}, \tag{4.47}$$

$$\|z\|_{[0,t]} \leq e^{D}\|z_0\|_\infty + e^{D(\mu+1+\varepsilon)} \max_{0\leq s\leq t}\left(|d(s)|e^{\delta s}\right). \tag{4.48}$$

Proof : Exponential L^2-stabilization results of parabolic PDEs is adressed thanks to the backstepping method in [59]. The system

$$\begin{cases} y_t(x,t) = py_{xx}(x,t) + cy(x,t) \\ y(0,t) = U(t) \\ y(1,t) = 0 \\ y(x,0) = y_0(x) \end{cases} \tag{4.49}$$

or equivalently (with the change of unknown $x \mapsto 1-x$)

$$\begin{cases} y_t(x,t) = py_{xx}(x,t) + cy(x,t) \\ y(1,t) = U(t) \\ y(0,t) = 0 \\ y(x,0) = y_0(x) \end{cases} \tag{4.50}$$

is transformed into the system

$$\begin{cases} w_t(x,t) = pw_{xx}(x,t) - Kw(x,t) \\ w(0,t) = w(1,t) = 0 \\ w(x,0) = w_0(x) \end{cases} \tag{4.51}$$

throught the backstepping transformation:

$$y(x,t) = w(x,t) + \int_x^1 l(x,s)w(s,t)ds \qquad \forall (x,t) \in \mathbb{R}\times(0,1).$$

The eigenvalues and eigenfunctions of system (4.51) are

$$\lambda_n = K + pn^2\pi^2$$

$$\phi_n(x) = \sqrt{2}\sin(\pi n x) \quad \forall n \in \mathbb{N}.$$

Henceforth, using the L^2_r-stability result ([59, Theorem 5.3]) where

$$L^2_r(0,1) := \left\{ f : [0,1] \to \mathbb{R} / \|f\|_r^2 = \int_0^1 r(x)|f(x)|^2 dx \right\},$$

one has for $r(x) = p(x) = 1, K = 0, a_1 = b_1 = 1, a_2 = b_2 = 0, d_1(t) = 0$ and $d_0(t) = z(t,0)$,

$$\|w(\cdot,t)\|_{L^2} \leq e^{-\pi^2 t}\|w_0\|_{L^2} + G \max_{0\leq s\leq t}(|z(s,0)|), \tag{4.52}$$

with

$$G = \frac{p(0)}{b_1^2 + b_2^2}\sqrt{\sum_{n=1}^{\infty} \frac{1}{\lambda_n^2} \left|b_1\phi_n'(0) - b_2\phi_n(0)\right|^2} = \frac{\sqrt{2}}{\pi}\sqrt{\sum_{n=1}^{\infty} \frac{1}{n^2}} = \frac{1}{\sqrt{3}}.$$

Regarding the transport subsystem (4.44), we can derive an ISS estimate expressed in the sup-norm of the state by applying [59, Proposition 3.2] on ISS estimate in L^p-norm and take the limit since $\lim_{p\to\infty} \|z\|_p = \|z\|_\infty$ for

$$\varepsilon_1(x) \equiv \varepsilon_2(x) \equiv \varepsilon_3(x) \equiv 1, R(x) \equiv \varphi(x) \equiv 0, a = 0, f = 0$$

and $r(x) = e^{-D\sigma x}$ one gets for all $t \geq 0$ and $\mu > 0$:

$$\|z(\cdot,t)\|_\infty \leq e^{-\mu(t-D)+D}\|z_0\|_\infty + e^{D(1+\mu)} \max_{0\leq s\leq t}\left(|d(s)|\right). \tag{4.53}$$

Using the Fading memory inequality (see Appendix A.5) or [59, Lemma 7.1], we guarantee that there exists $\delta > 0$ such that the following fading memory estimates holds for all $t \geq 0$:

$$\|w(\cdot,t)\|e^{\delta t} \leq \|w_0\| + \tfrac{1}{\sqrt{3}}(1+\varepsilon)\max_{0\leq s\leq t}(\|z(s,\cdot)\|_\infty e^{\delta t}), \tag{4.54}$$

$$\|z(\cdot,t)\|_\infty e^{\delta t} \leq e^{D}\|z_0\|_\infty + e^{D(\mu+1)}(1+\varepsilon)\max_{0\leq s\leq t}\left(|d(s)|e^{\delta s}\right). \tag{4.55}$$

Using (4.54)-(4.55) and the definitions (4.45)-(4.46), we get for all $t \geq 0$ the inequalities (4.47) and (4.48). ♢

In the proof of the main stability result of this section, we will also need to estimate the deviation $d(t)$ defined in (4.23).We will prove the following estimate which is a conversion of an ISS-like inequality in the max-formulation to a Fading Memory Inequality.

Lemma 4.7:

For every $\varepsilon > 0, \beta > 0$, there exists a constant $\delta > 0$ such that the following inequality holds:

$$\max_{0\leq s\leq t}\left(|d(s)|e^{\delta s}\right) \leq \beta(1+\varepsilon)\|w\|_{[0,t]} + \beta(1+\varepsilon)\|z\|_{[0,t]}. \tag{4.56}$$

Proof : From Definition 4.3, events are triggered to guarantee, for all $t_j \geq 0$ and $t \geq t_j$.

$$|d(t)| \leq \beta \max_{t_j\leq s\leq t}(\|w(s,\cdot)\|_{L^2}) + \beta \max_{t_j\leq s\leq t}(\|z(s,\cdot)\|_\infty). \tag{4.57}$$

Notice that (4.57) can be read as e.g.,

$$|d(t)| \leq e^{(-\pi(t-t_j))}|d(t_j)| + \beta \max_{t_j\leq s\leq t}(\|w(s,\cdot)\|_{L^2}) + \beta \max_{t_j\leq s\leq t}(\|z(s,\cdot)\|_\infty),$$

knowing that that $|d(t_j)| = 0$. Using again the fading memory estimate ($|d(t)|$ being locally bounded which is indeed guaranteed by the triggering law), then the following inequality holds for all $t \geq 0$:

$$\begin{aligned} |d(t)| \leq\ & e^{-\delta t}|d(0)| + \beta(1+\varepsilon)\max_{0\leq s\leq t}(\|w(s,\cdot)\|_{L^2}e^{-\delta(t-s)}) \\ & + \beta(1+\varepsilon)\max_{0\leq s\leq t}(\|z(s,\cdot)\|_\infty e^{-\delta(t-s)}) \end{aligned} \tag{4.58}$$

with δ, ε as in (4.54)-(4.55). Hence, since $|d(0)| = 0$ we obtain the following estimate:

$$\begin{aligned} |d(t)|e^{\delta t} \leq & \beta(1+\varepsilon) \max_{0\leq s\leq t}(\|w(s,\cdot)\|_{L^2}e^{\delta s}) \\ & + \beta(1+\varepsilon) \max_{0\leq s\leq t}(\|z(s,\cdot)\|_{\infty}e^{\delta s}). \end{aligned} \tag{4.59}$$

Using definitions (4.45)-(4.46), we get (4.56). ♢

Now, we can state and prove the main result of this section.

Theorem 4.8: Exponential stability

Let $\beta > 0$ be a design parameter (involved in the triggering condition (4.28)) that is selected in such a way that the following condition is fulfilled:

$$\beta < \frac{e^{-D}}{\left(1+\frac{\sqrt{3}}{3}\right)}. \tag{4.60}$$

Then, the closed-loop system (4.20) with event-triggered boundary control (4.28)-(4.29) is globally exponentially stable. More specifically, there exist constants $M, \delta > 0$ such that:

$$\|u(\cdot,t)\| + \|v(\cdot,t)\|_{\infty} \leq Me^{-\delta t}\left(\|u_0\|_{L^2} + \|v_0\|_{\infty}\right), \tag{4.61}$$

for all $t \geq 0$.

Proof : By virtue of condition (4.60), there exist constants $\varepsilon, \mu > 0$, such that

$$\beta\left(\tfrac{1}{\sqrt{3}}(1+\varepsilon)^3e^{\mu D} + (1+\varepsilon)^2e^{\mu D})\right)e^{D} < 1. \tag{4.62}$$

Indeed, the existence of ϵ, and $\mu > 0$ is guaranteed since the function

$$h_1(\varepsilon,\mu) := \beta\left(\tfrac{1}{\sqrt{3}}(1+\varepsilon)^3e^{\mu D} + (1+\varepsilon)^2e^{\mu D})\right)e^{D}$$

is continuous at $(0,0)$ and satisfies $h_1(0,0) < 1$. Condition (4.62), in turn, implies the following condition:

$$\beta(1+\varepsilon)^2e^{D(\mu+1)} < 1. \tag{4.63}$$

Therefore, using (4.56) along with (4.47)-(4.48), we get

$$\|w\|_{[0,t]} \leq \|w_0\|_{L^2} + \tfrac{1}{\sqrt{3}}(1+\varepsilon)\|z\|_{[0,t]}, \tag{4.64}$$

and

$$\begin{aligned} \|z\|_{[0,t]} \leq & e^{D}\|z_0\|_{\infty} + e^{D(\mu+1)}(1+\varepsilon)^2\beta\|w\|_{[0,t]} \\ & + e^{D(\mu+1)}(1+\varepsilon)^2\beta\|z\|_{[0,t]}. \end{aligned} \tag{4.65}$$

From (4.65) and since (4.63) holds, we have

$$\|z\|_{[0,t]} \leq \left(1-\beta(1+\varepsilon)^2\phi\right)^{-1}e^{D}\|z_0\|_{\infty} + \beta(1+\varepsilon)^2\phi\left(1-\beta(1+\varepsilon)^2\phi\right)^{-1}\|w\|_{[0,t]}, \tag{4.66}$$

where

$$\phi := e^{D(1+\mu)}. \tag{4.67}$$

Then,

$$\begin{aligned}\|w\|_{[0,t]} \leq \|w_0\|_{L^2} + \tfrac{1}{\sqrt{3}}(1+\varepsilon)\left(1-\beta(1+\varepsilon)^2\phi\right)^{-1} e^D\|z_0\|_\infty \\ + \tfrac{1}{\sqrt{3}}\beta(1+\varepsilon)^3\phi\left(1-\beta(1+\varepsilon)^2\phi\right)^{-1}\|w\|_{[0,t]}.\end{aligned} \tag{4.68}$$

Therefore,

$$\begin{aligned}\|w\|_{[0,t]} \leq (1-\beta(1+\varepsilon)^3\psi)^{-1}\|w_0\|_{L^2} \\ + \tfrac{1}{\sqrt{3}}(1+\varepsilon)\left(1-\beta(1+\varepsilon)^3\psi\right)^{-1} \times \left(1-\beta(1+\varepsilon)^2\phi\right)^{-1} e^D\|z_0\|_\infty,\end{aligned} \tag{4.69}$$

where

$$\psi := \tfrac{1}{\sqrt{3}}\phi(1-\beta(1+\varepsilon)^2\phi)^{-1}. \tag{4.70}$$

On the other hand, from (4.64) and (4.66), we have

$$\begin{aligned}\|z\|_{[0,t]} \leq \left(1-\beta(1+\varepsilon)^2\phi\right)^{-1} e^D\|z_0\|_\infty + \beta(1+\varepsilon)^2\phi\left(1-\beta(1+\varepsilon)^2\phi\right)^{-1}\|w_0\|_{L^2} \\ + \tfrac{1}{\sqrt{3}}\beta(1+\varepsilon)^3\phi\left(1-\beta(1+\varepsilon)^2\phi\right)^{-1}\|z\|_{[0,t]},\end{aligned} \tag{4.71}$$

and since (4.62) holds, then

$$\begin{aligned}\|z\|_{[0,t]} \leq (1-\beta(1+\varepsilon)^3\psi)^{-1}\left(1-\beta(1+\varepsilon)^2\phi\right)^{-1} e^D\|z_0\|_\infty \\ + \beta(1+\varepsilon)^2(1-\beta(1+\varepsilon)^3\psi)^{-1}\phi\left(1-\beta(1+\varepsilon)^2\phi\right)^{-1}\|w_0\|_{L^2}.\end{aligned} \tag{4.72}$$

Combining (4.69) and (4.72), we get

$$\begin{aligned}\|w\|_{[0,t]} + \|z\|_{[0,t]} \leq (1-\beta(1+\varepsilon)^3\psi)^{-1}\left(1+\beta(1+\varepsilon)^2\phi\left(1-\beta(1+\varepsilon)^2\phi\right)^{-1}\right)\|w_0\|_{L^2} \\ + (1-\beta(1+\varepsilon)^3\psi)^{-1}\left(1-\beta(1+\varepsilon)^2\phi\right)^{-1}(1+\tfrac{1}{\sqrt{3}}(1+\varepsilon))e^D\|z_0\|_\infty.\end{aligned} \tag{4.73}$$

Hence,

$$\begin{aligned}\|w(\cdot,t)\| + \|z(\cdot,t)\|_\infty \leq M_0 e^{-\delta t}\|w_0\| \\ + M_0(1+\tfrac{1}{\sqrt{3}}(1+\varepsilon))e^D e^{-\delta t}\|z_0\|_\infty,\end{aligned} \tag{4.74}$$

with

$$M_0 := (1-\beta(1+\varepsilon)^3\psi)^{-1}\left(1-\beta(1+\varepsilon)^2\phi\right)^{-1}.$$

Furthermore

$$\|w(\cdot,t)\| + \|z(\cdot,t)\|_\infty \leq M_1 e^{-\delta t}\left(\|w_0\| + \|z_0\|_\infty\right), \tag{4.75}$$

with

$$M_1 := M_0(1 + \tfrac{1}{\sqrt{3}}(1+\varepsilon))e^D).$$

Next, we use the estimates of the backstepping transformations, i.e.,

$$\|w(\cdot,t)\| \leq \tilde{k}\|u(\cdot,t)\|, \tag{4.76}$$
$$\|u(\cdot,t)\| \leq \tilde{l}\|w(\cdot,t)\|, \tag{4.77}$$

$$\|z(\cdot,t)\|_\infty \leq \tilde{\gamma}\|u(\cdot,t)\| + \tilde{q}\|v(t,\cdot)\|_\infty, \tag{4.78}$$
$$\|v(\cdot,t)\|_\infty \leq \tilde{\delta}\|w(\cdot,t)\| + \tilde{p}\|z(t,\cdot)\|_\infty, \tag{4.79}$$

with

$$\tilde{k} := 1 + \left(\int_0^1 \left(\int_0^x |k(x,y)|^2 dy\right) dx\right)^{1/2},$$
$$\tilde{l} := 1 + \left(\int_0^1 \left(\int_0^x |l(x,y)|^2 dy\right) dx\right)^{1/2}$$

and

$$\tilde{\gamma} := \|\gamma(x,\cdot)\|_{L^2},$$
$$\tilde{q} := 1 + D \max_{0\leq x\leq 1} \int_0^x |q(x,y)| dy,$$
$$\tilde{\delta} := \|\delta(x,\cdot)\|_{L^2}$$

and

$$\tilde{p} := 1 + D \max_{0\leq x\leq 1} \int_0^x |p(x,y)| dy.$$

The proof of the inequalities (4.76),(4.77),(4.78) and (4.79) are quite similar and we will just provide the one of (4.77). It is based on the inverse backstepping transformation (4.8), the triangular inequality and the Cauchy-Schwarz inequality, and we have:

$$\int_0^1 |u(x,t)|^2 dx \leq \int_0^1 |w(x,t)|^2 dx + \int_0^1 \left[\left(\int_0^x |l(x,y)|^2 dy\right)^{1/2} \left(\int_0^1 |w(y,t)|^2 dy\right)^{1/2}\right]^2 dx,$$
$$\|u(\cdot,t)\|_{L^2}^2 \leq \|w(\cdot,t)\|_{L^2}^2 \left(1 + \int_0^1 \left(\int_0^x |l(x,y)|^2 dy\right) dx\right).$$

Taking the square root of both sides, we get:

$$\begin{aligned}\|u(\cdot,t)\|_{L^2} &\leq \sqrt{\|w(\cdot,t)\|_{L^2}^2} \cdot \sqrt{1 + \left(\int_0^1 \left(\int_0^x |l(x,y)|^2 dy\right) dx\right)} \\ &\leq \|w(\cdot,t)\|_{L^2} \cdot \left(1 + \sqrt{\left(\int_0^1 \left(\int_0^x |l(x,y)|^2 dy\right) dx\right)}\right) \\ &= \tilde{l}\|w(\cdot,t)\|.\end{aligned}$$

Hence, from (4.75), along with (4.76)-(4.77) and (4.78)-(4.79), we finally obtain, for all $t \geq 0$

$$\|u(\cdot,t)\|_{L^2} + \|v(\cdot,t)\|_\infty \leq Me^{-\delta t}\left(\|u_0\|_{L^2} + \|v_0\|_\infty\right), \tag{4.80}$$

with

$$M := M_1 M_2 M_3,$$

where M_1 is as in (4.75),

$$M_2 := \max\{(\tilde{l} + \tilde{\delta}), \tilde{p}\}$$

and

$$M_3 := \max\{(\tilde{k} + \tilde{\gamma}), \tilde{q}\}.$$

This concludes the proof. ♢

Remark 4.2 The small-gain condition $\beta\left(\frac{1+\sqrt{3}}{\sqrt{3}}\right)e^D < 1$ from (4.60) is a delay-dependent condition which involves also the parameter β of the triggering condition (4.28). Notice that the larger D, the smaller β should be chosen to preserve the theoretical guarantees. This implies sampling faster, thus the boundary control input is updated more often. It is worth mentioning, however, that larger values of β can be taken (eventually violating (4.60)) and may be used in practice since the obtained estimates are conservative.

4.3.6 Simulation example

We illustrate the results by considering the reaction-diffusion PDE (4.20)

$$\begin{cases} u_t(x,t) = u_{xx}(x,t) + \lambda u(x,t), \\ u(0,t) = 0, \\ u(1,t) = v(0,t), \\ v_t(x,t) = \frac{1}{D} v_x(x,t), \\ v(1,t) = U_d(t), \end{cases}$$

with $\lambda = 12$, input delay $D = 0.5$, and initial condition

$$u_0(x) = \sum_{n=1}^{3} \frac{\sqrt{2}}{n} \sin(n\pi x) + 3(x^2 - x^3), \quad v_0(x) = 0, x \in [0,1].$$

For the numerical simulations, we implement an implicit Euler scheme for the parabolic subsystem combined with the two-step Lax–Wendroff method for the hyperbolic subsystem. The discretization with respect to space and time is done with steps $\Delta_x = 1 \times 10^{-3}$ and $\Delta_t = 1 \times 10^{-4}$, respectively. We run simulations on a time horizon $T = 1$.

We stabilize the system on events under the event-triggered control (4.28)-(4.29) where the parameter $\beta = 0.05$ is selected according to (4.60) in Theorem 4.3.5. Conditions (4.62)-(4.63) (used just in the stability analysis) are also verified with e.g., $\varepsilon = 0.1$ and $\mu = 0.1$. Since the event-triggering condition is monitored in terms of the states of the target system (4.21) and the kernel of the inverse transformation, their numerical solutions are also found according to (4.3)-(4.4), along with the explicit expressions (4.11)-(4.13). In addition, using (4.42), we compute the minimal dwell-time $\tau = 7.3 \times 10^{-3}$.

Figures 4.1 and 4.2 show the numerical solution of the closed-loop system (4.20) with continuous-time boundary control (4.17) and with event-triggered control (4.28)-(4.29), respectively. The time-evolution of control functions under the continuous and event-triggered case is shown in Figure 4.3. The control value is kept constant between event times and updated according to the triggering law. We obtained in total 29 updates within the considered time horizon.

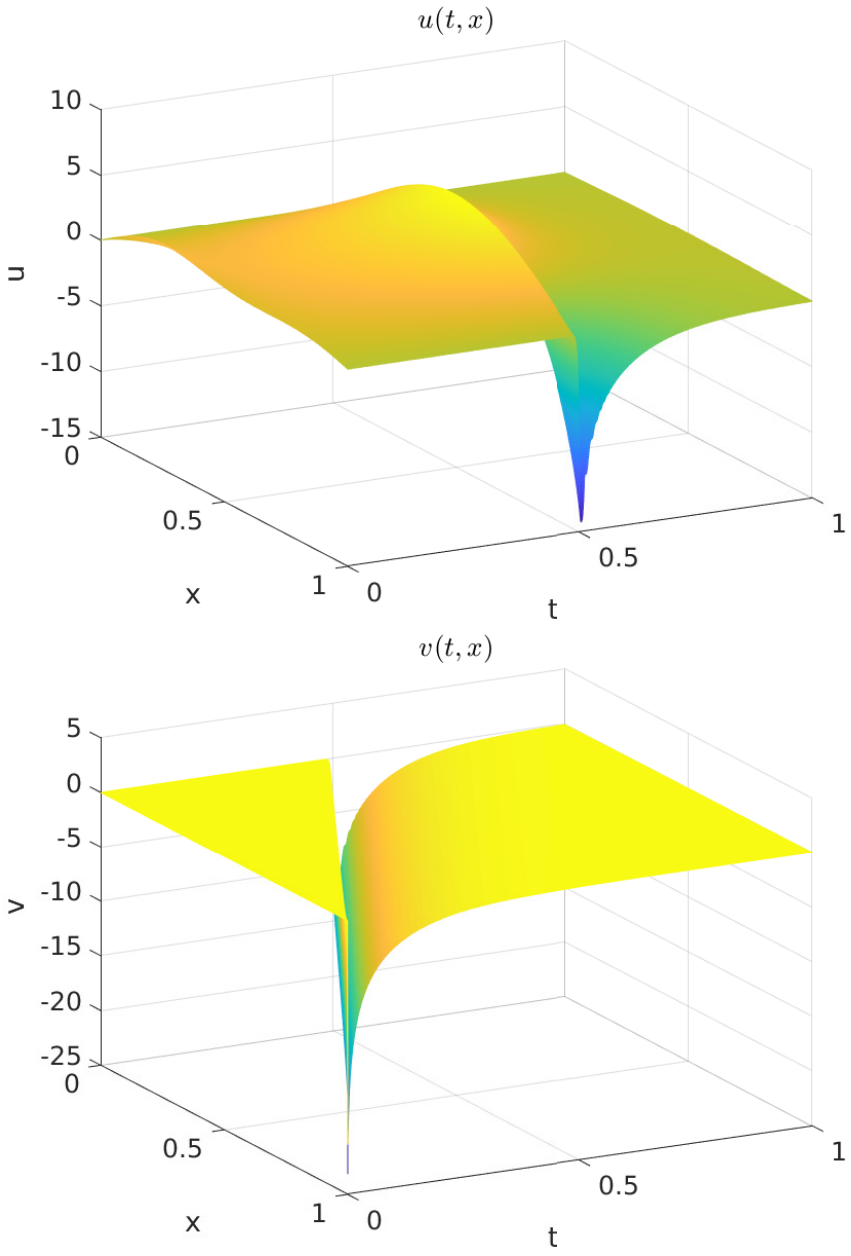

Figure 4.1: Numerical solutions of the closed-loop system with $\lambda = 12$, delay $D = 0.5$, initial condition $u_0(x) = \sum_{n=1}^{3} \frac{\sqrt{2}}{n} \sin(n\pi x) + 3(x^2 - x^3)$, $v_0(x) = 0$, $x \in [0,1]$ and under the continuous-time boundary control (4.17) . The parabolic subsystem is depicted on the top and the transport PDE is depicted on the bottom.

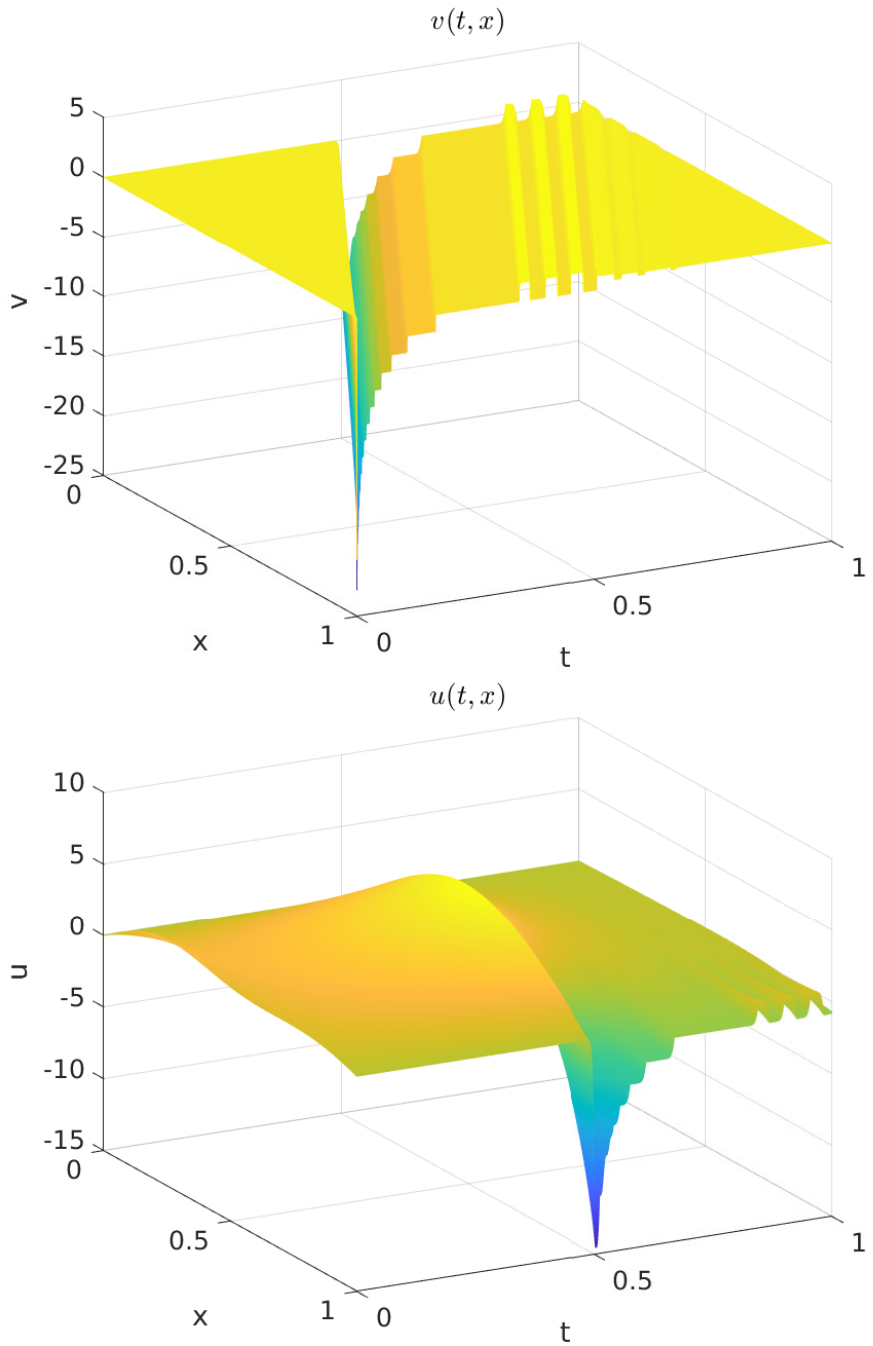

Figure 4.2: Numerical solutions of the closed-loop system with $\lambda = 12$, delay $D = 0.5$, initial condition $u_0(x) = \sum_{n=1}^{3} \frac{\sqrt{2}}{n} \sin(n\pi x) + 3(x^2 - x^3)$, $v_0(x) = 0$, $x \in [0, 1]$ and under the event-triggered control (4.28)-(4.29) with $\beta = 0.05$. The parabolic subsystem is depicted on the top and the transport PDE is depicted on the bottom. The piecewise-constant signal appears at the boundary and the discontinuities propagate along the spatial domain.

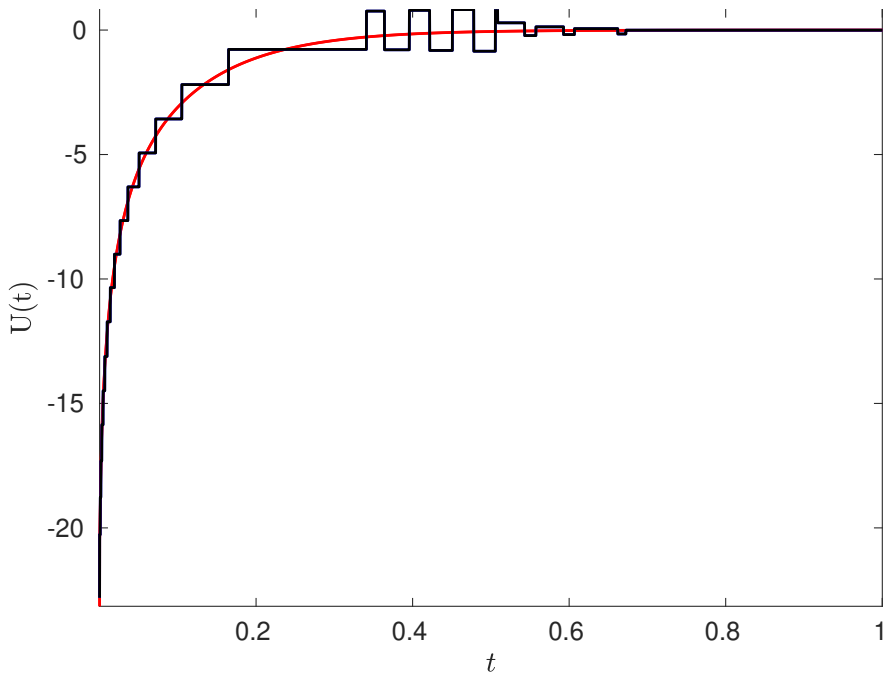

Figure 4.3: Time-evolution of the continuous-time boundary control (4.17) (red line) and the event-triggered boundary control(4.28)-(4.29) (black line).

4.4 Conclusion and perspectives

An event-triggered boundary control was proposed for the stabilization of a 1-D reaction diffusion equation with input delay. The delay is treated as a transport PDE, thus the problem is reformulated as a cascade PDE-PDE controlled system. We performed emulation on the backstepping control and proposed a state dependent event-triggering mechanism. The existence of a minimal dwell-time (independent of the initial conditions) between two triggering times is proved in order to exclude the Zeno phenomenon. Henceforth, we ensured the well-posedness of the closed-loop system and, thanks to the Input-to-State stability theory for PDEs and small-gain arguments, the global exponential stability is guaranteed.

In future work, one may design event-triggering mechanism for the following context.

• *Boundary stabilization of a class of linear parabolic partial integro-differential equations (PIDEs) in one dimension*

Using [109] and [69, Chapter 14] one can extend the results in this chapter to the following partial integro-differential equations defined in $(0,1)\times\mathbb{R}^+$:

$$\begin{cases} \partial_t z(x,t) = \partial_{xx} z(x,t) + b(x)\partial_x z(x,t) + \lambda_1(x) z(x,t) \\ +g_1(x) z(0,t) + \displaystyle\int_0^x f_1(x,y) z(y,t) dy \\ z(0,t) = 0 \\ z(1,x) = v(1,t) \\ z(0,t) = z_0(x) \\ \partial_t v(x,t) = \partial_x v(x,t) + \lambda_2(x) v(x,t) + g_2(x) v(0,t) + \int_0^x f_2(x,y) v(y,t) dy \\ v(1+D,t) = U(t), \end{cases}$$

where $b(x), \lambda_1(x), \lambda_2(x), g_1(x), g_2(x), f_1(x,y)$ and $f_2(x,y)$ are arbitrary continuous functions.

• *Delay compensated event-triggered gain scheduling for the reaction-diffusion system with time and space varying reaction coefficients*

Inspired by [56], an event-triggering mechanism can be design for the scalar reaction-diffusion system with time and space-varying reaction coefficient subject to input delay $D > 0$:

$$\begin{cases} u_t(t,x) = \varepsilon u_{xx}(t,x) + \lambda(t,x) u(t,x), & \text{for } (t,x) \in \mathbb{R}^+ \times (0,1), \\ u_x(t,0) = q u(t,0), & \text{for } t \in \mathbb{R}^+, \\ u(t,1) = U(t-D) \quad \text{or} \quad u_x(t,1) = U(t-D), & \text{for } t \in \mathbb{R}^+ \\ u(0,x) = u_0(x) & \text{for } x \in (0,1)\ . \end{cases}$$

where $\varepsilon > 0$, $q \in (-\infty, +\infty]$ (the case $q = +\infty$ is interpreted as the Dirichlet case), and $\lambda \in \mathscr{C}^0(\mathbb{R}^+ \times [0,1])$ with $\lambda[t] \in \mathscr{C}^1([0,1])$. Here, $u : [0,\infty) \times [0,1] \to \mathbb{R}$ represents the system state, and $U(t) \in \mathbb{R}$ is the control input. Moreover, the reaction coefficient $\lambda \in \mathscr{C}^0(\mathbb{R}^+ \times [0,1])$ is bounded and Lipschitz with respect to time.

• *Observer-based event-based control for more complex coupled reaction-diffusion systems with varying coefficients and subject to input/output delays*

Consider in $(0,1)\times\mathbb{R}^+$ the system

$$\begin{cases} \partial_t z(x,t) = \dfrac{1}{h(x)}\left(-\dfrac{d}{dx}\left(a(x)\dfrac{d\varphi(x)}{dx}\right)+b(x)\varphi(x\right)z(x,t) \\ \partial_x z(t,0)=0 \\ \partial_x z(t,1)=U(t-D) \\ z(0,t)=z_0(x) \\ U(\theta)=\phi(\theta), \quad \theta\in[-D,0]. \end{cases}$$

where $h(x)$, $a(x)$, and $b(x)$ are real-valued, sufficiently smooth functions defined on $[0,1]$, with $h(x)>0$ and $a(x)>0$. The function $U(t)$ represents the control input, and $D>0$ denotes a time lag.

- *Boundary stabilization of First-order Hyperbolic PIDEs*

For $(x,t)\in(0,1)\times\mathbb{R}^+$, and $g(x)$ and $f(x,y)$ two known coefficient functions belonging to $C[0,1]$, consider the first-order PIDE with an input delay $D>0$

$$\begin{cases} u_t(x,t)=u_x(x,t)+g(x)u(0,t)\displaystyle\int_0^x f(x,y)u(y,t)dy \\ u(t,1)=U(t-D) \\ u(0,x)=u_0(x) \end{cases} \tag{4.81}$$

which can be written as a couple of two transport equations:

$$\begin{cases} u_t(x,t)=u_x(x,t)+g(x)u(0,t)\displaystyle\int_0^x f(x,y)u(y,t)dy \\ u(1,t)=v(0,t), \\ Dv_t(x,t)=v_x(x,t), \\ v(t,1)=U(t) \\ u(0,x)=u_0(x), \\ v(0,x)=v_0(x). \end{cases} \tag{4.82}$$

Thanks to the backstepping method, in [91] the following delay-compensated controller

$$U(t)=\int_0^1\gamma(1,y)u(y,t)\,dy+D\int_0^1 q(1-y)v(y,t)\,dy$$

allows to design a delay-adaptive feedback control (D is replaced with estimated delay $\hat{D}(t)$)

$$U(x,t)=\int_0^1\gamma(1,y,\hat{D}(t))u(y,t)\,dy+\hat{D}(t)\int_0^1 q(1-y,\hat{D}(t))v(y,t)\,dy$$

in order to exponentially stabilizes the system under the delay-adaptive control. One may design an event-triggering mechanism $U(x,t):=U(x,t_k)\quad\forall t\in(t_k,t_{k+1})$, where (t_k) is appropriately chosen according to some event/threshold.

5
Conclusion

In this thesis, we addressed the problem of exponential stability for different types of linear partial differential equations under event-triggering mechanisms. Specifically, we design static and dynamic event-triggered control to some linear PDE: the wave equation, the Schrödinger equation and a 1-D reaction-diffusion equation with input delay.

For the wave equation, we derived a sufficient matrix inequality-based condition for exponential stability under a static event-triggered damping (and anti-damping) controller using an adequate Lyapunov functional. Moreover, we ensured the avoidance of Zeno behavior by showing the absence of accumulation points in the update sequence and consequently we guarantee the well-posedness of the system.

Concerning the Schrödinger equation, a static and a dynamic event-triggering mechanism was proposed to determine when the stabilizing control needs to be updated in digital implementations, while reducing the use of computational resources. The event-triggering conditions are such that the exponential stability and well-posedness are maintained while the occurence of Zeno behavior is avoided.

For the 1-D reaction-diffusion equation with input delay, we formulated a cascade PDE-PDE controlled system by treating the delay as a transport PDE. We introduced a static event-triggering mechanism, ensuring the existence of a minimal dwell-time between triggering times to prevent the Zeno phenomenon. Thanks to the Input-to-State stability theory for PDEs and small-gain arguments, we guarantee global exponential stability for the closed-loop event-triggered control system.

This work paves the way for forthcoming research endeavors within the event-triggering control. Future works may consider the event-based control in the context of aeroelastic systems (modeling flow-induced vibration), dynamic boundary control, and boundary damping control for both linear and nonlinear wave and Schrödinger equations. The small-gain approach together with ISS technique used in the context of the 1-D reaction-diffusion equation with input delay could be extended to more general parabolic and hyperbolic systems including integro-differential systems, as well as reaction-diffusion PDE with time and space-varying reaction coefficients. Furthermore, it could be interesting to consider observer-based event-triggered control problem for the wave and the Schrödinger equations and apply event-based control techniques to other PDEs, such as nonlinear transport equations, Kuramoto-Sivashinsky equations, Navier–Stokes equations, Euler-Bernoulli equations, Ginzburg-Landau equations, and Beam's equation, among others. Event-based control in interconnected control systems and PDEs subject to input nonlin-

earity, such as saturation, could also be explored. Finally, one should consider the implementation of our algorithms in real-world examples, which requires precision in the PDEs, including coefficients, physical parameters, etc., and adaptation of the proposed event-triggered law.

A

Appendix: Useful inequalities and identities

Lemma A.1: Cauchy-Schwarz's inequality

For any $u, v \in L^2(\Omega)$ it holds

$$\int_\Omega u(x)v(x)dx \leq \|u\|\|v\|.$$

Lemma A.2: Poincaré's inequality [34]

Let Ω be a bounded, connected, open subset of $\mathbb{R}^n$, of class C^1. There exists a constant $C_\Omega > 0$, depending only on the dimension n and on the diameter of the domain Ω, such that for each function $z \in H_0^1(\Omega)$,

$$\|z\| \leq C_\Omega \|\nabla z\|.$$

Lemma A.3: Green's formula

Let $\Omega \subset \mathbb{R}^n, n \geq 2$, be a bounded domain with Lipschitz boundary, then for all $u \in H^2(\Omega)$ and $v \in H^1(\Omega)$,

$$\int_\Omega \nabla v(x) \cdot \nabla u(x)dx = -\int_\Omega v(x)\Delta u(x)dx + \int_{\partial\Omega} u(s)\,\nabla v(s) \cdot \nu(s)ds,$$

where ν is the outward pointing unit normal vector field.

Lemma A.4: Gronwall's Inequality

Let u be a real-valued continuous functions defined on an interval I of the form $[a, \infty)$ or $[a, b]$ or $[a, b)$ with $a < b$.
• **Differential form.** If u is differentiable in (a, b) and satisfies the differential inequality $\dot{u}(t) \leq \beta(t)u(t) \forall t \in I$, then

$$u(t) \leq u(a) \exp\left(\int_a^t \beta(s)ds\right), \forall t \in I.$$

• **Integral form.** If β is non-negative, α is non-decreasing and if u satisfies $\forall t \in I$, the integral inequality $u(t) \leq \alpha(t) + \int_a^t \beta(r)u(r)dr$ then

$$u(t) \leq \alpha(t) \exp\left(\int_a^t \beta(r)dr\right), \forall t \in I.$$

Lemma A.5: Fading memory Lemma [59, Lemma 7.1, page 186]

For every $\sigma > 0, M \geq 0, \varepsilon > 0$, there exists a constant $\delta \in (0, \sigma)$ with the following property: If $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ and $y : \mathbb{R}_+ \to \mathbb{R}_+$ are locally bounded functions for which there exists a constant $\gamma \geq 0$ such that the following inequality holds for all $t_0 \geq 0$ and $t \geq t_0$

$$\varphi(t) \leq M \exp\left(-\sigma(t - t_0)\right) \varphi(t_0) + \gamma \sup_{t_0 \leq t \leq t} (y(s)), \tag{A.1}$$

then the following inequality holds for all $t \geq 0$

$$\varphi(t) \leq M \exp(-\delta t)\varphi(0) + \gamma(1 + \varepsilon) \sup_{0 \leq t \leq t} (y(s) \exp(-\delta(t - s)). \tag{A.2}$$

Lemma A.6: Schur complement Lemma [10]

The Hermitian block matrix $\begin{pmatrix} Q & S \\ S^\top & R \end{pmatrix}$ is negative definite if and only if

$$Q \prec 0 \text{ and } R - S^\top Q^{-1} S \prec 0 \tag{A.3}$$

which is equivalent to

$$R \prec 0 \text{ and } Q - S Q^{-1} S^\top \prec 0. \tag{A.4}$$

Lemma A.7: S-Procedure, [10]

Let $F_0, \cdots, F_p$ be quadratic functions of the variable $\eta \in \mathbb{R}^n$:

$$F_i(\eta) = \eta^\top T_i \eta + 2uT_i\eta + v_i, \text{ where } T_i = T_i^\top, i = 0, \cdots, p.$$

We consider the following condition:

$$F_0(\eta) \geq 0 \text{ subject to } F_i(\eta) \geq 0, i = 1, \cdots, p. \tag{A.5}$$

If there exist $\tau_1 \geq 0, \cdots, \tau_P \geq 0$ such that for all η,

$$F_0(\eta) - \sum_{i=1}^{p} \tau_i F_i(\eta) \geq 0 \tag{A.6}$$

then (A.5) holds. In the case $p = 1$, the condition is necessary and sufficient.

Lemma A.8: Finsler's Lemma [10, 18]

Let $x \in \mathbb{R}^n, Q \in \mathbf{S}^n$ and $\mathcal{H} \in \mathbb{R}^{n \times m}$ such that $rank(\mathcal{H}) < n$. The following statements are equivalent:

- $\eta^\top Q \eta < 0,\ H\eta = 0,\ \forall \eta \neq 0$
- ${H^\perp}^\top Q H^\perp \prec 0$
- $\exists \mu \in \mathbb{R} : Q - \mu \mathcal{H}^\perp \mathcal{H}^\perp \prec 0$
- $\exists \mathcal{X} \in \mathbb{R}^{n \times m} : Q + \mathcal{X}\mathcal{H} + \mathcal{H}^\perp \mathcal{X}^\perp \prec 0$

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